BIPARTITE PLANES

by

Andrew Granville
Department of Mathematics and Statistics
University of Toronto, Toronto, Ontario, M5S-1A1

Alexandros Moisiadis
Department of Mathematics and Statistics
Queen's University, Kingston, Ontario, K7M-3N6

Rolf Rees
Department of Mathematics and Computer Science
Mount Allison University, New Brunswick, E0A-3C0

ABSTRACT

It is well-known that the biclique partition number of the complete graph on \( n \) vertices (i.e. the smallest number of complete bipartite graphs required to partition the edge set of \( K_n \)) is \( n-1 \).

In this paper we address the following problem: For which integers \( s, t \) and \( n \) with \( st=n/2 \) does the complete graph \( K_n \) admit a decomposition into \( (n-1) K_{s,t} \)’s?

1. INTRODUCTION

Let \( n>0 \) be an integer, \( K_n \) denote the complete graph on \( n \) vertices and \( \mathcal{G} \) be a class of graphs where \( K_n \in \mathcal{G} \). By a \( \mathcal{G} \)-plane of size \( n \) we will mean a decomposition \( D \) of the edge set of \( K_n \) into copies of a fixed graph \( G \in \mathcal{G} \) with the property that, for any decomposition \( D' \) of \( K_n \) into (not necessarily isomorphic) graphs from \( \mathcal{G} \), \( |D| \leq |D'| \).

For example, if \( n \) is of the form \( k^2 + k + 1 \) and \( \mathcal{C} \) is the class of all complete graphs except \( K_n \), then a \( \mathcal{C} \)-plane of size \( n \) is equivalent to a projective plane of order \( k \) (whenever this exists) since the smallest non-trivial decomposition of \( K_n \) into complete subgraphs always consists of exactly \( n \) graphs (the minimum decompositions are either near-pencils or projective planes). We will herein be concerned with the class \( \mathcal{C} \) of all complete bipartite graphs (bicliques).

It is well-known (see [3], [5] and [6]) that the biclique partition number of \( K_n \) is \( n-1 \), and we will denote by \( B(s,t;n) \) a decomposition of \( K_n \) into \( n-1 \) copies of \( K_{s,t} \). Thus we pose the following

**PROBLEM:** For which \( s \), \( t \) and \( n \) does there exist a \( B(s,t;n) \)?

The above problem was motivated by the following similar question posed by D. de Caen [1]: For which \( s \) and \( t \) with \( st = n-1 \) can the complete symmetric directed graph \( \tilde{K}_n \) be decomposed into \( n \) directed \( K_{s,t} \)'s (i.e. all arcs oriented from one bipartition to the other)? (This has some interesting applications to matrix decompositions, see D. de Caen and D. Gregory [2]). This latter problem admits to a simple solution.

**THEOREM 1.1:** Given any positive integers \( s \), \( t \) and \( n \) with \( st = n-1 \) there is a (cyclic) decomposition of \( \tilde{K}_n \) into directed \( K_{s,t} \)'s.

**PROOF:** Label the vertices of \( \tilde{K}_n \) with the elements of \( Z_n \).

Develop the following directed biclique \((S,T)\) modulo \( n \):

\[
S = \{ t, 2t, \ldots, st \} \quad \text{and} \quad T = \{ 0, 1, \ldots, t-1 \}.
\]
The undirected analogue, which is the problem that we are herein addressing, appears to be much more difficult. It is easy to see that in a $B(s,t;n)$, $n$ must be even. Furthermore, by considering the bicliques containing a given vertex $x \in V(K_n)$, we see that the g.c.d. $(s,t)$ must divide $n-1$. On the other hand since $st=n/2$ we clearly have that the g.c.d. $(s,t)$ divides $n$. This means that $s$ and $t$ must be relatively prime. We record these simple observations as:

**Lemma 1.2:** If there exists a $B(s,t;n)$ then

(i) $n$ is even, and

(ii) $s$ and $t$ are relatively prime.

Notwithstanding the trivial design $B(1,1;2)$ we can therefore assume that $0 < s < t < n$ in our notation $B(s,t;n)$. A $B(s,t;n)$ with $s=1$ will be called a *claw plane*. We will show that for each (even) $n$ there exists a claw plane of size $n$. We will also prove the somewhat surprising result that, there does not exist a $B(s,t;n)$ with $s=2$, for any $n$.

2. The Results

**Theorem 2.1:** For each even integer $n \geq 0$ there exists a claw plane of size $n$.

**Proof:** A claw plane of size $n$ is a $B(1,n/2;n)$. Label the vertices of $Z_n$ with $(=) \cup Z_{n-1}$. Develop the following biclique $(X,Y)$ modulo $(n-1)$: $X = \{ 0 \}$ and $Y = \{ \equiv, 1, 2, \ldots, (n/2) - 1 \}$. 

243
Before proceeding we shall have to look a little more carefully at the relationships between the vertices and bicliques in a \( B(s,t;n) \). We will assume from here on that \( s \geq 2 \). For each vertex \( x \) in \( V(K_n) \) and each \( i=s,t \) let \( x_i \) denote the number of \( K_{s,t} \)'s whose bipartition of size \( i \) contains \( x \); we will then say that vertex \( x \) has type \( (x_s,x_t) \).

Now we clearly have

\[
sx_t + tx_s = n - 1 = 2st - 1
\]

whence

\[
\begin{align*}
x_s &= -(1/t) \pmod{s} \\
x_t &= -(1/s) \pmod{t}
\end{align*}
\]

For ease of expression let \( \alpha(a,b) \) denote the least positive residue of \( -(1/a) \pmod{b} \), where \( a \) and \( b \) are relatively prime. Then:

**Lemma 2.2:** For any relatively prime integers \( a \) and \( b \), where \( a,b \geq 1 \), we have that \( a\alpha(a,b) + b\alpha(b,a) = ab - 1 \).

**Proof:** Consider the expression \( \left[ 1 + b\alpha(b,a) \right]/a \). From the definition of \( \alpha(b,a) \) it follows immediately that this expression is an integer between 1 and \( b-1 \), whence so is \( \left[ ab - 1 - b\alpha(b,a) \right]/a \). But this latter expression is clearly congruent to \( -(1/a) \pmod{b} \); that is, \( \left[ ab - 1 - b\alpha(b,a) \right]/a = \alpha(a,b) \). Rearranging we get \( a\alpha(a,b) + b\alpha(b,a) = ab - 1 \), as desired. \( \blacksquare \)

**Lemma 2.3:** In a \( B(s,t;n) \) with \( s \geq 2 \) there are exactly \( 2s\alpha(s,t) + 1 \) vertices of type \( (s+\alpha(t,s),\alpha(s,t)) \) and \( n - 1 - 2s\alpha(s,t) \) vertices of type \( (\alpha(t,s),t+\alpha(s,t)) \).
PROOF: From expressions (2.1), (2.2) and Lemma 2.2 it follows that for any vertex \( x \), either

(i) \( x_s = \alpha(t,s) \) and \( x_t = t+\alpha(s,t) \), or

(ii) \( x_s = s+\alpha(t,s) \) and \( x_t = \alpha(s,t) \).

Let \( y \) be the number of vertices of type (i) and \( z \) be the number of vertices of type (ii). By noting that

\[
\sum_{x \in V(K_n)} x_t = t(n-1)
\]

we obtain the system

\[
\begin{align*}
(t+\alpha(s,t))y+\alpha(s,t)z &= t(n-1) \\
y+z &= n = 2st
\end{align*}
\]

which yields \( y = n-1-2s\alpha(s,t) \) and \( z = 2s\alpha(s,t)+1 \) as asserted. ■

REMARK: By using Lemma 2.2 we can rewrite \( y = n-1-2s\alpha(s,t) \) as \( y = 2t\alpha(t,s)+1 \). In particular there are vertices of both types represented; since \( s=t \) a \( B(s,t;n) \) can therefore never be balanced (in the sense of Huang and Rosa [4]).

We are now ready to prove the following.

THEOREM 2.4: There does not exist a \( B(s,t;n) \), with \( s = 2 \), for any \( n \).

PROOF: Suppose if possible that we have a \( B(2,n/4;n) \). From Lemma 2.3 there are \( 4\left(\frac{(n/4)-1}{2}\right)+1 = \frac{n}{2} \cdot 1 \) vertices of type \( (3,(n-4)/8) \) and \( (n/2)+1 \) vertices of type \( (1,(3n-4)/8) \).
Let $H$ denote the set of vertices of the former type and $J$ the set of vertices of the latter type. For each vertex $j \in J$ there is a unique biclique $B_j$ whose bipartition of size 2 contains $j$. Since the set \( \{ B_j : j \in J \} \) must pick up all edges joining pairs of vertices in $J$ it follows that:

(i) If $j_1 \neq j_2$ then $B_{j_1} \neq B_{j_2}$, else the edge joining $j_1$ to $j_2$ could not be covered, and

(ii) For each $j \in J$ the bipartition of size $n/4$ in $B_j$ is a subset of $J$, because \( \left\lfloor \frac{|J|}{2} \right\rfloor = \left(\frac{n}{2}+1\right)(n/4) = |\{ B_j : j \in J \}|(n/4) \).

From (ii) we see that a vertex in $H$ can be contained in the bipartition of size 2 in at most two $B_j$'s. On the other hand, since $|J| > |H|$, (i) implies that there is a vertex $h \in H$ which is contained in the bipartition of size 2 in exactly two $B_j$'s.

Let $G$ be that subgraph of $K_n$ obtained by removing all edges covered by the $B_j$'s. Then the edges of $G$ are being partitioned by the remaining $(n/2)$-2 bicliques $C_1, \ldots, C_{(n/2)-2}$ in the $B(2,n/4;n)$. But $G$ contains all the edges joining pairs of vertices in $H$, so that by the Graham-Pollack theorem,

(iii) Each biclique $C_1, \ldots, C_{(n/2)-2}$ contains at least one edge joining a pair of vertices in $H$.

Now, in $G$, $h$ is adjacent to exactly one vertex $j_0 \in J$. Without loss of generality let $C_1$ be the biclique containing the edge $hj$ and let $(h, h')$ be the bipartition of size 2 in $C_1$, with $h' \in H$. Note that $(h, j_0)$ must have been the bipartition of size 2 in some $B_j$, so the same cannot be true of $(h', j_0)$ as $j_0$ has type $(1,(3n-4)/8)$. This means that $h'$ is adjacent to either $(n/4)+1$ or $(n/2)+1$ vertices of $J$ in $G$, depending on whether it was contained in the bipartition of
size 2 in one or no $B_j$'s. In the first case the $n/4$ edges joining $h'$ to vertices in $J$ which remain after removing $C_1$ from $G$ must be covered by bicliques from $C_2, \ldots, C_{(n/2)-2}$, each with the property that its bipartition of size 2 contains $h'$. But $h'$ has type $(3, (n-4)/8)$ so that there can be only one such biclique, say $C_2$. Then the bipartition of size $n/4$ in $C_2$ must consist of the $n/4$ vertices in $J$ to which $h'$ is still adjacent and this means that $C_2$ contains no edges joining pairs of points in $H$, contradicting (iii). A similar argument rules out the second case. Thus no $B(2, n/4; n)$ can exist. \[ \]

Finally, an immediate consequence of Lemma 1.2 and Theorem 2.4 is

**Corollary 2.5:** Let $n = 2q$ or $4q$ where $q$ is a prime power. Then the only bipartite planes of size $n$ are the claw planes.

### 3. SUMMARY

We do not at present know of a single example of a bipartite plane that is not a claw plane. From Corollary 2.5 the smallest possible example would be a $B(3,4;24)$.

We would also like to mention a similar problem, posed by D. de Caen.

**Problem:** For which integers $k$ can the complete graph $K_n$, with $n = \binom{k}{2} + 1$, be decomposed into $n-1$ complete bipartite subgraphs, each containing a total of $k$ vertices?

It so happens that the existence of such a decomposition is a necessary one in order that a signed symmetric $(n,k,2)$-BIBD exists.
REFERENCES


graph into complete bipartite subgraphs, Ars Comb. 23(B) (198
139-146.

squashed cubes, Lect. Notes in Math. 303 (Springer, New York
1973) 99-110.

designs, Utilitas Math. 4 (1973) 55-75.


[6] H. Tverberg, On the decomposition of $K_n$ into complete bipartite

248