Lagrangian Quantum Structures II: Geometric and Numerical Rigidity.

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Outline

1. Background.
2. The “narrow - wide” dichotomy.
3. Aspects of Geometric Rigidity.
4. Numerical rigidity.
5. Further questions.

Setting

- We work with Lagrangians $L^n \hookrightarrow (M^{2n}, \omega)$ so that
$$\mu, \omega : \pi_2(M, L) \to \mathbb{Z} \times \mathbb{R}$$
are proportional with positive proportionality constant.
- Minimal Maslov class $N_L \geq 2$.
-
$$\Lambda^+ = \mathbb{Z}_2[t] \ , \ |t| = -N_L \ , \ \Lambda = \mathbb{Z}_2[t^{-1}, t]$$

As explained by Paul, for generic: $f : L \to \mathbb{R}$ Morse, g r. metric, J almost complex structure

$$\exists\ (\mathcal{C}(L; f, g, J), \partial) \ \ \text{pearl complex}$$

and additional algebraic structures: product, module structure, duality, augmentation etc.

$$QH_*(L) = H_*(\mathcal{C}(L; f, g, J) \otimes \Lambda) \cong HF(L)$$

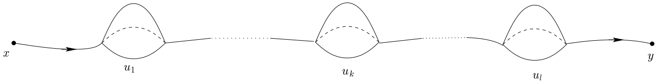


Figure: The differential ∂ counts elements as above.

A dichotomy.

monotone Lagr.	**narrow**	**wide**
Definition:	$QH_*(L) = 0$	$QH_*(L) \cong H_*(L;\mathbb{Z}_2) \otimes \Lambda$
Ham. action:	"displaceable"	not displaceable
Geom. Rigidity:	$Gr(L)$ is small	$Gr(M \setminus L)$ is small
Examples:	$L \subset \mathbb{C}^n$	$T_{Cliff} \subset \mathbb{C}P^n$, $\mathbb{R}P^n \subset \mathbb{C}P^n$

As we shall see, wide Lagrangians also present some numerical rigidity.

Theorem (A)

Assume $H_*(L;\mathbb{Z}_2)$ is generated as an algebra by $H_{\geq n-k}(L;\mathbb{Z}_2)$. If $k+1 \leq N_L$, then L is either narrow or wide. If $k+1 < N_L$, then L is wide.

Remark

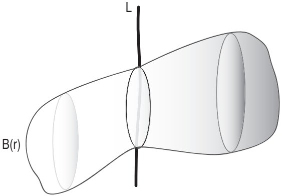
a. Theorem (A) is likely to be true for more general $H_*(L;\mathbb{Z}_2)$.

b. There is no example of monotone Lagrangian which is neither narrow nor wide.

c. However, there are Lagrangians - like T_{Cliff} - which are both wide and narrow with particular choices of coefficients.

Recall:

$$Gr(L) = \sup\{r \geq 0 \ : \ \exists \ e : (B^{2n}(r), \omega_0) \hookrightarrow (M, \omega) \\ e^{-1}(L) = \mathbb{R}^n \cap B^{2n}(r) \ \}$$



Geometric Rigidity: some statements.

Theorem (B)

i. *If* L *is narrow, then* $Gr(L) < \infty$.

ii. *If* L *is narrow and* $H_*(L;\mathbb{Z}_2)$ *is generated as an algebra by* $H_{\geq n-k}(L;\mathbb{Z}_2)$, $N_L = k+1$, *then*

$$\frac{\pi}{2} Gr(L)^2 \leq \tau N_L \quad , \ \tau = \ \textit{monotonicity cst} \ .$$

iii. *Normalize so that* $Gr(\mathbb{C}P^n) = 1$. *If* $L \subset \mathbb{C}P^n$ *is wide, then*

$$Gr(\mathbb{C}P^n \backslash L)^2 \leq \frac{n}{n+1} \quad , \ \frac{Gr(L)^2}{2} + Gr(\mathbb{C}P^n \backslash L)^2 \leq 1 \ .$$

Some proofs.

Idea of Proof of Theorem B: Algebraic data $\Rightarrow\ \exists$ J-disks through each point of L (or of $M\backslash L$) of bounded area $\Rightarrow$ radius bounds by area-radius inequalities.

Remark

Assume f has a single maximum. - $[L]$, the unit for the q. product, verifies $d[L]=0$ in $\mathcal{C}$

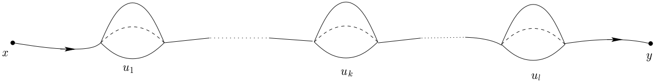


Figure: What if $x = max$?

B.ii. L narrow. Assume L admits a perfect Morse function f $\Rightarrow$ the generators of $\mathcal{C}(L;f)$ are identified with $H_*(L)$.

- if $\forall x \in H_{n-k}(L;\mathbb{Z}_2),\ dx \neq [L]t \ \Rightarrow\ [L] \notin Im(d)$ so L not narrow. Thus, $\exists x \in H_{n-k}(L;\mathbb{Z}_2),\ dx = [L]t$.

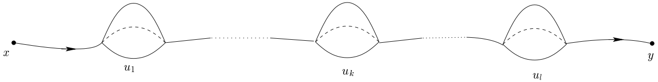


Figure: What if $y = max$?

$\Rightarrow \exists\ J-$disk through max of Maslov N_L.

B.iii L wide. Uses the module structure of $QH(L)$ over $QH(\mathbb{C}P^n)$; $h : \mathbb{C}P^n \to \mathbb{R}$ perfect Morse function, $f : L \to \mathbb{R}$ perfect Morse as before.

- $[L]t^{(n+1)/N_L} = \alpha^{*_{n+1}} * [L] = \alpha^{\cap_n} * (\alpha * [L]) = min(h) * y$ with $y = \alpha * [L]$ and α the class of the hyperplane.

$\Rightarrow \ \exists J - \text{disk through min(h) of Maslov } < n+1$

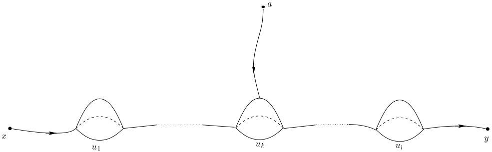


Figure: What if a is the minimum of h ?

The second inequality in B.iii:

$$\frac{Gr(L)^2}{2} + Gr(\mathbb{C}P^n \backslash L)^2 \leq 1$$

is more difficult.
What if we require the two balls involved to be disjunct ? In other words, *mixed packing: embeddings of* relative *balls of radii r_i in (M, L) and of balls of radii ρ_i in $M \backslash L$ all disjoint.*

Example

i. $Gr(\mathbb{T}^n_{\text{clif}})^2 = \frac{2}{n+1}$ (+ Buhovsky).
ii. $Gr(\mathbb{C}P^2, \mathbb{T}^n_{\text{clif}})^2 = \frac{n}{n+1}$.
iii. For every mixed symplectic packing of $(\mathbb{C}P^2, \mathbb{T}^2_{\text{clif}})$ by two balls of radii $(r; \rho)$ we have $\frac{1}{2}r^2 + \rho^2 \leq \frac{2}{3}$. In particular, if the two balls are assumed to have the same radius $r = \rho$ then $r^2 \leq \frac{4}{9}$.

Main idea for numerical rigidity.

L wide "$\Rightarrow$" counts of J-disks are invariant "$\Rightarrow$" GW-invariants defined.

Reason. Assume again L admits a perfect Morse function (a minimal model type construction allows one to drop this).

Then L wide, f perfect $\Rightarrow$ in $\mathcal{C}(L; f, J)$, $d = 0$.

But quantum product, module structure, etc are defined by counting "flow-disk" configurations at the chain level and are invariant in hlgy.

But $d = 0$ means these counts should be invariant at the chain level.

Difficulties.

i. For (f, J), (f', J') with both f, f' perfect there is a canonical chain isomorphism:

$$\phi : \mathcal{C}(L; f, J) \longrightarrow \mathcal{C}(L; f', J')$$

but this is not base preserving.

ii. The definition of all our operations is in terms of configurations made out of disks and flow lines $\Rightarrow$ need to interpret the counts in meaningful geometric terms.

An example.

Focus on a 2-torus $\mathbb{T} \hookrightarrow M^4$; J generic. Let $\Delta = ABC$ be a triangle on $\mathbb{T}$.

i. $n_A = \#_2$ of J-disks of Maslov 2 going through A and crossing (tranversely) the opposite edge.
ii. $n_\Delta = \#_2$ of J-disks of Maslov 4 going through A, B, C (in this order).

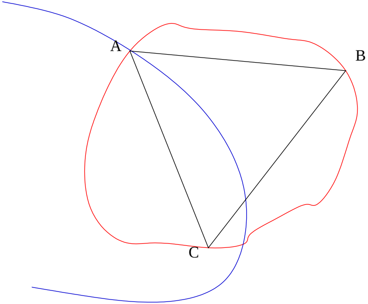


Figure: n_A in blue, n_Δ in red.

Proposition

i. *The sum* $s_1 = n_A + n_B + n_C$ *is independent of* J, Δ.
ii. *If* $s_1 = 1$, *then* $s_2 = n_A n_B + n_\Delta$ *is also an invariant.*
iii. *If* $\mathbb{T} = \mathbb{T}^2_{\text{clif}} \subset \mathbb{C}P^2$, *then* $s_1 = 1$, $s_2 = 1$.

Sketch of proof. By assumption $\mathbb{T}$ is wide:

$$Q^+ H(\mathbb{T}) = H_*(\mathbb{T}; \mathbb{Z}_2) \otimes \Lambda^+ \ .$$

Pick as generators for $Q^+ H(L)$: $w \neq 0 \in H_2(\mathbb{T})$, $a, b \in H_1(\mathbb{T})$ linearly independent, $m \in H_0(\mathbb{T})$ so that m, wt span $Q^+ H_0(L)$. Compute:

$$m * m = s' mt + s'' wt^2 \ , \Lambda = \mathbb{Z}_2[t^{-1}, t] \ , \ |t| = -2 \ , s', \ s'' \in \mathbb{Z}_2 \ .$$

Step 1: Changing basis.

- We first notice that s' is independent of the choice of generators. If m', a', b', w' is any different set of generators with the same properties $\Rightarrow \exists$ change of basis $\phi : Q^+H \to Q^+H$ with: $\phi(w) = w'$, $\phi|_{H_1} \to H_1$ an iso ,

$$\phi(m) = m' + \epsilon t w' \ , \ \epsilon \in \mathbb{Z}_2 \ .$$

If $\epsilon = 0$, $m' * m' = s'w't + s''w't^2$.
If $\epsilon = 1$, then

$$m' * m' = (\phi(m) - tw') * (\phi(m) - tw') = \phi(m * m) + w't^2$$

and so

$$m' * m' = (s'm' + s'w't)t + s''w't^2 + w't^2 = s'm't + (s' + s'' + 1)w't^2 \ .$$

Thus, if $s' = 1$, then s'' is basis independent.

Step 2: Interpreting s' and s''.

Lemma

$s' = s_1$, $s'' = s_2$.

Sketch of Proof for $s' = s_1$. Pick perfect Morse functions $f, g|_{\mathbb{T}}$ with critical points, respectively, x_0, a_1, b_1, x_2 and y_0, a_1', b_1', y_2. In both $\mathcal{C}(L; f)$ and $\mathcal{C}(L; f')$ we have $d = 0$. At the chain level $x_0 * y_0 = kx_0 t + hx_2 t^2$ $h, k \in \mathbb{Z}_2$; k is given by config.:

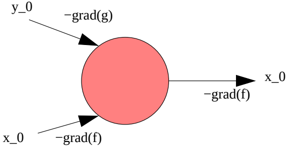


Figure: The Maslov index is 2.

Oops - no such things exist for generic J !

....but also by the configurations:

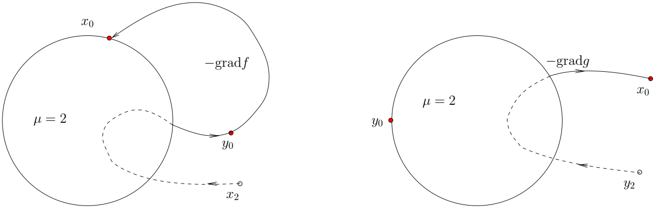


Figure: Contributors to k

We then take: $\Delta = x_2 y_0 x_0$ with edges given by flow lines: of $-grad(f)$ from x_2 to y_0, of $-grad(g)$ from $y_2 \approx x_2$ to x_0 and of $-grad(f)$ from y_0 to x_0. Left configurations $= n_{x_0}$; right ones $= n_{y_0} \Rightarrow k = n_{y_0} + n_{x_0}$. Finally, $y_0 = x_0 + n_{x_2} x_2 t$ (in hlgy) so that: $x_0 * x_0 = x_0 * y_0 + n_{x_2} x_0 t + \ldots$. and

$$s' = n_{x_0} + n_{y_0} + n_{x_2}$$

.

Further questions.

Question:

Is it true that any monotone Lagrangian is either wide or narrow ?

Conjecture:

Any (closed) Lagrangian in $\mathbb{C}^n$ verifies $Gr(L) < \infty$.

Question:

View $m * m - s_1 mt - s_2 wt^2 = 0$ as an equation in the variables s_1 and s_2 with coefficients in $QH(-)$. In general, how to detect such *invariant* equations ? When working over other fields or over $\mathbb{Z}$ what number theoretic equations come up ?

Going beyond the monotone case.

One of the key roles of $N_L \geq 2$:

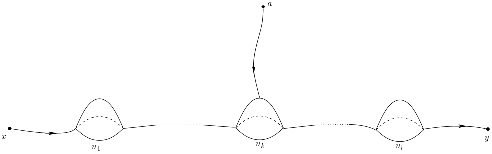


Figure: Configurations giving the module structure.

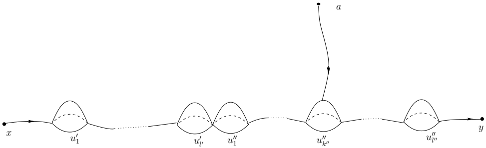


Figure: Possible bubbling.

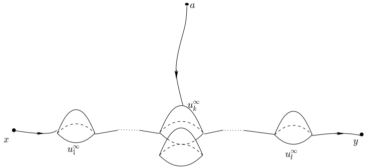


Figure: Impossible bubbling in 1-dim. moduli spaces.

$\Rightarrow$ trees used to model configurations in this case do not increase complexity (the maximal valence of vertices does not change) !

In general this is not the case. Algebra (either A_∞ of Fukaya-Oh-Ohta-Ono or cluster of Lalonde - C.)

$$\approx \mathbb{Q} + A + A \otimes A + \ldots A^{\otimes_k} \ldots$$

In case boundaries of disks in a htpy. class of negative Maslov number represent a non-vanishing hlgy. class, then $1 \in \mathbb{Q}$ becomes a boundary $\Rightarrow$ all hlgical invariants vanish !

However:
- (Lalonde -C.): If L is, oriented, relatively spin, $L \subset \mathbb{C}^n$ and $H_{2k}(L;\mathbb{Q}) = 0$ for $2k \neq 0, \dim(L)$, then $Gr(L) < \infty$.
- (Fukaya): If $L \subset \mathbb{C}^n$ is a $K(\pi, 1)$, oriented, relative spin, then $Gr(L) < \infty$.