Additional material for Risks & Rewards February 2018 paper titled Hedging variable annuities: How often should the hedging portfolio be rebalanced?

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This document justifies the expression for Δ_t given in the paper. The formula is reproduced below with a small correction in red.

$$\Delta_t = \frac{\partial V_t}{\partial S_t} = \underbrace{\Delta_t^{\text{put}}}_{\text{term 1}} - \underbrace{\left[(1-\alpha/252)^t - (1-\alpha/252)^T\right]}_{\text{term 2}}, \tag{1}$$

where

$$\begin{aligned}\Delta_t^{\text{put}} &= -(1-\alpha/252)^T \Phi(-d_1),\\ d_1 &= \frac{\log\left(\frac{A_t(1-\alpha/252)^{T-t}}{G}\right) + (r+\sigma_t^2/2)(T-t)/252}{\sigma_t\sqrt{(T-t)/252}}.\end{aligned}$$

The correction reflects the fact that r and σ_t^2 are annualized quantities, whereas time is expressed in trading days in the paper.

To justify the above expression, first recall that the GMAB rider creates a liability for the insurer in the form of a long-term put option guarantee. Term 1 in Eq. (1) is defined as the delta of this guarantee (with respect to S_t).

The fair value of the guarantee at time t, denoted by Π_t, is given by

$$\Pi_t := e^{-r(T-t)/252}\,\mathrm{E}^{\mathbb{Q}}\left[\max(G - A_T, 0) \mid \mathcal{F}_t\right],$$

where $\mathbb{Q}$ denotes the risk-neutral measure, and $\mathcal{F}_t$ represents the available market information up to time t. Since

$$A_t = S_t(1-\alpha/252)^t, \quad t \geq 0,$$

we can write

$$\begin{aligned}\Pi_t &= e^{-r(T-t)/252}\,\mathrm{E}^{\mathbb{Q}}\left[\max(G - S_T(1-\alpha/252)^T, 0) \mid \mathcal{F}_t\right]\\ &= (1-\alpha/252)^T \times \underbrace{e^{-r(T-t)/252}\,\mathrm{E}^{\mathbb{Q}}\left[\max(G(1-\alpha/252)^{-T} - S_T, 0) \mid \mathcal{F}_t\right]}_{=\text{Black-Scholes put price with strike } K=G(1-\alpha/252)^{-T}}.\end{aligned}$$

Term 1 in Eq. (1) is defined as the partial derivative of Π_t with respect to S_t, that is,

$$\frac{\partial \Pi_t}{\partial S_t} = (1-\alpha/252)^T \times \underbrace{\frac{\partial}{\partial S_t}\left(e^{-r(T-t)/252}\,\mathrm{E}^{\mathbb{Q}}\left[\max(G(1-\alpha/252)^{-T} - S_T, 0) \mid \mathcal{F}_t\right]\right)}_{\text{Black-Scholes put delta}}. \quad (2)$$

Notice that the underlying put option in Eq. (2) is written on asset S_t and has strike $K = G(1-\alpha/252)^{-T}$. From standard results on the Black-Scholes model, we therefore have that

$$\frac{\partial \Pi_t}{\partial S_t} = (1-\alpha/252)^T \times -\Phi(-d_1),$$

where $-\Phi(-d_1)$ is the well-known expression for the Black-Scholes put delta, and

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_t}{K}\right) + (r+\sigma_t^2/2)(T-t)/252}{\sigma_t\sqrt{(T-t)/252}} \\ &= \frac{\log\left(\frac{A_t(1-\alpha/252)^{-t}}{G(1-\alpha/252)^{-T}}\right) + (r+\sigma_t^2/2)(T-t)/252}{\sigma_t\sqrt{(T-t)/252}} \\ &= \frac{\log\left(\frac{A_t(1-\alpha/252)^{T-t}}{G}\right) + (r+\sigma_t^2/2)(T-t)/252}{\sigma_t\sqrt{(T-t)/252}}. \end{aligned}$$

It remains to justify term 2 in Eq. (1). This term is defined as the partial derivative with respect to S_t of the expected present value (under $\mathbb{Q}$) of fees that will be collected by the insurer from time t. Mathematically, this expected present value, denoted by Ψ_t, is equal to

$$\begin{aligned} \Psi_t &:= e^{-r(T-t)/252}\,\mathrm{E}^{\mathbb{Q}}\left[\sum_{n=t}^{T-1} A_n(\alpha/252)e^{r(T-n)/252} \,\middle|\, \mathcal{F}_t\right] \\ &= (\alpha/252)\,\mathrm{E}^{\mathbb{Q}}\left[\sum_{n=t}^{T-1} S_n(1-\alpha/252)^n e^{-r(n-t)/252} \,\middle|\, \mathcal{F}_t\right] \\ &= (\alpha/252)\sum_{n=t}^{T-1}(1-\alpha/252)^n \underbrace{\mathrm{E}^{\mathbb{Q}}\left[S_n e^{-r(n-t)/252} \mid \mathcal{F}_t\right]}_{=S_t \text{ by martingale property}} \\ &= S_t(\alpha/252) \times (1-\alpha/252)^t \left(\frac{1-(1-\alpha/252)^{T-t}}{(\alpha/252)}\right) \\ &= S_t\left[(1-\alpha/252)^t - (1-\alpha/252)^T\right]. \end{aligned}$$

Therefore, term 2 is given by

$$\frac{\partial \Psi_t}{\partial S_t} = \left[(1-\alpha/252)^t - (1-\alpha/252)^T\right].$$