Measuring Lagrangians.

Octav Cornea

(Université de Montréal)

I. Symplectic manifolds and Lagrangian submanifolds.

(M^{2n}, ω) symplectic $\Leftrightarrow$ ω 2-form, $d\omega = 0$, ω non-degenerate.

$L^n \hookrightarrow M$ submanifold, *closed*

$$L \text{ Lagrangian } \iff \omega|_L \equiv 0 \ .$$

Examples.

a. $\mathbb{C}$; $\omega_0 = dx \wedge dy$; $\mathbb{R} \subset \mathbb{C}$ or $S^1 \subset \mathbb{C}$.
b. $\mathbb{C}^n$; $\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$; $\mathbb{R}^n \subset \mathbb{C}^n$.
c. $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$.

Remark.
Pairs $L \hookrightarrow (M, \omega)$ appear in mechanics, algebraic geometry complex analysis etc...

Relation with complex analysis:

(M, ω) symplectic $\Rightarrow$ $\exists\ J : TM \to TM$ almost complex structure compatible with ω.

Example. $i : \mathbb{C}^n \to \mathbb{C}^n$.

Notation:

$$B(r) = \{x \in \mathbb{C}^n : ||x|| \leq r\} \ .$$

Proposition (Darboux, Weinstein).

$\forall\ L \hookrightarrow (M, \omega), \forall\ x \in L\ \Rightarrow \exists$ coordinate chart $\phi : U_x \longrightarrow B(r)$ so that:

$$(U_x, U_x \cap L, x, \omega) \overset{\phi}{\cong} (B(r), \mathbb{R}^n \cap B(r), 0, \omega_0) \tag{1}$$

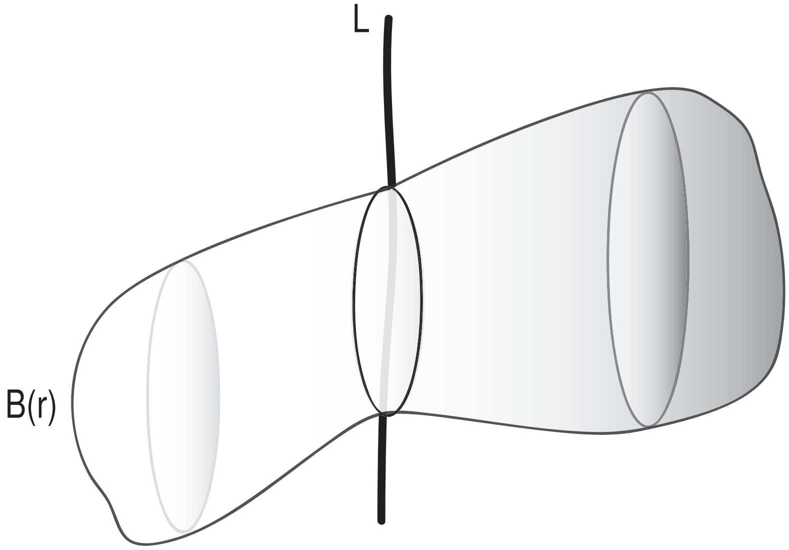
II. Problem.

Measure the "size" of $L \hookrightarrow (M, \omega)$.

Natural, naive geometric measure:

$$Gr(L) = \sup\{r :\ \exists\ \ \phi : U_x \to B(r) \text{ as in } (1)\} \ .$$

Easy to see this does not depend on x.



Question: Estimate $Gr(L)$.

Remark. $Gr(L)$ = Gromov radius of L. Essentially introduced by Gromov without the Lagrangian restriction (1985). The present form appeared out of joint work with Jean-François Barraud (2003).

Conjecture.

$$\forall \text{ closed } L \hookrightarrow (\mathbb{C}^n, \omega_0) \Rightarrow \; Gr(L) < \infty \tag{2}$$

Example. $S^1 \hookrightarrow \mathbb{C}$, $Gr(L) = \sqrt{2}$.

Joint with François Lalonde (2004): the conjecture is true if

- L monotone with $\mu_{min} \geq 2$ or
- L relative spin, orientable with

$$H_*(L;\mathbb{Q}) = 0 \text{ for } * = 2k,\ * \notin \{0, dim(L)\}\ .$$

III. Method of proof.

Originating in the work of Gromov: if there is a surface

$$u : D^2 \longrightarrow \mathbb{C}^n$$

so that:

$$u(S^1) \subset L,\ x \in u(S^1),\ u \ \underline{\text{holomorphic}}$$

then:

$$E(u) = \int_{D^2} u^*\omega_0 \geq \frac{\pi r^2}{2}\ .$$

Enough to show existence of J_0-holomorphic disks through $x \in L$.

Idea. Inspired by Gromov-Witten invariants:

- fix $\lambda \in \pi_2(\mathbb{C}^n, L)$.

- for $\forall$ J almost complex str. compatible with ω_0 define:

$$\mathcal{M}(\lambda, J) = \{u : (D^2, S^1) \to (M, L) : \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0;\ [u] = \lambda\}$$

Strategy:

Task 1 : For generic J $\mathcal{M}(\lambda, J)$, is a manifold of dimension

$$n + \mu(\lambda) - 3$$

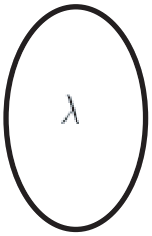
where $\mu(\lambda) =$ Maslov index - purely topological.

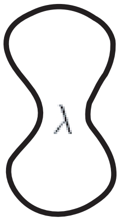
Task 2: Define:

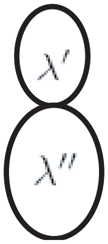
$$\mathcal{M}(\lambda, J) \times S^1 \xrightarrow{ev} L$$

and show $\exists$ λ so that ev is of degree 1.

Task 3: Make $J \longrightarrow J_0$ and show there still is a curve through x.







Difficulty A: $\mathcal{M}$ is not compact due to energy concentration (bubbling off).

Solution to A. Gomov's compactness theorem: add to $\mathcal{M}(\lambda)$ all "broken" configurations. With an appropriate topology obtain compactification $\overline{\mathcal{M}}(\lambda)$.

Conversely, $\forall$ broken configuration $\subset \overline{\mathcal{M}}(\lambda)$ by a gluing theorem.

The machinery in this step is due to: Gromov, Floer, Hofer Salamon.

Remark. Gromov compactness also takes care of Task 3.

Difficulty B: Under best of circumstances $\overline{\mathcal{M}}(\lambda)$ is a compact manifold but

$$\partial\overline{\mathcal{M}}(\lambda) \neq \emptyset \ .$$

so no "degree 1" map.

Difficulty C: Very rarely are we in the best of circumstances: in general, it can happen that

$$dim(\overline{\mathcal{M}}(\lambda)\backslash\mathcal{M}(\lambda) \geq dim(\mathcal{M}(\lambda)$$

Solution to C. Use solutions of

$$\partial u/\partial s + J\partial u/\partial t = \nu(u)$$

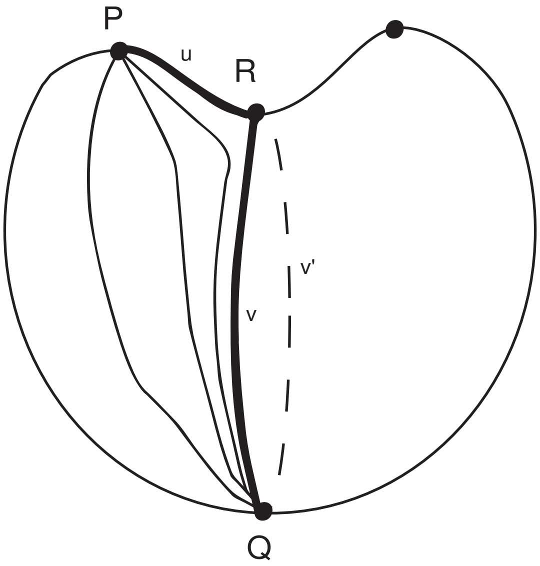
where $\nu(u)$ are small coherent perturbations.

This machinery due to: Liu-Tian, Hofer-Wysocki-Zehnder (still under development).

Solution to B : Cluster Homology (C.-Lalonde 2004).

Remark. Alternative way to deal with bubbling of disks due to Fukaya-Oh-Ono-Ohta (2000).

In other terms:



IV. Quantization of Morse theory.

Recall: Morse theory ⟷ combinatorics of gradient flow lines.

Fix Morse function $f : L \to \mathbb{R}$; fix g Riemannian metric on L.

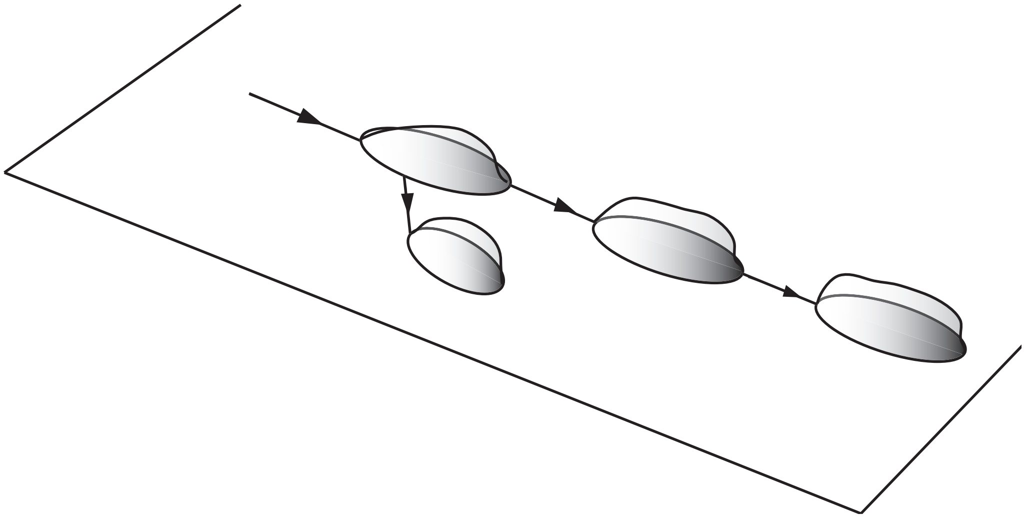
Morse complex:

$$C(f) = (\mathbb{Q} < \mathrm{Crit}(f) >, d)$$

Main fact:

$$d^2 = 0 \qquad \Longleftrightarrow$$

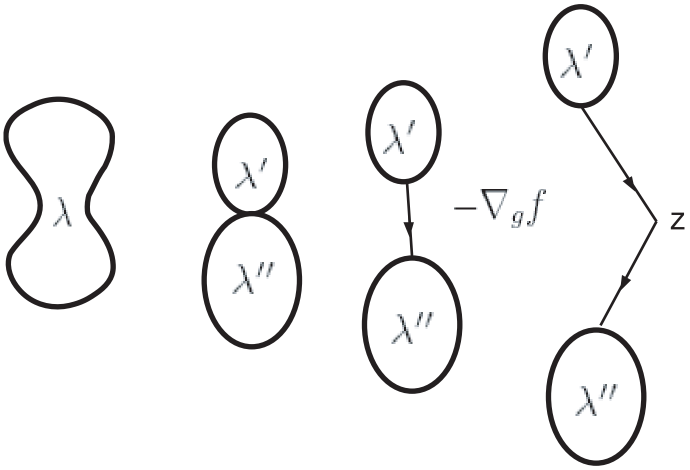
broken trajectories in bijection with ends of 1-dim. moduli spaces of flow lines of $-\nabla_g(f)$



J-holomorphic disks are quantified: $\exists$ smallest energy; only finitely homotopy classes containing disks below any positive energy value.

New objects combine negative gradient flow lines and J-holomorphic disks (and spheres).

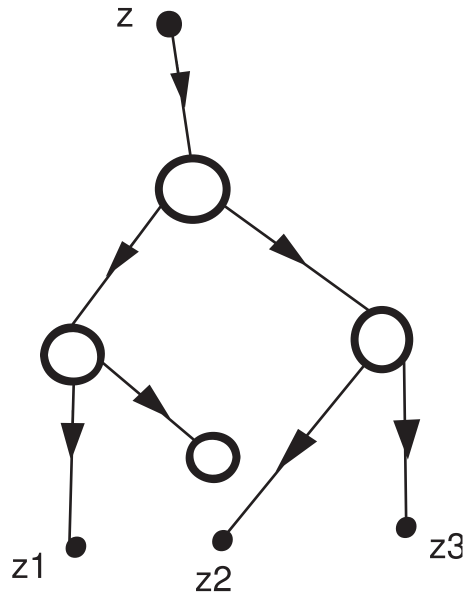
Remark. "Linear" such objects have been considered by Oh (following an idea of Fukaya) in the 1990's. They were viewed at that time as obtained by degeneration of Floer trajectories. These linear objects suffice in the monotone case.



Key new idea here: *use flow lines to deal with bubbling.*

This is used to localize the “breaks” at the critical points of f.

Are defined *cluster* moduli spaces which assemble objects of the type:



Where $z, z_1, z_2, z_3 \in \mathrm{Crit}(f)$.

The Morse complex is replaced by a (graded) commutative rational DGA:

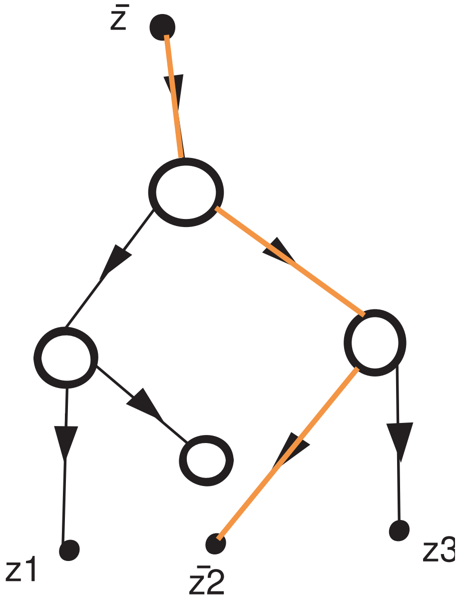
$$\mathcal{C}l(L, J, f) = (S(\mathbb{Q} < \mathrm{Crit(f)} >) \otimes \Lambda, d)$$

- $S(V)$ is the free commutaive DGA on the vector space V, Λ is an appropriate Novikov ring.

- d counts elements in 0-dimensional cluster moduli spaces:

$$dz = \sum_{\lambda, z_1, \ldots, z_k} m^{z}_{z_1, \ldots, z_k}(\lambda) z_1 \ldots z_k e^{\lambda}$$

and verifies the Leibniz formula.



V.Idea of proof of existence of disks.

Follow the Floer comparison principle:

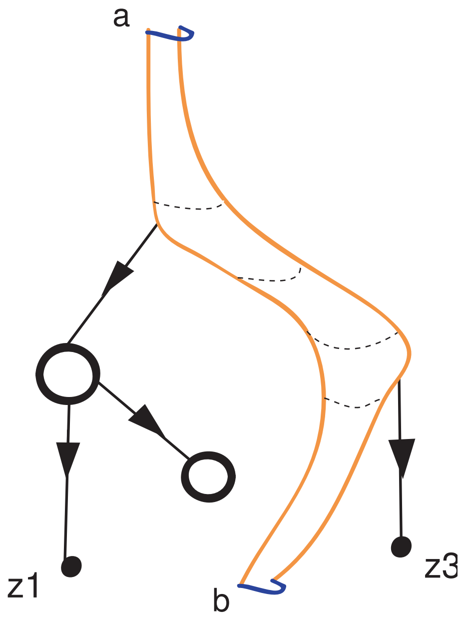
A. Consider a version of the construction before:

$$\widetilde{\mathcal{C}l}_*(L, J, f) = (\mathcal{C}l_*(L, J, f) \otimes \mathbb{Q} < Crit(f) >, D)$$

is a differential module over

$$\mathcal{C}l_*(L, J, f)$$

with D counting elements of moduli spaces as above.



B. For a (generic) hamiltonian $H : M \times [0,1] \to \mathbb{R}$ define the *fine Floer complex*

$$\mathbb{F}C_*(L,J,H,f) = (\mathcal{C}l_*(L,J,f) \otimes \mathbb{Q} < P_0^H >, D')$$

where P_0^H are the time 1-contractible orbits of X_H with ends on L.

- it is a differential module over $\mathcal{C}l_*(L,J,f)$

- D' counts objects whose "spine" verifies Floer's equation:

$$\partial u/\partial s + J\partial u/\partial t + \nabla H(u,t) = 0\ .$$

C. Prove that

$$\mathbb{F}H_*(L) \cong H_*\widetilde{\mathcal{C}l}(L)$$

by constructing comparison morphisms along an idea of Piunikin-Salamon-Schwarz.

Notice that

$$L \subset \mathbb{C}^n \;\Rightarrow\; \mathbb{F}H_*(L) = 0$$

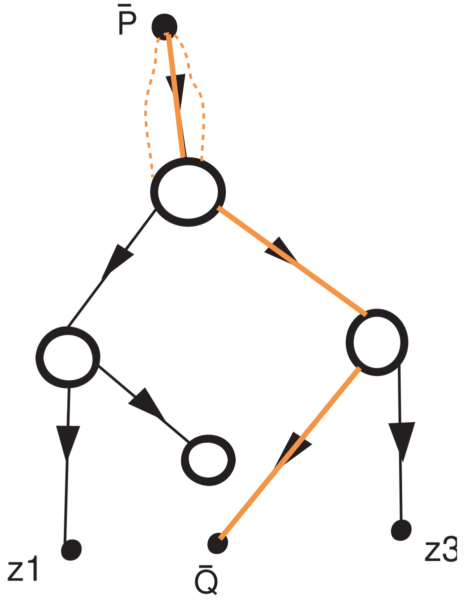
Assume that f has a single maximum, P.

Inspecting the definition of the moduli spaces $\mathcal{M}^{\bar{P}}_{\bar{Q};z_1,z_3,\ldots}$ that give

$$D(\bar{P}) \in \widetilde{\mathcal{C}l}(L, f)$$

notice that

$$\mathcal{M}^{\bar{P}}_{\bar{Q};z_1,z_3,\ldots} \neq \emptyset \;\Rightarrow\; \dim(\mathcal{M}^{\bar{P}}_{\bar{Q};z_1,z_3,\ldots}) \geq 1$$



Thus

$$D(\bar{P}) = 0$$

D.

$$H_*(\widetilde{\mathcal{C}l}(L,f)) = 0 \Rightarrow \bar{P} \text{ is a boundary}$$

Under the conditions mentioned at the beginning (even more generally) - but not in the presence of high free terms - this is only possible if:

$$\exists \bar{Q}\ ,\quad D(\bar{Q}) = qPe^{\lambda} + \ldots\ ,\ \lambda \in \Lambda\ ,\ q \in \mathbb{Q}$$

In that case

$$\mathcal{M}^{\bar{x}}_{\bar{P};z_1,z_3,\ldots} \neq \emptyset$$

but this means that ∃ disk through P