Lagrangian submanifolds, from physics to number theory.

Octav Cornea
joint work with Paul Biran

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Outline

Lagrangian topology (q. homology)

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Hochschild cohomology ← Free loop spaces

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Number theory (quadratic forms)

Motivation

- A variety of problems coming from physics: classically stability problems in classical mechanics (that gave rise to Hamiltonian dynamics); more recently, quantum mechanics (inspiring various quantization approaches in geometry); string theory etc.
- Lagrangian topology is a generalization which provides a setting for problems with boundary.
- Detect numerical invariants for Lagrangians - extension of classical problems in enumerative geometry. Results in a different direction by: Welschinger followed by J. Solomon and others in the *real* case.

Setting.

- (M^{2n}, ω) closed symplectic manifold - ω closed 2-form, ω^n volume form.
- $L^n \hookrightarrow M^{2n}$ Lagrangian - submanifold, $\omega|_L = 0$.
- Examples: $\mathbb{R}^n \subset \mathbb{C}^n$, $\mathbb{R}P^n \subset \mathbb{C}P^n$.
- Some properties:
    i. any closed Lagrangian in $\mathbb{C}^n$ is not simply connected (Gromov).
    ii. if $\omega|_{\pi_2(M,L)} = 0$ and $L' = \phi(L)$, ϕ Hamiltonian diffeo, then $\#(L \cap L') \geq \sum_{i=0}^{n} \mathrm{betti}_i(L)$ (Floer). Notice: the smooth bound is $\chi(L)$.

Quantum Homology of L.

Remark. Construction is prototype for modern sympl. top: mix J-holomophic curves - fundamental idea of Gromov + ideas from Morse theory - approach initiated by Floer.

$J : TM \to TM$, $J^2 = -id$, ω-compatible almost complex structure on M, $((X, Y) \to \omega(X, JY)$ defines a r. metric).

$f : L \to \mathbb{R}$ Morse ($\forall,\ x \in \mathrm{Crit(f)}$, $det(\partial f / \partial x) \neq 0$) , g generic riemannian metric.

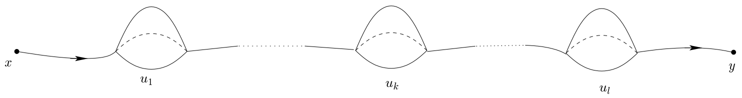
$\exists \quad (\mathcal{C}(L; f, g, J), \partial)$ – *pearl complex* – with associated homology:

$$QH_*(L) = H_*(\mathcal{C}(L; f, g, J))$$

Here $\mathcal{C}(L; f, g, J) = \mathbf{k} < Crit(f) > \otimes \mathbf{k}[t]$, $\mathbf{k}$=field or $\mathbb{Z}$, and the differential ∂ is $\mathbf{k}[t]$-linear, of degree -1 and is written:

$$\partial x = \sum_y a^x_{y,r} y t^r \ , \ a^x_{y,r} \in \mathbf{k} \ .$$

$|x| = \mathrm{ind}_f(x)$, $|t| = -N_L =$ minimal Maslov number of L. The coefficiencts $a^x_{y,r}$ count objects:



which are obtained from moduli spaces:

$$\mathcal{M}(\lambda, J) = \{u : (D^2, S^1) \to (M, L) : \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0\}$$

via evaluation maps:

$$t \in S^1 \ , \ ev_x : \mathcal{M}(\lambda, J) \to L \ , \ ev_t(u) = u(t) \ ;$$

the curves in the picture are flow lines of $-\nabla_g f$; $\sum \mu(u_i) = r$.

Technical conditions required:

i. $\mu, \omega : \pi_2(M, L) \to \mathbb{Z} \times \mathbb{R}$ are proportional with positive proportionality constant (μ is the Maslov class); minimal Maslov class $N_L \geq 2$.

ii. L -relatively spin.

<u>Remark</u> Pearl complex proposed by Oh and Fukaya, detailed by Biran -C. involving technical steps due to Lazarinni.
Additional algebraic structures: *product* and others.

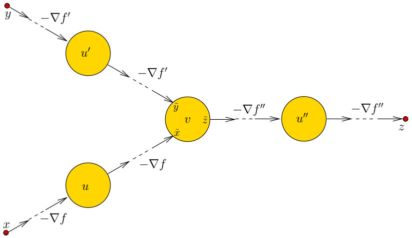


Figure: The elements counted by the product.

Narrow-wide dichotomy.

All known Lagrangians of the type considered here fit into one of the following two classes (when $\mathbf{k}$ =field):

i. Narrow: $QH(L;\mathbf{k}[t,t^{-}])=0$. Example: Lagrangians displaceable by Ham. isotopies.

ii. Wide: $QH(L;\mathbf{k}[t,t^{-}])\cong H(L;\mathbb{Z})\otimes\mathbf{k}[t,t^{-}]$. Here

$$QH(L;\Lambda)=H(\mathcal{C}(L;f,g,J)\otimes_{\mathbf{k}[t]}\mathbf{k}[t,t^{-1}])\ .$$

Example: $\mathbb{R}P^n\subset\mathbb{C}P^n$, the Clifford torus.

For many diffeo types of L this dichotomy is satisfied for each field $\mathbf{k}$ - independently of M - by results of Biran - C., Buhovsky.
It can happen that L is wide for one field but narrow for another.

Search for invariants.

From now on focus on the wide case and assume $\mathbf{k} = \mathbb{Z}$. To fix ideas we also assume $N_L = 2$ and so $|t| = -2$. We have:

$$\eta : H(L;\mathbb{Z}) \otimes \mathbb{Z}[t] \cong QH(L)$$

η is an iso of *graded groups* only and is non-canonical !
Look for numerical invariants in the wide case and over $\mathbb{Z}$ *!*
We use an identification η to write the quantum product in $QH(L)$:

For $x, y \in H(L;\mathbb{Z})$,

$$x * y = x \cdot y + \sum_{z,s>0} m_z^{x,y}(s) z t^s = x \cdot y + m_1^{\eta}(x,y)t + m_2^{\eta}(x,y)t^2 + \dots$$

$*$ is of degree $-n$, $m_z^{x,y}(s) \in \mathbb{Z}$, $z \in H(L;\mathbb{Z})$,

$$m_i : H(L;\mathbb{Z}) \otimes H(L;\mathbb{Z}) \to H(L;\mathbb{Z}) \text{ is of degree } -n + 2i \; .$$

The $m_z^{x,y}$'s and the m_i's depend on η - (and thus on J, f, g) !

Natural problem: find numerical invariants which are polynomial in the structural constants $m_z^{x,y}$.

First step: understand the dependency on η.
If $\eta' : H(L;\mathbb{Z}) \otimes \mathbb{Z}[t] \cong QH(L)$ is a second such isomorphism, then

$$\phi = \eta^{-1} \circ \eta = id + \phi_1 t + \phi_2 t^2 + \dots \ .$$

<u>Remark:</u> Changes in (J, f, g) may lead to $\phi \neq id$.

In essence: $(H(L;\mathbb{Z}) \otimes \mathbb{Z}[t], *)$ is a *deformation* of $(H(L;\mathbb{Z}), \cdot)$ and we need to understand it up to *deformation equivalence*.

Hochschild Homology is precisely the adequate tool for this !

Hochschild cohomology.

A associative algebra,

$$CH^{k,*}(A, A) = \hom(A^{\otimes_k}, A)$$

with $df(x_1, x_2, \ldots x_{k+1}) = x_1 f(x_2, \ldots x_{k+1}) + \ldots \pm f(x_1 + \ldots, x_i x_{i+1}, \ldots x_{k+1}) + \ldots \pm f(x_1, \ldots, x_k) x_{k+1}$.

Hochschild cohomology of A: $HH(A, A) = H^*(CH(A, A))$.

A deformation of A is an associative product structure on: $A \otimes \mathbf{k}[[t]]$ so that

$$x \odot y = xy + m_1(x, y)t + \ldots m_s(x, y)t^s + \ldots \ .$$

In our case $A = H(L; \mathbf{k})$ is graded and $|t| = -N_L = -2$.

By results of Gerstenhaber: m_1 is a Hochschild cycle
$\Rightarrow [m_1] \in HH^2(A, A)$; $[m_1]$ only depends on the equivalence class of the deformation.

Thus, if $A = H(L; \mathbf{k})$, the quantum product $*$ on $H(L; \mathbf{k}) \otimes \mathbf{k}[t]$

$$x * y = x \cdot y + m_1^{\eta}(x, y)t + m_2^{\eta}(x, y)t^2 + \dots$$

gives a class:

$$m_1^L \in HH^{2,*}(H(L; \mathbf{k}), H(L; \mathbf{k}))$$

which is independent of

$$\eta : H(L; \mathbf{k}) \otimes \mathbf{k}[t] \cong QH(L) \ .$$

Quadratic forms

By restriction we have:

$$m_1^{\eta} : H_{n-1}(L; \mathbf{k}) \otimes H_{n-1}(L; \mathbf{k}) \to H_n(L; \mathbf{k})$$

We put:

$$q_L : H_{n-1}(L; \mathbf{k}) \to \mathbf{k} \ , \ \ q_L(x)[L] = m_1^{\eta}(x, x)$$

This is a quadratic form and it has the property that it only depends on $[m_1^{\eta}] = m_1^L$.

Indeed, $m_1^{\eta} - m_1^{\eta'} = df \quad \Rightarrow m_1^{\eta}(x, x) - m_1^{\eta'}(x, x) =$
$x \cdot df(x) \pm f(x \cdot x) \pm f(x) \cdot x = 0 \quad \forall \ \ x \in H_{n-1}(L; \mathbf{k})$.

<u>Remark</u> This argument only depends on A and t being graded.

Theorem

There exist wide Lagrangian tori $T^n = S^1 \times S^1 \times \dots S^1$ *so that the map* $(H(T^n;\mathbb{Z}) \otimes \mathbb{Z}[t], *)/\mathrm{geom.def.equiv} \to Quad(H_{n-1}(T^n;\mathbb{Z}))$ *is a bijection.*

- $Quad(V)$ are the quadratic forms on V.
- *Geometric deformation equivalence* means the equivalences $\phi = \eta^{-1} \circ \eta'$ obtained by changing the geometric data (J, f, g).

An example of such a torus is the Clifford torus in $\mathbb{C}P^n$.

The arguments for the Clifford torus are related to work of Cho who also considered some variants of the quadratic forms discussed here.

Corollary

There is just one integral, "universal" numerical invariant of L which is polynomial in the structural constants of the quantum product and is non-trivial on tori: the discriminant, Δ_L, of q_L.

Argument:

- By the theorem such an invariant should be a polynomial invariant in the coefficients of the quadratic form q_L up to Gauss equivalence
- It is well-known in number theory (after Hilbert) that the discriminant is the single such invariant !

Remark There are other interesting relations with the theory of quadratic forms that go beyond this talk. Also there are some geometric interpretations of the discriminant.

Free loop spaces.

Representation of disks (of Maslov class 2) in the free loop space $\Lambda L = \{f : S^1 \to L : f \text{ continuous}\}$:

$$\psi_{\lambda,J} : \mathcal{M}(\lambda, J) \to \Lambda L \quad \Rightarrow \quad \alpha_{\lambda,J} \in H_n(\Lambda L; \mathbb{Z})$$

There exists a map constructed by Jones:

$$J : H_*(\Lambda L) \to HH^{\bullet,*-n}(C(L), C(L))$$

In favourable cases (for tori in particular)
$HH^{s,*}(C(L), C(L)) \cong HH^{s,*}(H(L;\mathbb{Z}), H(L;\mathbb{Z}))$ and then

$$J(\sum \alpha_{\lambda,J})|_{HH^{2,*}(H(L;\mathbb{Z}),H(L;\mathbb{Z}))} = m_1^L \ .$$

Thus the loop representation suffices to determine q_L.
Remark. i.This result is related to some work of Fukaya.
ii. It is also possible to write the representation above as a (Landau-Ginsburg) "potential" and then the discriminant can be interpreted as the determinant of the Hessian of this potential.

An example.

Focus on a 2-torus $T^2 \hookrightarrow M^4$; J a.c str. (generic). Let $\Delta = ABC$ be a triangle on $\mathbb{T}$.

i. $n_A = \#_2$ of J-disks of Maslov 2 going through A and crossing (tranversely) the opposite edge.

ii. $n_\Delta = \#_2$ of J-disks of Maslov 4 going through A, B, C (in this order).

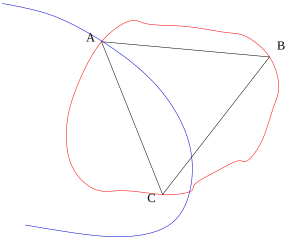


Figure: n_A - blue, n_Δ - red;

$\Delta_L = n_A^2 + n_B^2 + n_C^2 - 2(n_A n_B + n_A n_C + n_C n_B) + 4n_\Delta$