MAT6480 - Assignment 4

Due: 10 November 2008, 10:00 AM

In everything that follows, unless otherwise specified A and B are sets in some commutative group $(Z,+)$. Recall that the doubling constant $\sigma(A) = |A+A|/|A|$, and the difference constant $\delta(A) = |A-A|/|A|$. We denote by $\operatorname{stab}(A)$ the group $\{h \in Z \;: Zh + A = A\}$.

1. Prove that $A + A = A$ if and only if A is a subgroup of Z.
2. Let $d \geq 1$ be an integer. Give an example of a set A of integers such that $|A+A| = 6^d$ and $A - A = 7^d$. (Hint: in $\mathbb{Z}^d$, $\{0,1,3\}^d$ works.)
3. Show that a non-empty subset of A can have doubling constant at most $\sqrt{\sigma(A)|A|/2}$.
4. (a) For each $N \in \mathbb{N}$, give examples of finite non-empty sets A, B of integers such that $\sigma(A), \sigma(B) \leq 2$, but $\sigma(A \cup B) \geq N/2$.
   (b) Show that $\sigma(A \cup B) \leq \sigma(A) + |B|$.
5. Show that for all $k \geq 1$ and $n \in \mathbb{N}$, $\sum_{i=0}^{n} \binom{k+i}{i} = \binom{k+n+1}{n}$.
6. (a) Prove that $Z/\operatorname{stab}(A)$ is a group.
   (b) If $\phi : Z \to Z'$ is a homomorphism between groups, show that the range $\phi(Z)$ is a group which is isomorphic to $Z/\phi^{-1}(0)$.
7. Show that Kneser's theorem implies the Cauchy-Davenport inequality.
8. Show that if $|A+B| < |A| + |B|$ then
$$|A+B| = |A + \operatorname{stab}(A+B)| + |B + \operatorname{stab}(A+B)| - |\operatorname{stab}(A+B)|.$$