

NEW OBSTRUCTIONS TO THE THICKENING OF CW -COMPLEXES

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ABSTRACT. In this note we use Morse theory to produce new obstructions to the existence of thickenings of CW -complexes in low codimension. The obstructions are expressed as nonexistence of solutions x to an equation of type $\Sigma^k L = \Sigma^s x$ with L a Ganea-Hopf type invariant.

1. INTRODUCTION

Let X be a connected space with the homotopy type of a CW -complex and let M^n be a manifold. A thickening of X in M is a homotopy equivalence $X \simeq W$ where $W \subset M$ is a compact submanifold with boundary of dimension $n = \dim(M)$. In this paper we shall work in the smooth category (and thus our thickenings will be smooth), but the concept exists for PL -manifolds as well. Thickenings were introduced by C.T.C. Wall in [12]. We shall produce here some new obstructions to the existence of thickenings. This problem is closely related to that of embedding CW -complexes up to homotopy, and it was brought to my attention by a discussion with the authors of the recent paper [7].

The approach of the present note is based on a Morse-theoretic result from [4] and is new to this problem. For this reason, we do not attempt here for the most general results possible, and rather prefer to focus on the main geometric idea. In particular, we shall only consider here the case $M = S^n$ and thickenings that are *regular* in the sense that $\pi_1(\partial W) \rightarrow \pi_1(W)$ is required to be an isomorphism.

1.1. Statement. We work in the category of pointed spaces with the homotopy type of CW -complexes. Fix a connected CW -complex X . Assume that

$$(1) \quad \mathcal{C}_1 : S^{q-1} \xrightarrow{u} X' \rightarrow X'', \quad \mathcal{C}_2 : S^{p-1} \xrightarrow{v} X'' \rightarrow X'''$$

are two cofibration sequences, $p \geq q \geq 1$, and suppose that $X'' = X^{(p-1)}$ and that X''' is a subcomplex of $X^{(p)}$, the p -skeleton of X . Consider the composition

$$H : S^{p-1} \xrightarrow{v} X'' \xrightarrow{\nabla} S^q \vee X'' \hookrightarrow S^q \vee X$$

where ∇ is the co-action associated to \mathcal{C}_1 and the last map is the inclusion. Let $p_2 : S^q \vee X \rightarrow X$ be the obvious projection. Clearly, $p_2 \circ H \simeq *$. Recall that the homotopy fiber of p_2 is canonically homotopy equivalent to $S^q \wedge ((\Omega X)^+)$ where A^+ is the space A with an added disjoint point. Moreover, the fiber inclusion

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$j : S^q \wedge (\Omega X)^+ \rightarrow S^q \vee X$ is monomorphic in homotopy. Therefore, the homotopy class $[H]$ lifts uniquely to a homotopy class $H(v, u) \in \pi_{p-1}(\Sigma^q(\Omega X)^+)$, which is called the *Hopf invariant of v relative to u* .

Theorem 1. *Assume the setting and the notation above. Moreover, suppose that X and X' are simply-connected and fix $n \in \mathbb{N}$, $n > p + 2$. If $\Sigma^{n-p+1}H(v, u)$ is not a suspension of order $q + 1$, then X does not admit a regular thickening in S^n .*

The main idea of the proof appears in a particular case and is as follows. Assume that N is a thickening of X in S^n , and suppose that u, v have the special property that there exists a Morse-Smale function $f : N \rightarrow \mathbb{R}$ with two critical points Q and P of indexes respectively q and p such that u, v are the handle-attaching maps corresponding to Q and P by classical Morse theory. Under these assumptions, a Morse-theoretic result from [4] shows that $H(v, u)$ is equal to the homotopy class of a certain Thom-Pontryagin map associated to the framed moduli space $Z_f(P, Q)$ of negative gradient flow lines of f that connect P to Q (see §2.1). But $-f$ is also a Morse-Smale function and the moduli space $Z_{-f}(Q, P)$ is also defined, as well as the corresponding Thom-Pontryagin map $S^{n-q-1} \rightarrow S^{n-p} \wedge (\Omega X)^+$. These two moduli spaces coincide as sets, and we shall see that, after suspending both maps the number of times needed so as to obtain maps defined on S^n , the two Thom-Pontryagin maps coincide up to sign. Therefore, in this particular case, $\Sigma^{n-p+1}H(v, u)$ de-suspends $q + 1$ times.

We comment on the theorem below. The impatient reader may want to go directly to the proof in the next section. A relevant example is discussed in the last section of the paper.

1.2. Comments.

1.2.1. *Relation to other results in the literature.* There are other thickening criteria which are expressed in terms of the existence of solutions x to an equation of the type

$$(2) \quad \Sigma^k L = \Sigma^s x$$

for an appropriate homotopy class L . Probably, the first such condition that appears in the literature is due to Connolly and Williams in [1] (and is related to earlier work of Cooke [2]). With our notation, it shows that if

- (i) $X' = *$, $X = X'''$, $L = v$ (with v from (1)),
- (ii) the suspension orders satisfy $k = n - p - 1$ and $s = q$, and
- (iii) $n > 2p - 2q$, $p \geq q \geq 2$, $n > p + 2$,

then X admits a *PL*-thickening in S^n iff (2) has a solution. A more recent result (restated) for *PL*-thickenings is that of Lambrecht, Stanley and Vandembroucq [7]. They provide a *sufficient* condition for thickening which applies to two-cones (these are spaces that are homotopy equivalent to the cofiber of a map between two suspensions) and not only to two-cell complexes. This reference is also the first where the role of Hopf invariants is recognized (in the particular case of two-cell complexes). Indeed, with our notation, the authors show that if $X' = *$, $X = X'''$, $L = \gamma^t(v)$ is the t -th James-Hopf invariant of v , $k = n - p - 2$, $s = q$, then the existence of a solution of (2) for all t 's implies that X thickens in S^n (again, under the additional assumption $p \geq q \geq 2$, $n \geq p + 3$).

Clearly, the result of Theorem 1 is a *necessary* condition and, not surprisingly, the difference $s - k = p + q - n$, which tracks how many desuspensions are needed with respect to the number of suspensions, is smaller in our case than in the sufficient criterion mentioned above. Obviously, the main interest of Theorem 1 comes from the fact that it applies to arbitrary CW-complexes, and not only to two-cones. From this perspective, our result provides a bridge between the results valid for 2-cell complexes (or 2-cones) and the classical obstructions to thickening like those discovered by Thom [10] or Peterson and Stein [9] and which are based on homology operations. It is likely that all these results are particular instances of some more general statement yet to be discovered.

1.2.2. *The relative Hopf invariant and the Ganea-Hopf invariant.* The invariant $H(v, u)$ is closely related to the Hopf-type invariant introduced by Ganea in [6] and defined as follows.

Consider the relative attaching map $\delta(v, u) = p_1 \circ \nabla \circ v$ (where $p_1 : S^q \vee X \rightarrow S^q$ is the projection on the first factor) and the homotopy class

$$h' = [\nabla \circ v - i_2 \circ v - i_1 \circ \delta(v, u)] \in \pi_{p-1}(S^q \vee X'')$$

(where i_j is the inclusion of the j -th term in the wedge $S^q \vee X''$, $j = 1, 2$). There exists a homotopy fibration

$$\Omega S^q * \Omega X'' \xrightarrow{r} S^q \vee X'' \xrightarrow{l} S^q \times X'',$$

where l is the inclusion. It has the property that $\pi_*(r)$ is monomorphic. Clearly, $l(h') = 0$. Therefore, h' lifts to a unique homotopy class $\mathcal{H}(v, u) \in \pi_{p-1}(\Omega S^q * \Omega X'')$ which is, by definition, the Ganea-Hopf invariant of (v, u) .

Now let $k : S^q \hookrightarrow \Sigma^q(\Omega X)^+ = S^q \vee \Sigma^q \Omega X$ be the obvious inclusion, and let

$$\overline{H}(v, u) = H(v, u) - [k \circ \delta(v, u)] \in \pi_{p-1}(\Sigma^q(\Omega X)^+).$$

To see the relation between $\mathcal{H}(v, u)$ and $H(v, u)$, consider the commutative diagram

$$\begin{array}{ccc} S^q \vee X'' & \xrightarrow{id \vee incl} & S^q \vee X \\ \downarrow l & & \downarrow p_2 \\ S^q \times X'' & \longrightarrow & X \end{array}$$

The commutativity of the diagram provides an induced map $s : \Omega S^q * \Omega X'' \rightarrow \Sigma^q(\Omega X)^+$ between the homotopy fibers of the vertical arrows.

The images of h' and $\overline{H}(v, u)$ inside $\pi_{p-1}(S^q \vee X)$ coincide. Therefore, we have

$$\overline{H}(v, u) = s \circ \mathcal{H}(v, u)$$

and $\Sigma H(v, u) = \Sigma s(\mathcal{H}(v, u)) + \Sigma \delta(v, u)$.

2. PROOF OF THEOREM 1

We start in §2.1 by recalling some well-known elements of Morse theory, as well as the result from [4], which is the key of the proof that is contained in §2.2.

2.1. Some elements of Morse theory.

2.1.1. *Setting.* Assume that $f : N \rightarrow \mathbb{R}$ is a Morse function defined on a smooth, connected, compact manifold N of dimension n . In the case $\partial N \neq \emptyset$, we assume that f is constant and regular on each component of ∂N . Assume also that α is a Riemannian metric on N , and let ∇f be the corresponding gradient vector field induced by f . Denote by γ the flow induced on N by $-\nabla f$.

Recall that the unstable and stable manifolds of a critical point R of f are defined respectively by

$$W^u(R) = \{x \in \text{Int}(N) : \lim_{t \rightarrow -\infty} \gamma_t(x) = R\},$$

$$W^s(R) = \{x \in \text{Int}(N) : \lim_{t \rightarrow +\infty} \gamma_t(x) = R\}.$$

The sets $W^u(R)$ and $W^s(R)$ admit canonical smooth manifold structures, and they are diffeomorphic to open disks of dimension respectively $\text{ind}_f(R)$ and $n - \text{ind}_f(R)$. We now assume that (f, α) satisfies the Morse-Smale condition. By definition this requires the transversality of the intersection between $W^u(P)$ and $W^s(Q)$ for all the pairs of critical points P, Q of f . For f (or α) fixed this condition is satisfied for generic choices of α (respectively f).

Consider two critical points P and Q of f , and assume that $f(P) > f(Q)$ and no critical point R of f satisfies $f(R) \in (f(Q), f(P))$. Fix also the indexes of P and Q to be respectively p and q . Let $a \in \mathbb{R}$ be a regular value of f such that $f(Q) < a < f(P)$. Let $V = f^{-1}(a)$, and let $S_a^u(P) = W^u(P) \cap V$ and $S_a^s(Q) = W^s(Q) \cap V$. It is easy to see that $S_a^u(P)$ and $S_a^s(Q)$ are manifolds diffeomorphic to spheres of dimensions equal to $p-1$ and $n-q-1$ respectively. Let $Z_f(P, Q) = S_a^u(P) \cap S_a^s(Q)$. The Morse-Smale transversality assumption implies that $Z_f(P, Q)$ is a compact manifold of dimension $p-q-1$, and by translating $Z_f(P, Q)$ along the flow γ it is immediate to see that its diffeomorphism type does not depend on a . Following [5], we call $Z_f(P, Q)$ the connecting manifold of P and Q .

2.1.2. *Additional structure of connecting manifolds.* The facts above are standard. They are carefully presented in Franks' paper [5] where the significance of the existence of a canonical normal framing to $Z_f(P, Q)$ was first noticed. This framing will play an important role here too; so we now describe it. To fix terminology, recall that a *framing* of a fiber bundle ξ , of base B and of rank k , is a choice of k everywhere linearly independent sections of ξ modulo the equivalence relation that identifies two such choices if they are the 0 and 1 ends of a choice of k linearly independent sections for the bundle $p^*\xi$ where $p : B \times [0, 1] \rightarrow B$ is the projection. Given an embedding $A \hookrightarrow N$, a *normal framing* of A is a framing of the associated normal bundle.

Pick a normal framing of $W^s(Q)$ associated to the embedding $W^s(Q) \subset N$ (since $W^s(Q)$ is contractible there is a unique such framing). By transversality this framing induces a normal framing of $Z_f(P, Q)$ inside $S_a^u(P) \approx S^{p-1}$, denoted by $\eta_f(P, Q)$.

There is a second structure naturally associated to $Z_f(P, Q)$. This is a map $j_f(P, Q) : Z_f(P, Q) \rightarrow \Omega N$, which is defined as follows. Fix a point $o \in Z_f(P, Q)$ and denote by ω the path in N which coincides geometrically with the flow line of γ that passes through o but is reparametrized by the interval $[1/2, 1]$ and goes from Q to P . The map $j_f(P, Q)$ associates to each point $x \in Z_f(P, Q)$ the path in N which follows from P to Q the flow line of γ that passes through x and then returns

from Q to P along the fixed path ω . Of course, this is slightly imprecise since first all the γ -flow lines joining P to Q need to be reparametrized in a uniform way by a finite interval, say $[0, 1/2]$. This is however possible and immediate because close to the critical points ∇f is almost linear. With this definition it is immediate that the map $j_f(P, Q)$ is continuous.

We apply the Thom construction to the framed embedding

$$Z_f(P, Q) \xrightarrow{\eta_f(P, Q)} S^{p-1}$$

and the map

$$j_f(P, Q) : Z_f(P, Q) \rightarrow \Omega N,$$

thus getting a homotopy class $[Z_f(P, Q)] \in \pi_{p-1} \Sigma^q((\Omega N)^+)$.

2.1.3. *Relation between Hopf invariants and connecting manifolds.* By classical Morse theory, in the setting above, the critical points P and Q correspond to cell attachments of cells of dimensions respectively p and q . In other words, we have cofibration sequences $S^{q-1} \xrightarrow{\alpha(Q)} N' \rightarrow N''$ and $S^{p-1} \xrightarrow{\alpha(P)} N''' \rightarrow N''''$ with $N' = f^{-1}(-\infty, f(Q) - \epsilon]$, $N'' \simeq f^{-1}(-\infty, f(P) - \epsilon]$, $N'''' = f^{-1}(-\infty, f(P) + \epsilon]$ where the attaching maps $\alpha(P)$ and $\alpha(Q)$ are represented by the inclusions

$$\alpha(P) : S^u_{(f(P)-\epsilon)}(P) \rightarrow N'', \quad \alpha(Q) : S^u_{(f(Q)-\epsilon)}(Q) \rightarrow N'$$

and ϵ is small. Clearly, we also have an inclusion $N'''' \hookrightarrow N$ and, as described in §1.1, we define the Hopf invariant $H_f(P, Q) = H(\alpha(P), \alpha(Q)) \in \pi_{p-1}(\Sigma^q(\Omega N)^+)$. For further use we let $\delta_f(P, Q)$ be the relative attaching map of the cells corresponding to P and Q , $\delta_f(P, Q) = l_*(H_f(P, Q))$, with $l : \Sigma^q(\Omega N)^+ \simeq \Sigma^q N \vee S^q \rightarrow S^q$ the projection. The result from [4] that we shall use is that, up to sign, and when N is simply connected (such that $[Z_f(P, Q)]$ does not depend on the choice of ω) we have

$$(3) \quad [Z_f(P, Q)] = H_f(P, Q) .$$

Notice that this equality implies the stable equality up to sign of $\delta_f(P, Q)$ and the framed cobordism class of $Z_f(P, Q)$ (a result first proved in [5]).

2.2. **The proof.**

2.2.1. *Consequence of time reversion on Hopf invariants.* Time reversion for a flow simply replaces the flow γ by the flow $-\gamma_t(x) = \gamma_{-t}(x)$. In our case, when γ is induced by $-\nabla f$ for some smooth function f , this simply corresponds to replacing the function f by its opposite $-f$. Clearly, if the pair (f, α) satisfies the Morse-Smale condition, the same is true for the pair $(-f, \alpha)$. Obviously, the critical points Q, P have respectively indexes $n - q$ and $n - p$ with respect to the function $-f$, and the manifold $Z_{-f}(Q, P)$ coincides as a set with $Z_f(P, Q)$. The map $j_{-f}(Q, P)$ is simply $j_f(P, Q)$ composed with the loop reversion $\Omega N \xrightarrow{-1} \Omega N$. The framings $\eta_f(P, Q)$ and $\eta_{-f}(Q, P)$ may also be compared and, as a byproduct, it was seen in [3] (thus extending again a result from [5]) that the relative attaching maps, $\delta_f(P, Q)$ and $\delta_{-f}(Q, P)$, coincide stably up to a twisting depending on the stable normal bundle of N .

In the present paper we shall need a slightly different result than those mentioned above.

Lemma 2.1. *Assume the setting and notation above, and suppose also that N is a compact, n -dimensional submanifold of S^n . We have*

$$\Sigma^{n-p+1}H_f(P, Q) = \pm\Sigma^{q+1}H_{-f}(Q, P).$$

Proof. Fix the regular value $a \in (f(Q), f(P))$ and $V = f^{-1}(a)$. To simplify notation, let $S^u = S_a^u(P)$ and let $S^s = S_a^s(Q)$, $Z = Z_f(P, Q)$. Let $s_i, i \in \{1, \dots, q\}$, be the linearly independent sections of the normal bundle of $W^s(Q)$ that induce the framing $\eta_f(P, Q)$ of $Z \hookrightarrow S^u$ by projection on TS^u . Similarly, let $l_j, j \in \{1, \dots, n-p\}$, be linearly independent sections of the normal bundle of $W^u(P)$ that induce (by projection on TS^s) the framing $\eta_{-f}(Q, P)$ of $Z \hookrightarrow S^s$. Let η be the framing of $Z \subset N \subset S^n$ given by $(s_1, \dots, s_q, l_1, \dots, l_{n-p}, -\nabla f)$. In view of the equality (3) and given the definition of the Thom map, the statement follows if we show that η is equivalent (up to orientation) to $\eta_f(P, Q) \oplus \epsilon^{n-p+1}$ as well as to $\eta_{-f}(Q, P) \oplus \epsilon^{q+1}$. We recall that if ξ is a normal framing of $M \hookrightarrow S^k$, then $\xi \oplus \epsilon^r$ is the normal framing of $M \subset S^k \hookrightarrow S^{k+r}$ which is obtained by completing ξ by the sections coming from the trivial normal framing of the standard embedding of $S^k \hookrightarrow S^{k+r}$ (addition of such a trivial framing suspends r times the corresponding Thom map). We shall only prove $\eta = \pm\eta_f(P, Q) \oplus \epsilon^{n-p+1}$, since the other equality then follows by symmetry. First, let s'_i be the projections of s_i on TS^u , $i = 1, \dots, q$. By definition, (s'_1, \dots, s'_q) represents the framing $\eta_f(P, Q)$. Let η' be the framing represented by $(s'_1, \dots, s'_q, l_1, \dots, l_{n-p}, -\nabla f)$. The first remark is that $\eta' = \eta$. Indeed, consider the q -plane bundle μ associated to the normal bundle of Z in S^n . The planes spanned by the normal sections (s_1, \dots, s_q) and (s'_1, \dots, s'_q) give two sections s and s' of this bundle. The fact that both s and s' are complementary at each point to the $(n-q)$ plane given by $(l_1, \dots, l_{n-p}, -\nabla f) \oplus TZ$ implies that s and s' are homotopic via sections that remain complementary to this $(n-q)$ -plane (because the space of q -planes in \mathbb{R}^n that are complementary to \mathbb{R}^{n-q} is contractible). It is then immediate to use precisely this homotopy to show that η and η' coincide. Therefore, the proof of the lemma is reduced to showing that $\eta' = \pm\eta_f(P, Q) \oplus \epsilon^{n-p+1}$. At this point it is useful to assume a particular form of α around P . Fix a Morse chart around P such that with respect to this chart $f(x_1, x_2, \dots, x_n) = f(P) - x_1^2 - \dots - x_p^2 + x_{p+1}^2 + \dots + x_n^2$ and the metric α is the flat Euclidean one in this chart, $\alpha = \sum_i dx_i^2$. This is not restrictive, since we may perturb the initial metric to one with this property. Of course, f remains Morse with respect to all the metrics appearing along this deformation, and thus our framing of $S^u(P)$ given by $(l_1, \dots, l_{n-p}, -\nabla f)$ is not changed by this modification. Let S_ϵ^{n-1} be a sphere around P , of radius ϵ , which is contained in our special chart, and let $S_\epsilon^{p-1} \subset S_\epsilon^{n-1}$ be the intersection $W^u(P) \cap S_\epsilon^{n-1}$. The embedding $S_\epsilon^{p-1} \hookrightarrow S_\epsilon^{n-1}$ is the standard one. Moreover, since $S_\epsilon^{p-1} = S_{f(P)-\epsilon^2}^u(P)$ we may assume $a = f(P) - \epsilon^2$ (otherwise we transport Z along the flow $-\gamma$ until it reaches $f^{-1}(f(P) - \epsilon^2)$), and so $Z \subset S_\epsilon^{p-1} = S^u$. On S_ϵ^{p-1} the framing represented by (l_1, \dots, l_{n-p}) is the trivial one, and so the normal framing of Z in S_ϵ^{n-1} is represented by $(s'_1, \dots, s'_q, l_1, \dots, l_{n-p})$ and coincides with $\eta_f(P, Q) \oplus \epsilon^{n-p}$. When restricted to S_ϵ^{p-1} , the vector field $-\nabla f$ coincides with the vector field normal to the sphere S_ϵ^{n-1} . Clearly, the embedding $S_\epsilon^{n-1} \hookrightarrow S^n$ is the trivial one. So, finally, η' coincides with $(\eta_f(P, Q) \oplus \epsilon^{n-p}) \oplus \epsilon^1$. \square

2.2.2. *The last step.* Notice that the non-desuspension condition in Theorem 1 implies that $q + p > n$, $p > q + 1$, and since $p + 2 < n$ we have $p - 1 > q > 3$,

$n \geq 9$. Clearly, if X admits a regular thickening in S^m , then it also admits a regular thickening in S^{m+1} . Therefore, the proof of Theorem 1 is now immediate, by contradiction, from the duality Lemma 2.1 and the result below.

Proposition 2.2. *Suppose that X, u, v are as in Theorem 1. Assume that N is a regular n -dimensional thickening of X inside S^n , $n \geq 6$. Suppose also that $p - 1 > q > 1$, $p + 2 < n$. There exists a Morse-Smale pair (f, α) on N with the property that there are critical points P_i, Q_i of f with $\text{ind}(P_i) = p$, $\text{ind}(Q_i) = q$ and there also exist coefficients $a_i \in \mathbb{Z}$ and a homotopy equivalence $\phi : N \rightarrow X$ such that*

$$\Sigma H(v, u) = \sum_i a_i (\Sigma(\Sigma^q \Omega \phi \circ H_f(P_i, Q_i))).$$

Proof. Because $n \geq 6$ and N as well as ∂N are simply-connected, it follows from classical Morse theory that there exists a Morse function $f : N \rightarrow \mathbb{R}$ such that f is maximal, constant and regular on ∂N , f is self-indexed (meaning that if R is a critical point of f of index r , then $f(R) = r$) and, moreover, f has no critical point of index $i \in (q, p) \cup \{1\}$. For this last statement we use $q \geq 2$ and $p < n - 2$ such that the cancellation technique appearing in the h -cobordism theorem (and based on the Whitney trick) works for cells of dimension $\leq p$ (see also [8], page 69). The attachments of the q -cells that correspond to the critical points of index q provide a cofibration sequence $\bigvee_{i_q} S^{q-1} \xrightarrow{w} N' \rightarrow N''$, and the attachment of the cells corresponding to the index- p critical points provides yet another cofibration sequence $\bigvee_{i_p} S^{p-1} \xrightarrow{z} N'' \rightarrow N'''$ where i_k is the number of critical points of index k of f . Notice that N', N'', N''' are all simply connected. We fix homotopy equivalences $\phi : N \rightarrow X$, $\phi^{-1} : X \rightarrow N$. Clearly, the whole problem comes from the fact that the cellular decomposition induced by f may be different from the decomposition given by the CW-structure of X (since $2p \geq n$, even by modifying f we might not recover this last cell decomposition [5]). To compare these two cell decompositions, we shall make use of the following lemma.

Lemma 2.3. *Suppose that we have two cofibration sequences and maps given by the solid arrows in the diagram*

$$(4) \quad \begin{array}{ccccc} V & \xrightarrow{a_1} & A_1 & \xrightarrow{a_2} & A_2 \\ \downarrow \text{dotted } t_0 & & \downarrow t_1 & & \downarrow t_2 \\ V' & \xrightarrow{b_1} & B_1 & \xrightarrow{b_2} & B_2 \end{array}$$

such that V and V' are wedges of $(q - 1)$ -dimensional spheres, the A_i 's and B_i 's are all simply connected, and the solid-arrow diagram commutes. There exists a map t_0 represented by the dotted arrow in the diagram, and there exists a homotopy $h : V \times [0, 1] \rightarrow B_1$ between $t_1 \circ a_1$ and $b_1 \circ t_0$ that induces by passage to cofibers a map homotopic to t_2 .

Proof. The proof of the lemma is rather standard, but since it is not completely trivial, we shall give it here in detail. We start with the existence of t_0 . To show this we may assume $q > 2$ because, otherwise, the maps a_1, b_1 are both null-homotopic and the existence of t_0 is immediate. The existence of t_0 can be deduced from the Blackers-Massey excision theorem or, alternatively, by using the following geometric argument. First we write $A_2 = A_1 \cup CV$ ($CV = (V \times [0, 1]) / (V \times \{0\} \cup * \times [0, 1])$),

$B_2 = B_1 \cup CV'$. Let $i : F \rightarrow A_1$ be the fibration obtained by pulling back the path loop fibration $\Omega A_2 \rightarrow PA_2 \rightarrow A_2$ (clearly, F is the homotopy fiber of a_2). Notice that by pulling back this same path-loop fibration over the cofibration $V \rightarrow A_1 \rightarrow A_2$ we obtain the existence of a push-out diagram

$$\begin{array}{ccc} V \times \Omega A_2 & \longrightarrow & F \\ p_2 \downarrow & & \downarrow \\ \Omega A_2 & \longrightarrow & PA_2. \end{array}$$

This shows that $\Sigma F \simeq \Sigma V \wedge (\Omega A_2)^+$. By the defining property of pull-backs, the null-homotopy of the composition $V \rightarrow A_1 \rightarrow A_2$ given by deforming V to a point inside $CV \subset A_2$ produces a map $a'_1 : V \rightarrow F$ that satisfies $i \circ a'_1 = a_1$ (and that equals the composition $V \times * \rightarrow V \times \Omega A_2 \rightarrow F$). Let $r : \Sigma F \rightarrow \Sigma V$ be the map induced by collapsing to a point F inside PA_2 and A_1 inside A_2 . The definition of a'_1 shows that $r \circ \Sigma a'_1 \simeq id$. Let $i' : F' \rightarrow B_1$, $r' : \Sigma F' \rightarrow \Sigma V'$, $b'_1 : V' \rightarrow F'$ be the same objects constructed from the second cofibration. Let $c' : \Sigma V \rightarrow \Sigma V'$ be the map induced by (t_1, t_2) by collapsing to a point A_1 and B_1 inside, respectively, A_2 and B_2 . Let $c : F \rightarrow F'$ be the comparison map also induced by (t_1, t_2) . Therefore, in the diagram

$$(5) \quad \begin{array}{ccccc} \Sigma V & \xrightarrow{\Sigma a'_1} & \Sigma F & \xrightarrow{r} & \Sigma V \\ \downarrow c' & & \downarrow \Sigma c & & \downarrow c' \\ \Sigma V' & \xrightarrow{\Sigma b'_1} & \Sigma F' & \xrightarrow{r'} & \Sigma V' \end{array}$$

the right square commutes up to homotopy and the horizontal compositions are homotopic to the respective identities. Given the fact that V and V' are wedges of $(q - 1)$ -spheres and since we have $\Sigma F \simeq \Sigma V \wedge (\Omega A_2)^+$ and $\Sigma F' \simeq \Sigma V' \wedge (\Omega B_2)^+$ with F, F', A_2, B_2 simply connected, we obtain that c' makes the left square of the diagram homotopy commutative, and moreover c' de-suspends to a map t_0 that has the property $b'_1 \circ t_0 \simeq c \circ a'_1$. Clearly, this last equality shows that the map t_0 makes the left square in (4) homotopy commutative. We now need to show the existence of the homotopy h . We first recall that the map $j_g : A_2 \rightarrow B_2$ induced by a homotopy $g : V \times [0, 1] \rightarrow B_2$, with $g_0 = b_1 \circ t_0$, $g_1 = t_1 \circ a_1$, is defined by $j_g(x) = t_1(x)$ if $x \in A_1$, $j_g(y, t) = (t_0(y), 2t)$ if $(y, t) \in CV$ and $t \leq 1/2$, and $j_g(y, t) = g(y, 2t - 1)$ if $(y, t) \in CV$, $t > 1/2$. Clearly, by passage to cofibers the pair (t_1, j_g) induces a map homotopic to Σt_0 . Moreover, since $j_g|_{A_1} = t_2|_{A_1}$, there is a map $a_g : \Sigma V \rightarrow B_2$ such that $t_2 \simeq (a_g \vee j_g) \circ \nabla$ where $\nabla : A_2 \rightarrow \Sigma V \vee A_2$ is the coaction associated to the top cofibration in (4). By collapsing to a point A_1 and B_1 inside A_2 and B_2 , respectively, this equation becomes $\Sigma t_0 \simeq \delta \circ a_g + \Sigma t_0$ with $\delta : B_2 \rightarrow \Sigma V'$ the connectant of the bottom cofibration sequence in (6). This means that $\delta \circ a_g \simeq *$. Since B_2, B_1 are simply connected, we have an isomorphism $\pi_q(B_2, B_1) \approx \pi_q(\Sigma V')$ induced by the collapsing of B_1 . So we obtain that there exists a map $a'_g : \Sigma V \rightarrow B_1$ such that $b_2 \circ a'_g \simeq a_g$. It is this element a'_g that we shall use to modify the homotopy g . At this point it is simpler to assume that V contains a single sphere S^{q-1} (this is not restrictive, since we can repeat our construction for each term of the wedge). We define a new homotopy $h : V \times [0, 1] \rightarrow B_2$ as follows. First let $k : V \times [0, 1/2] \rightarrow \Sigma V \vee (V \times [0, 1])$ be the

collapsing map obtained by contracting to a point a sphere S^{q-1} embedded in the interior of $V \times [0, 1/2] = S^{q-1} \times [0, 1/2]$. We now let $h(x, t) = g(x, 2t - 1)$ when $t \in [1/2, 1]$, and $h(x, t) = (a'_g \vee (b_1 \circ t_0)) \circ k$ if $t \in [0, 1/2]$. We consider the induced map j_h and notice that, by construction, $j_h \simeq ((b_2 \circ a'_g) \vee j_g) \circ \nabla$. Therefore, $j_h \simeq t_2$. \square

We now return to the proof of the proposition. By applying the lemma in our setting we obtain diagrams

$$(6) \quad \begin{array}{ccccc} \vee S^{q-1} & \xrightarrow{w} & N' & \xrightarrow{s_1} & N'' \\ j_0 \downarrow & & j_1 \downarrow & & j_2 \downarrow \\ S^{q-1} & \xrightarrow{u} & X' & \xrightarrow{l_1} & X'' \end{array}$$

and

$$(7) \quad \begin{array}{ccccc} S^{p-1} & \xrightarrow{v} & X'' & \xrightarrow{l_2} & X''' \\ t \downarrow & & j_2^{-1} \downarrow & & j_3 \downarrow \\ \vee S^{p-1} & \xrightarrow{z} & N'' & \xrightarrow{s_2} & N''' \end{array},$$

which are constructed as follows. We start with maps j_k , $k = 1, 2, 3$, obtained from ϕ, ϕ^{-1} by cellular approximation, and we let the s_i 's and l_i 's be inclusions. We notice that j_2 is a homotopy equivalence, because both X'' and N'' are $(p - 1)$ -skeleta of dimension q and $p - 1 > q$. We then use the lemma to construct the map j_0 and a homotopy h' making commutative the left square in (6) and such that $j_{h'} \simeq j_2$, as well as the map t and a homotopy h'' which makes the left square in (7) commutative and with $j_{h''} \simeq j_3$.

An immediate consequence of the fact that $j_2 \simeq j_{h'}$ is that the co-actions associated to the two cofibration sequences in (6) commute (up to homotopy):

$$(8) \quad \begin{array}{ccc} N'' & \xrightarrow{\nabla} & (\vee S^q) \vee N'' \\ j_2 \downarrow & & \downarrow \Sigma j_0 \vee j_2 \\ X'' & \xrightarrow{\nabla} & S^q \vee X'' \end{array}.$$

By combining the left square in (7) with (8) and after composition with the respective inclusions into X and N we obtain the next homotopy commutative diagram:

$$(9) \quad \begin{array}{ccccc} S^{p-1} & \longrightarrow & X'' & \longrightarrow & S^q \vee X \\ t \downarrow & & \updownarrow & & \uparrow \Sigma j_0 \vee \phi \\ \vee S^{p-1} & \longrightarrow & N'' & \longrightarrow & (\vee S^q) \vee N \end{array}$$

and this immediately implies the claim by the definition of the respective Hopf invariants. \square

3. AN EXAMPLE

It is not too difficult to build examples where Theorem 1 applies and previous results do not. Before describing the construction of such an example, notice that

the criterion in the theorem is only relevant if $n - p + 1 < q + 1 \Leftrightarrow n < p + q$. Our example will be one with $q = 8, p = 30$ and no thickening exists whenever $n < 37$.

3.1. A construction. Let α_1 be an element of order three in $\pi_6(S^3) \approx \mathbb{Z}/12$. Consider the space $T = S^3 \cup_{\alpha_1} e^7$, and let δ be the map $\delta : T \rightarrow S^7$ obtained by collapsing S^3 to a point in T . Let $i \in \pi_7(T)$ be an element of infinite order such that the image of i by δ is homotopic to three times a generator of $\pi_7(S^7)$. The existence of such an element follows from the exact sequence $\pi_7(T) \rightarrow \pi_7(S^7) \rightarrow \pi_6(L) \rightarrow \pi_6(T) \approx \mathbb{Z}/4 \rightarrow 0$ with L the homotopy fiber of $T \rightarrow S^7$, because

$$(10) \quad \pi_6(L) \approx \pi_6(S^3).$$

Before proceeding with our construction, we justify the isomorphism (10). We use δ to pull back the fibration $\Omega S^7 \rightarrow PS^7 \rightarrow S^7$ over the cofibration sequence $S^6 \rightarrow S^3 \rightarrow T$. We thus obtain a push-out square of the form

$$\begin{array}{ccc} S^6 \times \Omega S^7 & \xrightarrow{\alpha_1 \times id} & S^3 \times \Omega S^7 \\ \downarrow & & \downarrow \\ \Omega S^7 & \longrightarrow & L \end{array}$$

where the vertical map at the left restricted on $S^6 \times *$ equals the inclusion of the bottom cell $S^6 \hookrightarrow \Omega S^7$. We only need to consider the 7-skeleta involved, and we deduce the push-out square

$$\begin{array}{ccc} S^6 \vee S^6 & \longrightarrow & S^3 \vee S^6 \\ \downarrow & & \downarrow \\ S^6 & \longrightarrow & L^{(7)} \end{array}$$

which shows that $L^{(7)} \simeq S^3$, because the vertical map on the left is the folding map.

We return to our construction, and let $w : S^{14} \rightarrow S^8 \vee T$ be given by $w = [i_1, i]$ where $i_1 : S^8 \rightarrow S^8 \vee T$ is the inclusion of the first factor. Recall ([11], page 188) that there exists an element $\rho_{13} \in \pi_{28}(S^{13}) \approx \mathbb{Z}/480 \oplus \mathbb{Z}/2$ that is of order 32 and that does not de-suspend. On the other hand, the stabilization of $\rho_{13}, \rho \in \pi_{15}^S$, is still of order 32 and, in fact, $\pi_{15}^S \approx \pi_{28}(S^{13})$. Let $\rho_{14} : S^{29} \rightarrow S^{14}$ be the suspension of ρ_{13} .

We now let X be the cofiber of the map $v = w \circ \rho_{14} : S^{29} \rightarrow S^{14} \rightarrow S^8 \vee T$.

3.2. Absence of thickenings. The purpose now is to show the following consequence of Theorem 1.

Corollary 3.1. *The space X defined above does not admit a regular thickening in S^n for any $n < 37$.*

Proof. We consider the two cofibration sequences

$$(11) \quad S^7 \xrightarrow{u} T \longrightarrow S^8 \vee T, \quad S^{29} \xrightarrow{v} S^8 \vee T \xrightarrow{t} X$$

with $u \simeq *$, and the associated Hopf invariant $H(v, u)$. We consider the composition

$$\rho' : S^{29} \xrightarrow{\rho} S^{14} \xrightarrow{l} S^8 \wedge (\Omega X)^+ \xrightarrow{q} S^8 \wedge (\Omega S^7)^+ \xrightarrow{e} S^{14},$$

where:

- l is the unique lift of the composition $S^{14} \xrightarrow{w} S^8 \vee T \xrightarrow{id_{S^8} \vee incl} S^8 \vee X$ to the homotopy fiber of $S^8 \vee X \xrightarrow{p_2} X$.
- $q = \Sigma^8(\Omega q')^+$, with $q' : X \rightarrow S^7$ the composition of the map q'' from the cofibration sequence $S^8 \vee S^3 \rightarrow X \xrightarrow{q''} S^{30} \vee S^7$ with the projection $S^{30} \vee S^7 \rightarrow S^7$.
- e is the composition $\Sigma^8(\Omega S^7)^+ \simeq \Sigma^8 \Omega S^7 \vee S^8 \xrightarrow{p_1} \Sigma^8 \Omega S^7 \xrightarrow{\Sigma^7(ev)} S^{14}$, with $ev : \Sigma \Omega S^7 \rightarrow S^7$ the adjoint of the identity.

These maps possess the following two properties:

- (i) $H(v, u) \simeq l \circ \rho$.
- (ii) $e \circ q \circ l$ is a map of degree ± 3 .

We postpone the justification of (i) and (ii) and first describe how they imply the claim of the corollary. Indeed, if $H(v, u)$ would de-suspend 9 times after being suspended $n - 30 + 1 > 1$ times, then the same would be true for $\rho' = e \circ q \circ l \circ \rho$ but, after inverting 3, we see that $e \circ q \circ l$ induces a homotopy equivalence $S^{14}_{(3)} \rightarrow S^{14}_{(3)}$ and, in particular, ρ' remains of order 32. Since there are no elements of order 32 in $\pi_{27}(S^{12})$, it follows that $n - 29 \geq 8$, and by using Theorem 1, we obtain the statement.

We now discuss (i) and (ii). We start with (ii). This is immediate from the fact that $q \circ l$ is the unique lift of the Whitehead product $S^{14} \xrightarrow{[i_1, 3i_2]} S^8 \vee S^7$ to the homotopy fiber of the map $p_2 : S^8 \vee S^7 \rightarrow S^7$, which in turn follows from the commutativity of the diagram

$$\begin{array}{ccccc}
 S^{14} & \xrightarrow{[i_1, i_2]} & S^8 \vee S^7 & \xrightarrow{id \vee i} & S^8 \vee T & \xrightarrow{incl} & S^8 \vee X \\
 & & \searrow id \vee 3 & & \downarrow id \vee \delta & & \swarrow id \vee q' \\
 & & & & S^8 \vee S^7 & & .
 \end{array}$$

So, to conclude the proof we only have to show (i). For this we make use of the commutative diagram

$$\begin{array}{ccccccc}
 S^{29} & & & & & & \\
 \downarrow \rho & \searrow H' & & & & & \\
 S^{14} & \longrightarrow & S^8 \wedge (\Omega T)^+ & \longrightarrow & S^8 \wedge (\Omega(S^8 \vee T))^+ & \longrightarrow & S^8 \wedge (\Omega X)^+ \\
 & \searrow [i_1, i] & \downarrow & & \downarrow & & \downarrow j \\
 & & S^8 \vee T & \xrightarrow{id_{S^8} \vee t'} & S^8 \vee (S^8 \vee T) & \xrightarrow{id_{S^8} \vee t} & S^8 \vee X \\
 & & \downarrow p_2 & & \downarrow p_2 & & \downarrow p_2 \\
 & & T & \xrightarrow{t'} & S^8 \vee T & \longrightarrow & X
 \end{array}$$

where $t' : T \rightarrow S^8 \vee T$ is the obvious inclusion and t appears in the second cofibration in (11). We see from this diagram that $H' \simeq l \circ \rho$. The proof of $H(v, u) \simeq l \circ \rho$ now

follows from the commutativity of the diagram

$$(12) \quad \begin{array}{ccccc} S^{29} & \xrightarrow{\rho} & S^{14} & \xrightarrow{[i_1, i]} & S^8 \vee T \\ \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla \vee id_T \\ S^{29} \vee S^{29} & \xrightarrow{\rho \vee \rho} & S^{14} \vee S^{14} & \xrightarrow{[i_1, i] \vee [i_2, i]} & S^8 \vee S^8 \vee T \xrightarrow{id_{S^8} \vee t} S^8 \vee X . \end{array}$$

Indeed, $H(v, u)$ is uniquely defined as the lift of $h'' = (id_{S^8} \vee t) \circ (\nabla \vee id_T) \circ v$ to the fiber of $S^8 \vee X \rightarrow X$, and the commutativity of (12) implies that $[h''] = [t \circ v] + [j \circ H'] = [j \circ H']$, and so $H(v, u) \simeq H'$ (we have used here the obvious identity $t \circ v \simeq *$, and we recall that $j : S^8 \wedge (\Omega X)^+ \rightarrow S^8 \vee X$ is the inclusion of the fiber). Finally, in diagram (12), the left square is commutative because $\rho = \rho_{14}$ is a suspension, and the right square is commutative due to the following immediate commutativity of universal Whitehead products:

$$\begin{array}{ccc} S^{14} & \xrightarrow{[i_1, i_2]} & S^8 \vee S^7 \\ \downarrow \nabla & & \downarrow \nabla \vee id_{S^7} \\ S^{14} \vee S^{14} & \xrightarrow{[i_1, i_3] \vee [i_2, i_3]} & S^8 \vee S^8 \vee S^7 . \end{array}$$

□

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REFERENCES

1. F. X. Connolly and B. Williams, *Embeddings up to homotopy type and geometric suspensions of manifolds*, Quart. J. Math. Oxford Ser. (2) **29** (1978), 385–401. MR **83a**:57022
2. G. Cooke, *Embedding certain complexes up to homotopy type in Euclidean space*, Ann. of Math. **90** (1969), 144–156. MR **39**:3486
3. O. Cornea, *Homotopical Dynamics II: Hopf invariants, smoothings and the Morse complex*, Annales Scient. École Norm. Sup. (4) **35** (2002), 549–573.
4. O. Cornea, *Homotopical Dynamics IV: Hopf invariants and Hamiltonian flows*, Commun. Pure and Appl. Math. **LV** (2002), 1033–1088. MR **2003e**:37081
5. J. Franks, *Morse-Smale flows and homotopy theory*, Topology **18** (1979), 199–215. MR **80k**:58063
6. T. Ganea, *A generalization of the homology and homotopy suspension*, Comment. Math. Helv. **39** (1965), 295–322. MR **31**:4033
7. P. Lambrechts, D. Stanley, and L. Vandembroucq, *Embeddings up to homotopy of two-cones in Euclidean space*, Trans. Amer. Math. Soc. **354** (2002), 3973–4013. MR **2003h**:55012
8. J. Milnor, *Lectures on the h-cobordism theorem*, Princeton Math. Notes, Princeton University Press, Princeton, NJ, 1965. MR **32**:8352
9. F. P. Peterson and N. Stein, *The dual of a secondary cohomology operation*, Illinois J. of Math. **4** (1960), 397–404. MR **27**:1948
10. N. Steenrod and D. Epstein, *Cohomology operations*, Annals of Math. Studies **50**, Princeton University Press, Princeton, NJ, 1962. MR **26**:3056

11. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies **49**, Princeton University Press, Princeton, NJ, 1962. MR **26**:777
12. C. T. C. Wall, *Classification problems in differential topology IV*, Topology **5** (1966), 73–94. MR **33**:734

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