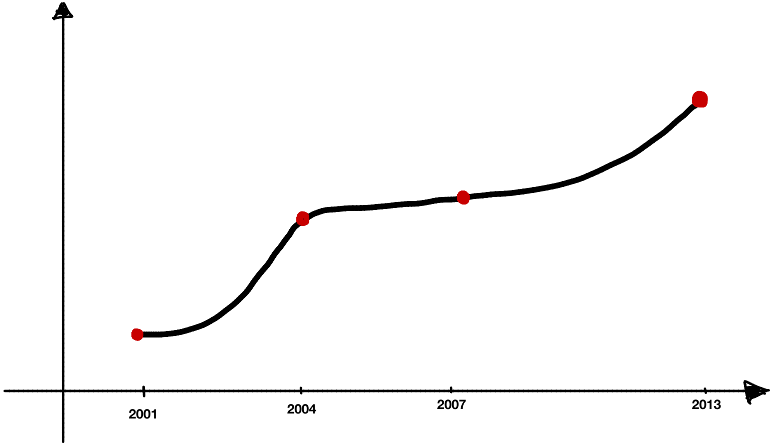
Lagrangian cobordism.

Octav Cornea
based on joint work with Paul Biran

D-Days, ETH

Happy Birthday Dietmar!

Mathematical debt to Dietmar



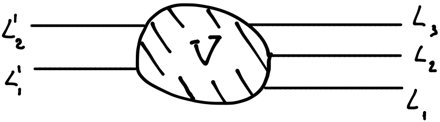
Definition (*Arnold '80*)

(M, ω) symplectic manifold; $(L_1, \ldots, L_k)$, $(L'_1, \ldots, L'_{k'})$ two families of closed, connected Lagrangian submanifolds $\subset M$.
A Lagrangian cobordism:

$V : (L_i) \to (L'_j)$ is a Lagrangian $V \subset (\mathbb{C} \times M, \omega_0 \oplus \omega)$ so that

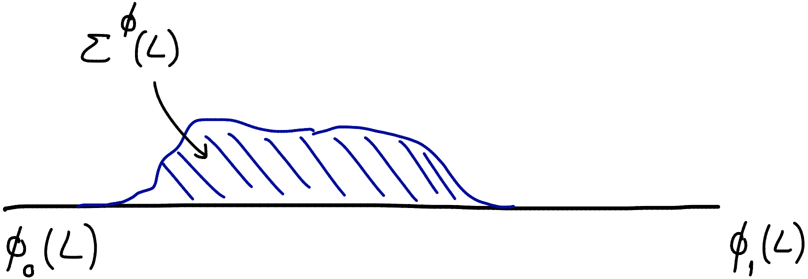
$$V|_{[1,\infty)\times\mathbb{R}\times M} = \cup_i [1, \infty) \times \{i\} \times L_i$$
$$V|_{(-\infty,0]\times\mathbb{R}\times M} = \cup_j (-\infty, 0] \times \{j\} \times L'_j \ .$$

If $\pi : \mathbb{C} \times M \to \mathbb{C}$ is the projection, $\pi(V)$ looks like this:

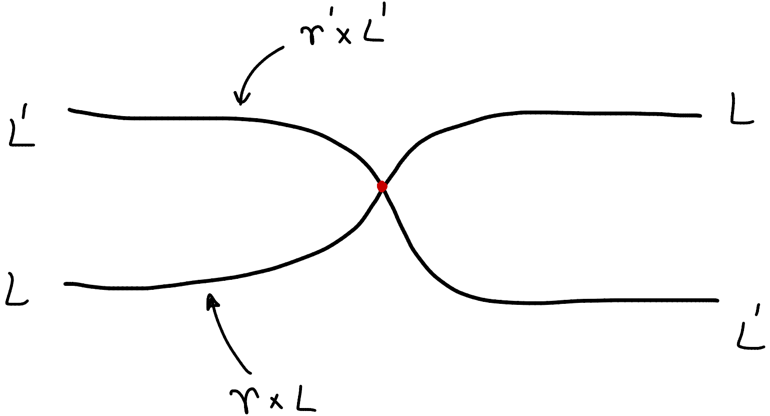


a. Lagrangian suspension: $\phi_t \in Ham(M)$, $t \in [0,1]$, $L \subset M$ Lagrangian.

$$\Sigma^{\phi}(L) = (t, H(t, \phi_t(x)), \phi_t(x))$$



b. Surgery: $L, L' \subset M$ Lagrangians, $L \cap L'$ transverse.



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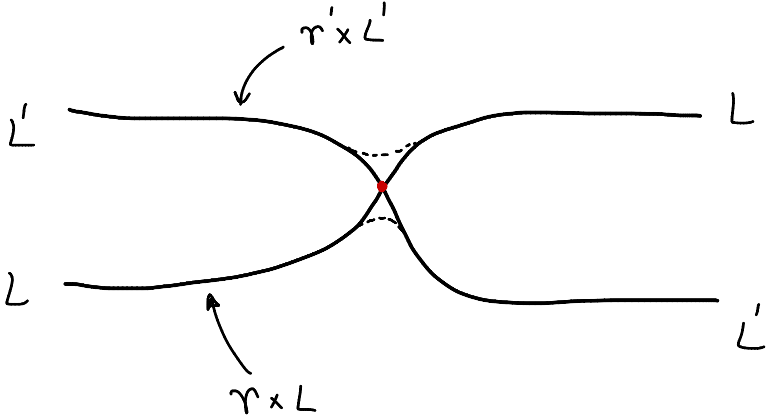

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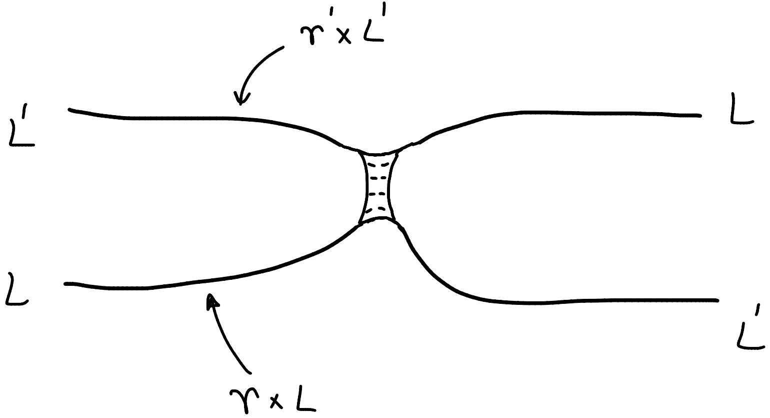
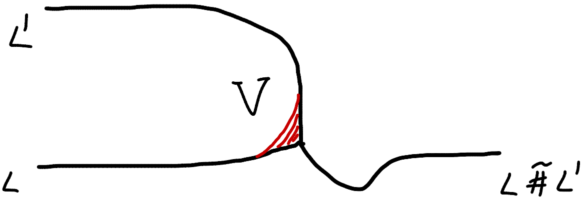


b. Surgery: $L, L' \subset M$ Lagrangians, $L \cap L'$ transverse.

In general, by surgery we can transform any *immersed* cobordism into an *embedded* one.

c. <u>Trace of surgery:</u> $L, L' \subset M$ Lagrangians

$L \cap L'$ transverse $\rightsquigarrow$ cobordism $V : L\widetilde{\#}L' \to (L, L')$.



Form a group: $G_{cob}(M) = \mathbb{Z}_2 < L \subset M, \ \text{Lagrangian} > /\mathcal{R}_{cob}$.

Relations $\mathcal{R}_{cob}$ generated by:

$$L_1 + \ldots L_k = 0 \text{ if } (L_1, \ldots L_k) \text{ is null} - \text{bordant.}$$

More rigid versions, denoted by: $G^*_{cob}(M)$.

$-^*$ means that the Lagrangians and the cobordisms are restricted.

Here, the restriction is that:

all Lagrangians and cobordisms are *monotone* in a uniform way

Flexibility: embedded (*non-restricted*) cobordism $\rightsquigarrow$ immersed cobordism (by surgery) $\rightsquigarrow$ governed by the *h*-principle $\rightsquigarrow$

$$\rightsquigarrow \;\; G_{cob}(M) \text{ computable}$$

For $M = \mathbb{C}^n$ computations are due to *Audin* '85 and *Eliashberg* '84.

Flexibility results of *Ekholm-Eliashberg-Murphy-Smith* '13 on numbers of double points of Lagrangian immersions are expected to shed further light.

Rigidity: From now on *only look at* $G^*_{cob}(M)$ (thus, assuming *uniform monotonicity*).

Early results: *Chekanov* '97 - number of Maslov 2 J-disks through a point is the same for each one of the ends of a connected cobordism (for generic J).

Theorem (*Biran-C.* '11 & '13)

$\exists$ *group morphism :*

surjective

$$\hat{\mathcal{F}} : G^*_{cob}(M) \longrightarrow K_0(D\mathcal{F}uk^*(M))$$

Purpose of talk: discuss this statement .... will need to recall

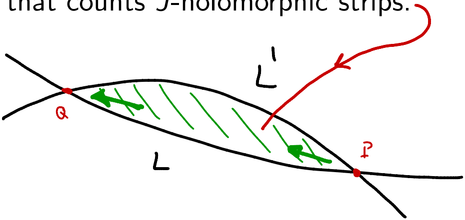
Floer homology and $D\mathcal{F}uk^*(M)$.

a. Floer Homology (*Floer* '88 using earlier work of *Gromov* '85 and followed by work of *Hofer*, *Salamon*, Oh, Fukaya-Oh-Ohta-Ono and others):

$L, L' \subset M$ Lagrangians, $L \cap L'$ transverse.

$$CF(L, L') = \mathbb{Z}_2 < L \cap L' > \quad \text{with differential}$$

$d : CF(L, L') \to CF(L, L')$ that counts J-holomorphic strips.

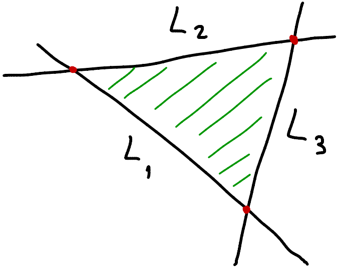


$\rightsquigarrow \quad HF(L, L') = H(CF(L, L'), d)$

b. The triangle product: $L_1, L_2, L_3 \subset M$ Lagrangians in general position.

$$* : CF(L_1, L_2) \otimes CF(L_2, L_3) \to CF(L_1, L_3)$$

given by counting J-holomorphic triangles.



Product *is associative in homology* $\rightsquigarrow$ (due to *Donaldson* '93) the Donaldson category, $\mathcal{D}on^*(M)$.

$\mathcal{D}on^*(M)$ has as objects $L \in Lag^*(M)$ and

$$\hom(L, L') = HF(L, L')\ ,\ \text{composition} = *$$

Fukaya '95: product and higher chain-level structures - counting J-holomorphic polygons with more edges $\rightsquigarrow$

$$\rightsquigarrow\ A_\infty - \text{category}\ \mathcal{F}uk^*(M)\ .$$

Kontsevich '97: Use $\mathcal{F}uk^*(M) \quad \rightsquigarrow$

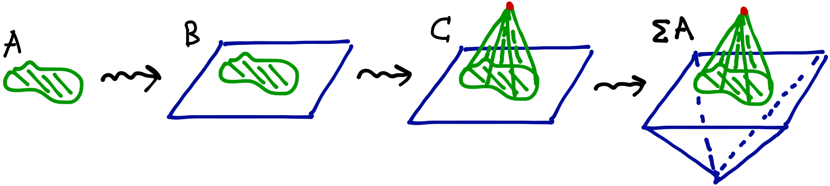
$$\rightsquigarrow\ \textit{triangulated completion of } \mathcal{D}on^*(M) =$$

$$= D\mathcal{F}uk^*(M)\ .$$

These structures described in *Fukaya-Oh-Ohta-Ono* '09 (and earlier) and *Seidel* '06.

A category $\mathcal{C}$ is triangulated (*Verdier* '63, *Dold-Puppe* '61) if it has a class of *exact (or distinguished) triangles* ... subject to axioms similar to the properties of cofibrations sequences in topology:

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A, \quad C = B \cup_f CA$$



$\mathcal{C}$ triangulated $\Rightarrow$

- can decompose objects by iterated triangles.

- Grothendieck group

$$K_0(\mathcal{C}) = \mathbb{Z}_2 < O \in \mathcal{O}b(\mathcal{C}) > /\mathcal{R}' \ .$$

Relations $\mathcal{R}'$ are generated by:

$$A \to B \to C \text{ exact triangle} \Rightarrow A - B + C \in \mathcal{R}' \ .$$

Proof of the Theorem - sketch.

Recall the claim:
$\exists$ surjective group morphism

$$\hat{\mathcal{F}} : G^*_{cob}(M) \longrightarrow K_0(D\mathcal{F}uk^*(M))$$

Remark

- $\hat{\mathcal{F}}$ is a sort of rigid version of the obvious morphism

$$G_{cob}(M) \to H_n(M) \ .$$

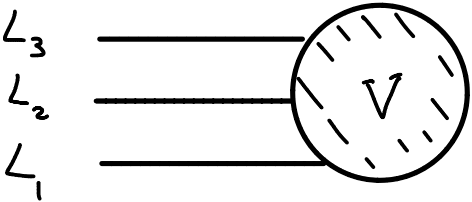
- $K_0(D\mathcal{F}uk^*(M))$ is known in some cases, mainly surfaces, by work of *Seidel*, *Abouzaid*. It can be "identified" by homological mirror symmetry (when this applies).
- For the 2-torus (a variant) of $\hat{\mathcal{F}}$ has been proven to be an isomorphism by *Haug* '13.

Define $\hat{\mathcal{F}}$ on generators:

$$[L] \in G^*_{cob}(M) \longrightarrow [L] \in K_0(D\mathcal{F}uk^*(M)) .$$

Need to show compatibility with relations.

Consider a null-cobordism $V : \emptyset \to (L_1, L_2, L_3)$ so that $L_1 + L_2 + L_3 \in \mathcal{R}_{cob}$.



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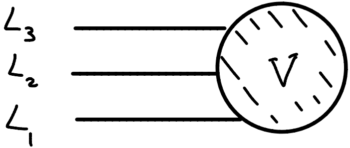


Lemma

There are exact sequences in $D\mathcal{F}uk^*(M)$

$$L_2 \to L_1 \to M_1 \quad , \quad L_3 \to M_1 \to M_2$$

and $M_2 \cong 0$. In particular, $L_1 + L_2 + L_3 \in \mathcal{R}'$.

Proof of Lemma.

Objects in $D\mathcal{F}uk^*(M)$ are (certain) *modules*, $\mathcal{M}$, over $\mathcal{F}uk^*(M)$.

By definition, these associate to each Lagrangian $N \in Lag^*(M)$ a chain complex $\mathcal{M}(N)$ + higher structures.

A Lagrangian $L \leftrightsquigarrow CF(N, L)$.

A example of exact triangle is:

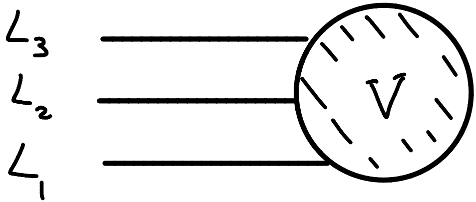
$$L \xrightarrow{\phi} L' \longrightarrow Cone(\phi)$$

So that $\phi_N : CF(N, L) \to CF(N, L')$ and

$$Cone(\phi)_N = CF(N, L) \oplus CF(N, L'), \; D = \begin{pmatrix} d_L & 0 \\ \phi_N & d_{L'} \end{pmatrix}$$

This holds for $\forall$ N, there are higher structures that add additional constraints - I neglect them here.

Returning to V.



We need to show - forgetting the higher structures - for each N:

$$CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1)$$

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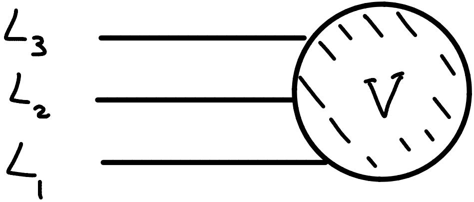

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$$CF(N, L_3) \xrightarrow{\phi_2} Cone(CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1))$$

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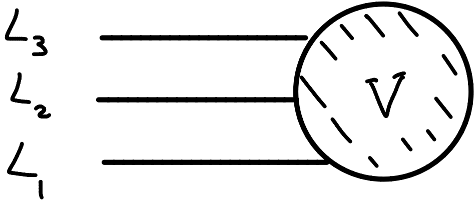

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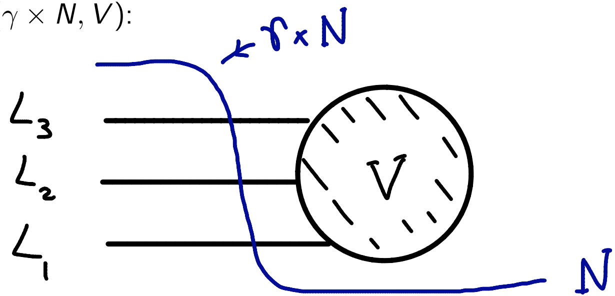
$$[Cone(CF(N, L_3) \xrightarrow{\phi_2} Cone(CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1)))] \simeq 0$$

- Define $CF(W, W')$ for any two cobordisms, W, W'.

- Show that $HF(W, W')$ only depends on the *horizontal* Hamiltonian isotopy type of W and W'.

Compactness is key for both points !

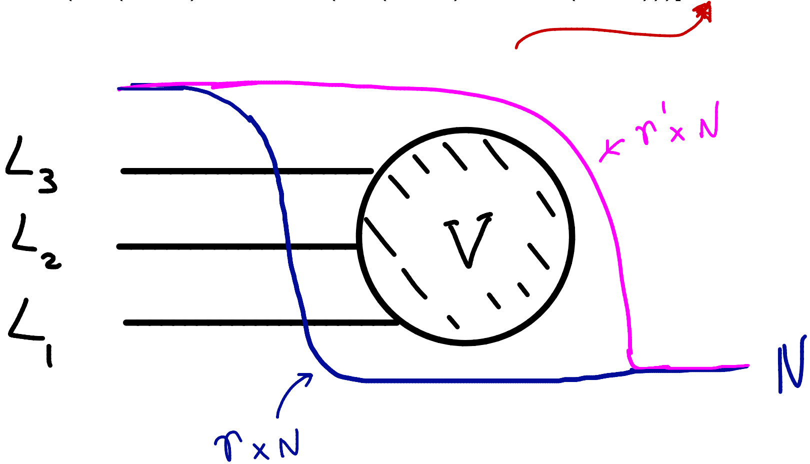
- Consider $CF(\gamma \times N, V)$:



Show:

$$CF(\gamma \times N, V) =$$

$$\simeq [Cone(CF(N, L_3) \xrightarrow{\phi_2} Cone(CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1)))] \simeq 0$$



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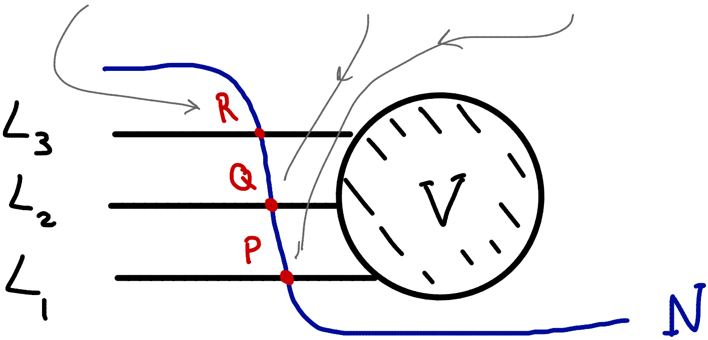

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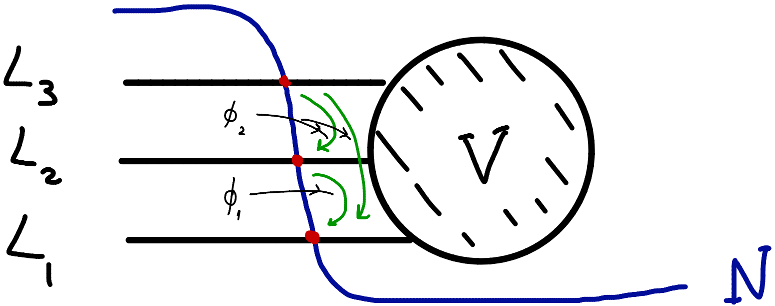
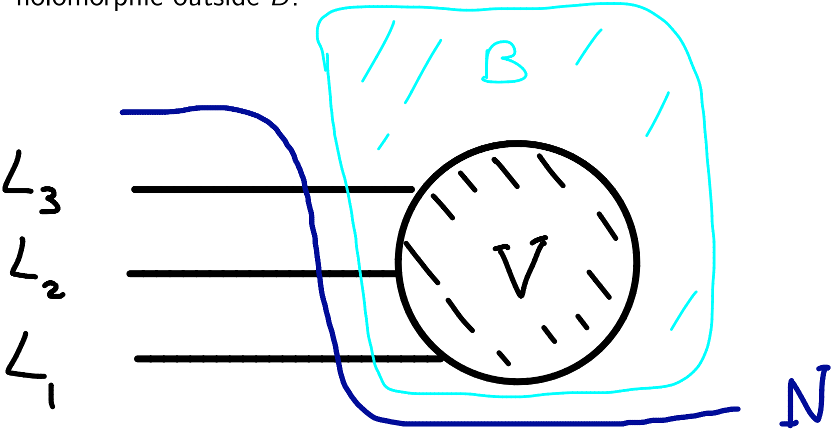


Key point is that Floer strips can only "*go down*" !

- For the structure of the strips giving the differential in $CF(\gamma \times N, V)$ use $J|_{\mathbb{C}\times M}$ so that $\pi : \mathbb{C} \times M \to \mathbb{C}$ is $J - i$ holomorphic outside B.



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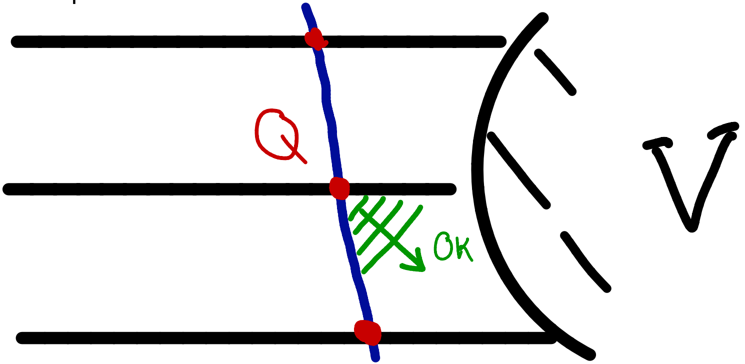
- Then use the open mapping theorem for curves $v = \pi \circ u$, $u =$ Floer strip.



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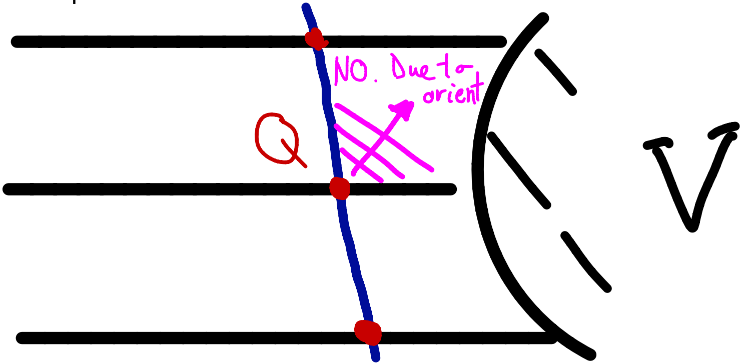

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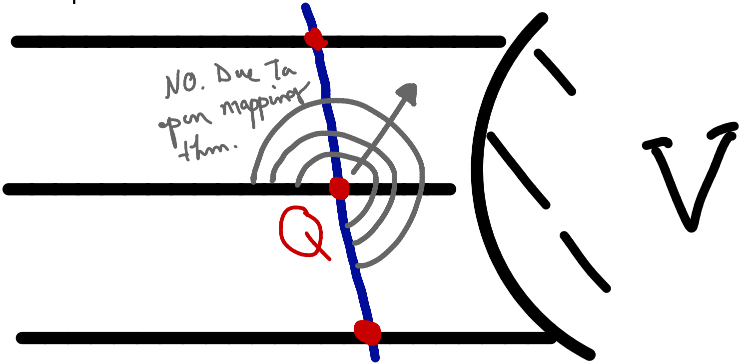
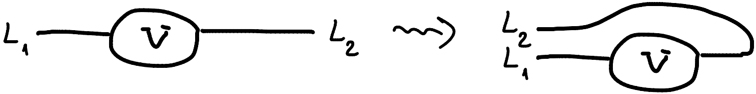


- $L_1, L_2 \in Lag^*(M)$, transverse; $L_1 \tilde{\#} L_2$ obtained by surgery. $\forall N$, $\exists$ long exact sequence :

$$\longrightarrow HF(N, L_1) \to HF(N, L_2) \to HF(N, L_1 \tilde{\#} L_2) \longrightarrow$$

(similar sequence by other methods by *Fukaya-Oh-Ohta-Ono* '09)

- For just two ends: if $V : L_2 \rightsquigarrow L_1$ is a cobordism, then $HF(N, L_2) \cong HF(N, L_1) \ \forall N$.



Immediate Corollaries (of the Lemma)

- $L_1, L_2 \in Lag^*(M)$, transverse; $L_1 \tilde{\#} L_2$ obtained by surgery. $\forall N$, $\exists$ long exact sequence :

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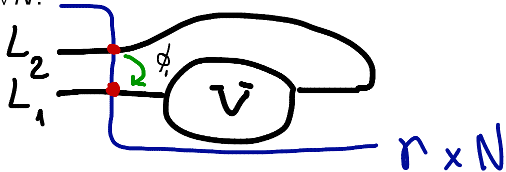

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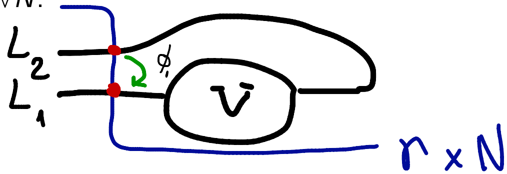

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- *Suarez* '13 (in progress) : if V, L_1, L_2 are exact and

$$\pi_1(L_1) \xrightarrow{\approx} \pi_1(V) \ , \ \ \pi_1(L_2) \xrightarrow{\approx} \pi_1(V) \ , \text{ then } V \cong [0,1] \times L_1 \ .$$

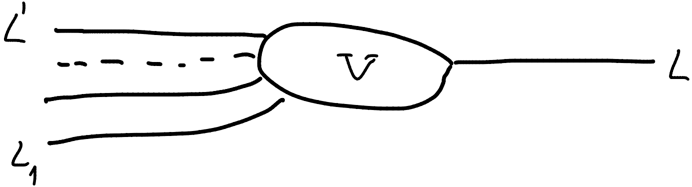
$$G^*_{cob}(M) \xrightarrow{\hat{\mathcal{F}}} K_0(D\mathcal{F}uk^*(M))$$

$$\begin{array}{ccc} G^*_{cob}(M) & \xrightarrow{\hat{\mathcal{F}}} & K_0(D\mathcal{F}uk^*(M)) \\ \uparrow & & \uparrow \\ S\mathcal{C}ob^*(M) & \xrightarrow{\mathcal{F}} & D\mathcal{F}uk^*(M) \end{array}$$

Cobordism category $S\mathcal{C}ob^*(M)$:

$$\mathcal{O}b(S\mathcal{C}ob^*(M)) = \{L \in Lag^*(M)\}$$

$$hom_{S\mathcal{C}ob^*(M)}(L, L') = \{V \subset \mathbb{C} \times M \text{ cobordism}\} \text{ mod isotopy.}$$



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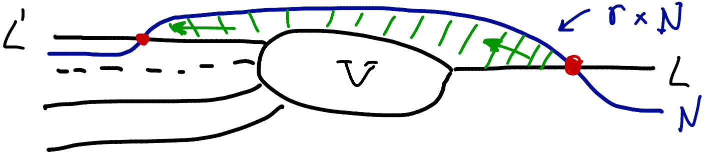

Functor: $\mathcal{F}(L) = L$, $\mathcal{F}(V) : CF({\color{red}N}, L) \to CF({\color{red}N}, L')$ counts the green strips.

a. Put $\mathcal{H}F_N = \hom(N, -) \circ \mathcal{F}$, then

$$\mathcal{H}F_N : S\mathcal{C}ob^*(M) \xrightarrow{\mathcal{F}} D\mathcal{F}uk^*(M) \xrightarrow{\hom(N,-)} \mathcal{V}ect_{\mathbb{Z}_2}$$

is a functor with:

$$\mathcal{H}F_N(L) = HF(N, L) \ .$$

Thus $\mathcal{H}F_N$ presents $HF(N, -)$ as a sort of TQFT.

b. The "triangulation" properties of $\mathcal{F}$ - extending the Lemma - are reflected in the existence of a refinement of $\mathcal{F}$:

$$S\mathcal{C}ob^*(M) \xrightarrow{\mathcal{F}} \qquad\qquad D\mathcal{F}uk^*(M)$$

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$$S\mathcal{C}ob^*(M) \qquad\qquad T^S D\mathcal{F}uk^*(M)$$

Objects in $T^S D\mathcal{F}uk^*(M)$ are famillies $(L_1, \ldots, L_k)$; morphisms are (iterated) cone-decompositions of some L with respect to the L_i's.

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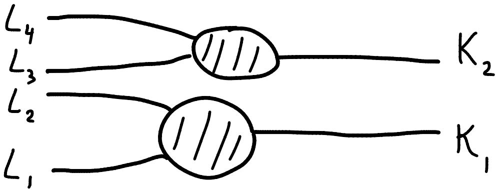
The objects in in $\mathcal{C}ob^*(M)$ are the same families; morphisms are as below:



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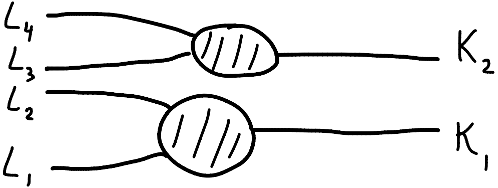

c. The value of $\mathcal{H}F_N$ on morphisms is related to an equivariance property of $\mathcal{F}$ relative to the action of the Hamiltonian group.

Theorem (joint with *Charette* '13)

There is a diagram that "categorifies" Seidel's representation:

$$\pi_1(Ham(M)) \xrightarrow{\quad S \quad} QH(M)^*$$

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$$
\begin{array}{ccc}
\pi_1(Ham(M)) & \xrightarrow{\quad S \quad} & QH(M)^* \\
\big\downarrow i & & \big\downarrow * \\
\Pi(Ham(M)) & \xrightarrow{\quad \widetilde{S} \quad} & \mathcal{E}nd(D\mathcal{F}uk^*(M)) \\
\big\downarrow \Sigma & & \big\downarrow \mathcal{F}^* \\
\mathcal{E}nd(S\mathcal{C}ob^*(M)) & \xrightarrow[\quad \mathcal{F}_* \quad]{} & fun(S\mathcal{C}ob^*(M), D\mathcal{F}uk^*(M))
\end{array}
$$

The categories and functors in the top square are strict monoidal as is Σ.

$\widetilde{S}$ = extension of the Lagrangian Seidel morphism (see *Hu-Lalonde-Leclercq* '11); $*$=closed-open map; Σ = extension of Lagrangian suspension.

In particular, if $V = \Sigma^{\phi}(L)$ is the Lagrangian suspension associated to a loop of Hamiltonian diffeomorphisms ϕ, then

$$\mathcal{H}F_N(V) : CF(N, L) \to CF(N, L)$$

verifies:

$$\mathcal{H}F_N(V)(\alpha) = S(\phi) * \alpha$$

where

$$* : QH(M) \otimes CF(N, L) \to CF(N, L)$$

is the module product and

$$S(\phi) \in QH(M)$$

is the element associated to ϕ by Seidel's representation.