Towards quantization of the Morse complex

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Abstract of a talk given at the workshop in Oberwolfach, July 23rd - July 29th 2006.

I. Recalls from Morse theory. Fix: L a compact, closed manifold; $f : L \to \mathbb{R}$ a Morse function, g a generic Riemannian metric on L. The Morse complex has the form

$$C(f) = (\mathbb{Z}/2 < \mathrm{Crit}(f) >, d) \ .$$

The main point is that $d^2 = 0$ which is due to the fact that broken trajectories between critical points of indexes different by 2 are in bijection with the ends of 1-dimensional moduli spaces of flow lines of $-\nabla_g(f)$. This complex computes the homology of L. One major advantage of the construction is that the algebraic structure is simple so that the notion extends easily leading to, for example, Floer theory. Simplicity becomes also a disadvantage as the algebra is *so* simple that many interesting phenomena can not be encoded in it.

Two such examples to be discussed here:

i. Higher dimensional moduli spaces of trajectories.
ii. Bubbling.

II. Higher Dimensional Moduli Spaces. Let the negative gradient flow be $\gamma : \mathbf{R} \times L \to L$. The stable/unstable manifold(s) of $x \in Crit(f)$ are $W^s(x) = \{y \in L : \lim_{t\to\infty} \gamma_t(y) = x\}$, and $W^u(x) = \{y \in M : \lim_{t\to-\infty} \gamma_t(y) = x\}$. Generically, $M(P,Q) = W^u(P) \cap W^s(Q) \cap f^{-1}(a)$ is a manifold of dimension $ind(P) - ind(Q) - 1$.

Natural problem 1. "Measure" the connecting manifolds $M(P,Q)$, $ind(x) - ind(y) > 1$.

A few results:

- Starting point due to Franks [5]: $M(P,Q)$ is a framed manifold. If P, Q consecutive in the flow, then $M(P,Q)$ is a closed manifold whose cobordism class is computable.

- Loop representation of moduli spaces of flow lines (O.C., 1998): there is a natural map $l_{P,Q} : M(P,Q) \to \Omega L$ obtained as follows (ΩX is the space of based Moore loops). First, identify all critical points of f to a single base point (by contracting to a point a simple path that goes through all critical points). The resulting quotient space has the same homotopy type as L. Then associate to each flow line from P to Q the closed loop it defines in this quotient space. A framed bordism class is now defined $[M(P,Q)] \in \Omega_*^{fr}(\Omega L)$ and is again computable when $M(P,Q)$ is closed [2].

Natural problem 2. How to measure $M(P,Q)$ if P and Q are not consecutive?

Due to broken orbits $M(P,Q)$ is, in general, non-compact but there is a *natural compactification* $\overline{M}(P,Q)$ which is a (compact) manifold so that:

$$\partial\overline{M}(P,Q) = \bigcup_R \overline{M}(P,R) \times \overline{M}(R,Q) \quad (1)$$

Moreover, (1) is compatible with the maps $l_{-,-}$ so that these maps provide a representation of the moduli spaces $M(-,-)$ inside ΩL. A key idea at this point (J.-F. Barraud, O.C. [1]) is to *enlarge the ring over which the Morse complex is defined.* Take $\mathcal{R} = S_*(\Omega M)$ where $S_*(-)$ are cubical chains and define the *extended* Morse complex:

$$\mathcal{C}(f) = (\mathcal{R} \otimes \mathbb{Z}/2 < Crit(f) > \ , \ \delta) \ , \ \delta x = \sum_y a_{xy} \otimes y$$

where, essentially, the a_{xy} represent the fundamental classes of $M(x,y)$ rel boundary so that if we put $A = (a_{xy})$, then (1) gives $dA = A^2$ and so $\delta^2 = 0$. There is a natural filtration

$$F^k\mathcal{C} = \mathcal{R} \otimes \mathbb{Z}/2 < Crit_{\leq k}(f) >$$

which leads to a spectral sequence E^r. The remarkable property of this is that E^r is invariant for $r \geq 2$ - these terms are in fact identified with the terms of the Serre spectral sequence of the path-loop fibration over L - and the differentials represent higher dimensional moduli spaces. Moreover, the construction is "robust" and carries over to Lagrangian Floer theory (J-strips instead of flow lines) *when* $\omega|_{\pi_2(M,L)} = 0$ which leads to symplectic applications.

III. Difficulties with relative Gromov-Witten invariants. Let now (M^{2n}, ω) be symplectic, $L^n \hookrightarrow M$ a *closed* Lagrangian in M. Fix $\lambda \in \pi_2(M,L)$. For $\forall$ J almost complex structure compatible with ω consider the moduli spaces of J-disks:

$$\mathcal{M}(\lambda, J) = \{u : (D^2, S^1) \to (M, L) : \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0; \ [u] = \lambda\}$$

Recall that J-holomorphic disks are quantified: $\exists$ smallest energy and there are only finitely homotopy classes containing J-disks below any fixed, positive energy value.

Natural problem 3. Count J-disks through cycles !

Under good circumstances $\mathcal{M}(\lambda, J)$, is a manifold of dimension $n + \mu(\lambda) - 3$ where $\mu(\lambda) =$ Maslov index. However, $\mathcal{M}(\lambda, J)$ compactifies to a manifold *with boundary* and due to bubbling the count is not invariant.

Instead, one may construct a homology theory - *cluster homology* (O.C., F. Lalonde, 2004 [3]). The main idea in this construction is to *use "quantized" flow lines which combine negative gradient flow lines and J-holomorphic disks (and spheres)* in an arrangement modeled on trees where the vertices are replaced by pseudo-holomorphic objects and the edges by flow lines of f. Such objects are assembled in *cluster moduli* spaces.

Remark. a. "Linear" such objects have been considered by Oh [6] (following an idea of Fukaya) in the 1990's. These linear objects suffice in the monotone case when $\mu_{min} \geq 2$.

b. An alternative way to deal with the bubbling of disks is due to Fukaya-Oh-Ono-Ohta (2000) [4] and also leads to a homology theory.

c. The transversality required for the regularity of cluster moduli spaces is work in progress along two different methods, the first due to Hofer - Wysocki - Zehnder and, the second to Cieliebak - Mohnke.

We then define a differential graded commutative rational algebra: $\mathcal{C}l(L, J, f) = (S(\mathbb{Q} < \mathrm{Crit(f)} >) \otimes \Lambda, d)$ (rational coefficients are necessary here !) where $S(V)$ is the free commutative DGA on the vector space V, Λ is an appropriate Novikov ring and d counts elements in 0-dimensional cluster moduli spaces. We use $\mathcal{R}' = \mathcal{C}l(L, J, f)$ as a "rich" ring which encodes bubbling and can then define Morse-Floer theory over it (for oriented, relative spin Lagrangians):

$$\mathbb{F}C_*(L, J, H, f) = (\mathcal{R}' \otimes \mathbb{Q} < P_0^H >, D')$$

where $H : M \times [0, 1] \to \mathbb{R}$ hamiltonian, P_0^H are time-1 contractible orbits of X_H with ends on L.

It is likely that a theory dealing simultaneously with the phenomena described in II, and III is possible and will lead to interesting applications.

A few applications of various parts of this machinery and related structures have been mentioned in the talk. I will list here only two examples which are true for monotone Lagrangians L with minimal Maslov class at least 2 (P. Biran, O.C. 2006): in other words, $\exists \rho > 0$ so that $\omega(\lambda) = \rho\mu(\lambda)$, $\forall \lambda \in \pi_2(M, L)$) and $\mu_{min} \geq 2$.

i. If $L = T^n$, then $HF_*(L) = 0$ or $HF_*(L) = H_*(L; \mathbb{Z}/2) \otimes \Lambda$. In the first case the Gromov radius, R, of L verifies $\pi R^2/2 \leq 2\rho$ (here $H(; \mathbb{Z}/2)$ is singular homology; $HF(-)$ is Floer homology, Λ is the appropriate Novikov ring).

ii. Assume $L \subset \mathbb{C}P^n$ verifies $HF_*(L) \neq 0$. If there is a symplectic embedding of a standard ball $B(r) \hookrightarrow \mathbb{C}P^n \backslash L$, then

$$\pi r^2 \leq \frac{n}{n+1} \ .$$

(Normalization: the maximal symplectic ball in $\mathbb{C}P^n$ is so that $\pi r^2 = 1$.)

References

[1] J.-F. Barraud, O. Cornea, *Lagrangian Intersections and the Serre spectral sequence*, preprint 2003, to appear Annals of Mathematics.

[2] O. Cornea, *Homotopical Dynamics II: Hopf invariants, smoothings and the Morse complex*, Ann. Scient. Ec. Norm. Sup. **35**, (2002) 549–573.

[3] O. Cornea, F. Lalonde, *Cluster homology*, arXiv:math.SG/0508345 August 2005.

[4] K. Fukaya, Y-G. Oh, H. Ohta, K. Ono, *Lagrangian intersection Floer theory - anomaly and obstruction* Preprint

[5] J.Franks *Morse-Smale flows and homotopy theory*, Topology **18**, (1979), 199–215.

[6] Y-G. Oh, *Relative Floer and quantum cohomology and the symplectic topology of Lagrangian submanifolds* Contact and symplectic geometry (Cambridge, 1994), , Publ. Newton Inst., **8**, Cambridge Univ. Press, Cambridge, 1996, 201–267