

LAGRANGIAN COBORDISM

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1

CONTENTS

1. Introduction	2
2. Preliminaries	5
2.1. Monotonicity	5
2.2. A quick review of Lagrangian Floer theory	6
2.3. The Fukaya category, derived and not	12
2.4. Cone decompositions over a triangulated category	13
3. Cobordism: definitions and geometric aspects	18
3.1. Basic definitions	18
3.2. Some constructions	21
3.3. The category $\mathcal{Cob}^d(M)$	22
4. Construction of $\tilde{\mathcal{F}}$	24
4.1. The functors \mathcal{F} and $\tilde{\mathcal{F}}$	26
4.2. Some technical ingredients	33
5. Quantum homology and the morphisms in $\mathcal{Cob}^d(M)$	41
5.1. Quantum homology for Lagrangians with cylindrical ends	42
5.2. Elementary cobordisms	48
5.3. Two ended splitting	49
6. Examples	54
6.1. Topologically non-trivial monotone cobordisms	54
6.2. The trace of surgery as Lagrangian cobordism	55
6.3. Cobordant Lagrangians that are not isotopic	59
References	61

Date: September 26, 2011.

The second author was supported by an NSERC Discovery grant and a FQRNT Group Research grant.

¹Early versions of this work have been presented by one or both authors at an MSRI workshop (Spring 2010), Oberwolfach meeting (Summer 2010), Kyoto Symplectic workshop (February 2011) and the Geometry and Physics conference at the Fields Institute (May 2011). We thank the organizers of all these meetings for giving us the opportunity to present our work.

1. INTRODUCTION

Lagrangian cobordism is a natural notion, initially introduced by Arnold [Arn1, Arn2] at the beginnings of symplectic topology. This notion was studied by Eliashberg [Eli] and Audin [Aud] who showed that, in full generality, this is a very flexible notion that can be translated to purely algebraic topological constraints. By contrast, the work of Chekanov [Che] points out a certain form of rigidity valid in the case of monotone cobordisms.

The present paper is a review of work in progress on the properties of Lagrangian cobordism. Ideas of proof are only sometimes sketched, detailed arguments will appear in a more ample manuscript currently in preparation.

Our focus here is to explain how cobordism naturally organizes the Lagrangian submanifolds of a fixed symplectic manifold (M, ω) in a category and to describe some functors relating this cobordism category to the derived Fukaya category. The data is organized in the following diagram

$$(1) \quad \begin{array}{ccc} \mathcal{C}ob^*(M) & \xrightarrow{\mathcal{F}} & \Sigma D\mathcal{F}uk^*(M) \\ & \searrow \tilde{\mathcal{F}} & \nearrow \mathcal{P} \\ & T^S D\mathcal{F}uk^*(M) & \end{array}$$

and most of the paper will be spent explaining the different objects in this diagram.

We start by describing the right side: $\mathcal{F}uk^*(M)$ stands for the Fukaya category of M , $D\mathcal{F}uk^*(M)$ stands for the derived Fukaya category of M , the decoration $*$ restricts the Lagrangians considered to a special class to be described in more detail later (see Theorem A). We assume the reader is familiar with the definition of the Fukaya category as described in Seidel's book [Sei3] but we will shortly recall some of the relevant properties later in the paper. The category $D\mathcal{F}uk^*(M)$ is triangulated and $\Sigma D\mathcal{F}uk^*(M)$ is the stabilization of $D\mathcal{F}uk^*(M)$ in the sense that the morphisms of $D\mathcal{F}uk^*(M)$ are enriched by those morphisms that shift “degree” (see §2.4).

Interesting computations are generally easier to understand in the *derived* Fukaya category, as this is a genuine triangulated category and not only an A_∞ one as the Fukaya category itself. However, to actually complete these computations one often needs to first work at the level of the Fukaya category and deduce the results of actual interest by passage to homology. One of the main points of the current paper is that in some interesting cases the existence of the functor \mathcal{F} provides a shortcut by going directly from a geometric category, the cobordism category $\mathcal{C}ob^*(M)$, to the derived Fukaya category. In this sense, the fundamental property of the functor \mathcal{F} is its factorization indicated at the bottom of diagram (1). Here the category $T^S D\mathcal{F}uk^*(M)$ is obtained from the category $D\mathcal{F}uk^*(M)$

by a general construction (apparently new) that associates to any triangulated category \mathcal{C} a new category $T^S\mathcal{C}$ - the category of (stable) triangular (or cone) resolutions over \mathcal{C} . The morphisms sets $\text{hom}(x, -)$ in this category parametrize the ways in which x can be resolved by iterated exact triangles (or cone attachments). The category $T^S\mathcal{C}$ comes with a canonical projection $\mathcal{P} : T^S\mathcal{C} \rightarrow \Sigma\mathcal{C}$. The morphisms $[V]$ in the cobordism category $\text{Cob}^*(M)$ are represented by a certain type of Lagrangian cobordisms V with possibly multiple ends. The factorization of \mathcal{F} through $\tilde{\mathcal{F}}$ is of interest because it associates to each such morphism $[V] \in \text{Mor}_{\text{Cob}^*(M)}$ a series of exact triangles in $D\mathcal{F}uk^*(M)$ that relate the ends of V .

Here are three consequences of our constructions:

- i) If two connected Lagrangians L, L' are monotone cobordant, then up to shift they represent the same object in the appropriate derived Fukaya category. In particular, for any third Lagrangian N , the two Floer homologies $HF(N, L)$ and $HF(N, L')$ are isomorphic (if defined).
- ii) A monotone Lagrangian cobordism $(V; L, L')$ with L and L' connected is a quantum h -cobordism in the sense that quantum homology verifies

$$QH(V, L) = QH(V, L') = 0 .$$

In particular, if V, L and L' are exact, then $(V; L, L')$ is an h -cobordism. See §5.2 for more details.

- iii) Let $(V; L, (L_1, L_2))$ be a monotone Lagrangian cobordism with one positive end and two negative ends. If $QH(L_1), QH(L_2) \neq 0$ then:

$$rk(QH_k(L)) \leq |rk(QH_k(L_1)) - rk(QH_k(L_2))| .$$

In particular, such a cobordism does not exist for $\dim L = 2$. See §5.3 for more details.

The paper also contains a discussion of some examples. In particular, we notice that the trace of Lagrangian surgery gives rise to a Lagrangian cobordism. We use this fact to produce examples of connected Lagrangians that are monotone cobordant but not isotopic even in the *smooth* category.

To conclude, Lagrangian cobordism - even in its monotone version - is somewhat more flexible than Hamiltonian isotopy even if, remarkably, it preserves Floer homology and all similar invariants. Thus, this notion appears to have the potential to shed new light both on these invariants as well as on the geometry of Lagrangians.

Here is an outline of the restrictions required for the constructions above (see §2). First, all Lagrangians have to verify a monotonicity condition in a uniform way. This is crucial for the transversality issues involving bubbling of disks to be approachable by the

methods in [BC2] and [BC4]. Assuming monotonicity, Floer homology is defined over the universal Novikov ring, denoted in the paper by \mathcal{A} . Another version applies to Lagrangians $L \subset M$ so that all elements in the image of the morphism $H_1(L; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$ induced by inclusion are torsion (see [Oh1]). With this additional constraint, Floer homology is defined over \mathbb{Z}_2 . In both cases the resulting homology theory is in general not graded. The universal Novikov ring and \mathbb{Z}_2 will be the two coefficient rings that will be used in the paper. It is over \mathbb{Z}_2 that the functoriality results above are valid as stated (assuming that the preceding condition on the H_1 's is imposed on all Lagrangians as well as on cobordisms). At the same time, our geometric constructions will be performed over \mathcal{A} .

The fact that the ring \mathcal{A} is a large ring is a significant advantage at times and it plays a crucial role in the proof of the points ii) and iii) above. However, the Floer chains and structures defined over \mathcal{A} are highly sensitive to perturbations and as a consequence functoriality takes a much more complicated form, one that we will not attempt to establish or further discuss here. The point i) above exemplifies this issue: the first part of the statement (concerning the identifications of objects in the Fukaya category) is claimed over \mathbb{Z}_2 (under the assumptions on the H_1 's) but the second part of the statement, concerning the isomorphism of $HF(N, L)$ and $HF(N, L')$, is true over \mathcal{A} without these assumptions. However, over \mathcal{A} the isomorphism between $HF(N, L)$ and $HF(N, L')$ is determined by the cobordism V relating L to L' only up to multiplication by a unit in \mathcal{A} . In contrast, over \mathbb{Z}_2 , this isomorphism is completely determined by V .

The structure of the paper is as follows. In §2 we fix our setting and recall some basic facts about Floer homology and the Fukaya category. All of this can be safely skipped by experts except for §2.4 which contains the definition of the category of cone resolutions over a triangulated category. Section 3 contains the main geometric definitions that are at the center of our study of cobordism as well as the definition of the cobordism category that is the central subject of interest in the paper. Section 4 is dedicated to discussing the functor $\tilde{\mathcal{F}}$. It starts with the basic ideas behind the construction of this functor and it pursues with a description of the main technical ingredients needed to prove the commutativity in (1). As mentioned before, the full details of the proof will appear in a later publication. In §5 we analyze some obstructions to the existence of morphisms in our cobordisms categories that are based on the interplay between the constructions involved in the definition of the functor $\tilde{\mathcal{F}}$ and the Lagrangian quantum homology. Finally, §6 contains the analysis of surgery through the prism of cobordism as well as the related examples of non-isotopic cobordant Lagrangians.

Acknowledgments. The first author would like to thank Dietmar Salamon and Ivan Smith for helpful discussions on Floer theory and Fukaya categories. The second author thanks

Mohammed Abouzaid, Denis Auroux, François Charette, Yasha Eliashberg, Paul Gauthier and Clément Hyvrier for useful discussions as well as the MSRI for its hospitality during the Fall of 2009, when the work presented here was initiated.

2. PRELIMINARIES

In this paper (M^{2n}, ω) is a connected symplectic manifold. We assume that M is compact but the constructions described have immediate adaptations to the case when M is only tame (see [ALP]). Lagrangian submanifolds $L^n \subset M^{2n}$ will be generally assumed to be closed unless otherwise indicated.

2.1. Monotonicity. Given a Lagrangian submanifold $L \subset M$ there are two canonical morphisms

$$\omega : \pi_2(M, L) \rightarrow \mathbb{R}, \quad \mu : \pi_2(M, L) \rightarrow \mathbb{Z}$$

the first given by integration of ω and the second being the Maslov index. We will say that the Lagrangian L is *monotone* if there exists a positive constant $\rho > 0$ so that for all $\alpha \in \pi_2(M, L)$ we have $\omega(\alpha) = \rho\mu(\alpha)$ and moreover the minimal Maslov number

$$N_L := \min\{\mu(\alpha) : \alpha \in \pi_2(M, L), \omega(\alpha) > 0\}$$

verifies $N_L \geq 2$.

In what follows we will use \mathbb{Z}_2 as the ground ring for homological considerations. We remark that most of the discussion below generalizes under additional assumptions on the Lagrangians to arbitrary rings. We therefore denote the ground ring by K , keeping in mind that in this paper $K = \mathbb{Z}_2$.

To each connected closed, monotone Lagrangian L there is an associated basic Gromov-Witten type invariant $d_L \in K$. In short, d_L is the number (in K) of J -holomorphic disks of Maslov index 2 going through a generic point $P \in L$ (for J a generic almost complex structure that is tamed by ω). Here is a more precise definition. Let $\mathcal{J} = \mathcal{J}(M, \omega)$ be the set of almost complex structures on M that are tamed by ω . Denote by $H_2^D = H_2^D(M, L)$ the image of the Hurewicz homomorphism $\pi_2(M, L) \rightarrow H_2(M, L)$ and consider $B \in H_2^D(M, L)$. For $J \in \mathcal{J}$ let $\widetilde{\mathcal{M}}(B, J)$ be the space of maps $u : (D, \partial D) \rightarrow (M, L)$ that are J -holomorphic and so that $u_*([D]) = B$. Denote by $G = \text{Aut}(D) \cong \text{PSL}(2, \mathbb{R})$ the group of biholomorphisms of the disk. Consider now the space of disks with one marked point on the boundary, $(\widetilde{\mathcal{M}}(B, J) \times \partial D)/G$, (where G acts as follows $\sigma \cdot (u, z) = (u \circ \sigma^{-1}, \sigma(z))$, for $\sigma \in G$). Assume now that $\mu(B) = 2$ and that J is generic. Then, by standard arguments, (see for instance [BC4]) it follows that $(\widetilde{\mathcal{M}}(B, J) \times \partial D)/G$ is a smooth compact manifold without boundary and of (real) dimension n . Consider the evaluation map

$$ev : (\widetilde{\mathcal{M}}(B, J) \times \partial D)/G \rightarrow L, \quad ev(u, z) = u(z).$$

We denote by $\nu(B) \in K$ the degree of this map. Standard arguments then show that $\nu(B)$ does not depend on J but only on B . Moreover, there can be at most a finite number of classes $B \in H_2^D$ with $\nu(B) \neq 0$ and we put:

$$(2) \quad d_L = \sum_{B \in H_2^D; \mu(B)=2} \nu(B) .$$

It is easy to see by the Gromov compactness theorem that this number is independent of the almost complex structure J .

A family of Lagrangian submanifolds $L_i, i \in I$ is called *uniformly monotone* if each L_i is monotone and the following condition is satisfied: there exists $d \in K$ so that for all $i \in I$ we have $d_{L_i} = d$ and, if $d \neq 0$, then there exists a positive real constant ρ so that the monotonicity constant of L_i equals ρ for all $i \in I$.

In the absence of other indications, all the Lagrangians used in the paper will be assumed monotone and, similarly, the Lagrangian families will be assumed uniformly monotone.

2.2. A quick review of Lagrangian Floer theory. In the sequel we will use various versions of Lagrangian Floer homology as well as Lagrangian quantum homology for monotone Lagrangian submanifolds which we now briefly recall. We refer the reader to [Oh1, Oh2, Oh3] for the foundations of Floer homology for monotone Lagrangians, and to [FOOO2, FOOO3] for the general case. For Lagrangian quantum homology see [BC2, BC4, BC3, BC1].

2.2.1. Lagrangian Floer homology. Let $L_0, L_1 \subset M$ be two monotone Lagrangian submanifolds with $d_{L_0} = d_{L_1} = d$. In case $d \neq 0$ we assume in addition that L_0 and L_1 have the same monotonicity constant (or in other words that the pair (L_0, L_1) is uniformly monotone).

Denote by \mathcal{A} the universal Novikov ring, i.e.

$$\mathcal{A} = \left\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} \mid a_k \in K, \lim_{k \rightarrow \infty} \lambda_k = \infty \right\},$$

endowed with the obvious multiplication. We do not grade \mathcal{A} .

Denote by $\mathcal{P}(L_0, L_1) = \{\gamma \in C^0([0, 1], M) \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}$ the space of paths in M connecting L_0 to L_1 . For every $\eta \in \pi_0(\mathcal{P}(L_0, L_1))$ we denote the path connected component of η by $\mathcal{P}_\eta(L_0, L_1)$.

Fix $\eta \in \pi_0(\mathcal{P}(L_0, L_1))$ and let $H : M \times [0, 1] \rightarrow \mathbb{R}$ be a Hamiltonian function with Hamiltonian flow ψ_t^H . We assume that $\psi_1^H(L_0)$ is transverse to L_1 . (We generally view H as a mean of possible perturbation of L_0 , and when not needed we will often use $H = 0$.) We denote by $\mathcal{O}_\eta(H)$ the set of paths $\gamma \in \mathcal{P}_\eta(L_0, L_1)$ which are orbits of the flow

ψ_t^H . Finally, we choose also a generic 1-parametric family of almost complex structures $\mathbf{J} = \{J_t\}_{t \in [0,1]}$ compatible with ω .

Using this data one can define in a standard way the Floer complex $CF(L_0, L_1; \eta; H, \mathbf{J})$ with coefficients in \mathcal{A} . Recall that the underlying module of this complex is generated over \mathcal{A} by the elements of $\mathcal{O}_\eta(H)$. The Floer differential $\partial : CF(L_0, L_1; \eta; H, \mathbf{J}) \rightarrow CF(L_0, L_1; \eta; H, \mathbf{J})$ is defined as follows. For a generator $\gamma_- \in \mathcal{O}_\eta(H)$ define

$$\partial(\gamma_-) = \sum_{\gamma_+ \in \mathcal{O}_\eta(H)} \sum_{u \in \mathcal{M}_0(\gamma_-, \gamma_+; H, \mathbf{J})} \varepsilon(u) T^{\omega(u)} \gamma_+.$$

Here $\mathcal{M}_0(\gamma_-, \gamma_+; H, \mathbf{J})$ stands for the 0-dimensional components of the space of Floer strips $u : \mathbb{R} \times [0, 1] \rightarrow M$ connecting γ_- to γ_+ , modulo the \mathbb{R} -action coming from translation in the s coordinate. The strips u are assumed to have finite energy and we denote by $\omega(u) = \int_{\mathbb{R} \times [0,1]} u^* \omega$ the symplectic area of u . Finally, each such strip u comes with a sign $\varepsilon(u) = \pm 1 \in K$. As mentioned before, in this paper we will mostly work with $K = \mathbb{Z}_2$ hence the signs $\varepsilon(u)$ are irrelevant. However, under additional assumptions on L_0, L_1 (such as orientability and prescribed spin structures) the moduli spaces of Floer strips can be canonically oriented which imply the signs $\varepsilon(u) \in \mathbb{Z}$, hence one can work with any ring K . See [FOOO3] for a detailed account on how to define orientations in Floer theory.

Under the preceding assumptions on L_0, L_1 we have $\partial^2 = 0$ hence one can define the homology

$$HF(L_0, L_1; \eta; H, \mathbf{J}) = \ker(\partial) / \text{image}(\partial).$$

Remark 2.2.1. In the general context of the paper, with CF defined over \mathcal{A} , the chain complex CF is not graded and hence HF has no grading too. In special situations one can endow CF with some grading though not always over \mathbb{Z} (e.g. when L_0 and L_1 are both oriented, then there is a \mathbb{Z}_2 -grading). See [Sei1] for a systematic approach to these grading issues. Even if in this paper we will stick geometrically to the general, non-graded situation, we will perform most algebraic constructions in such a way that adaptation to one of these graded settings is simple.

Standard arguments show that the homology $HF(L_0, L_1; \eta; H, \mathbf{J})$ is independent of the additional structures H and \mathbf{J} up to canonical isomorphisms. We will therefore omit H and \mathbf{J} from the notation.

We will often consider all components $\eta \in \pi_0(\mathcal{P}(L_0, L_1))$ together i.e. take the direct sum complex

$$(3) \quad CF(L_0, L_1; H, \mathbf{J}) = \bigoplus_{\eta} CF(L_0, L_1; \eta; H, \mathbf{J})$$

with total homology which we denote $HF(L_0, L_1)$. There is an obvious inclusion map $i_\eta : HF(L_0, L_1; \eta) \longrightarrow HF(L_0, L_1)$.

Remarks 2.2.2. (1) When L_0 and L_1 are mutually transverse we can take $H = 0$ in $CF(L_0, L_1; H, \mathbf{J})$ in which case the generators of the complex are the intersection points $L_0 \cap L_1$ and Floer trajectories $\mathcal{M}_0(\gamma_-, \gamma_+; 0, \mathbf{J})$ are genuine holomorphic strips connecting intersection points $\gamma_-, \gamma_+ \in L_0 \cap L_1$. When $H = 0$ we will omit it from the notation and just write $CF(L_0, L_1; \mathbf{J})$. We will sometimes omit \mathbf{J} too when its choice is obvious.

(2) The use of families of almost complex structures $\mathbf{J} = \{J_t\}_{t \in [0,1]}$ is needed for transversality reasons typically occurring in the construction of Floer homology. However, it is still possible to work with almost complex structure J that do not depend on t , provided the Hamiltonian H is chosen to be generic (see [FHS]).

2.2.2. Moving boundary conditions. As before assume that L_0 and L_1 are two transverse Lagrangians. Fix the component η and the almost complex structure \mathbf{J} . We also fix once and for all a path γ_0 in the component η . Now let $\varphi = \{\varphi_t\}_{t \in [0,1]}$ be a Hamiltonian isotopy starting at $\varphi_0 = \mathbb{1}$. The isotopy φ induces a map

$$\varphi_* : \pi_0(\mathcal{P}(L_0, L_1)) \longrightarrow \pi_0(\mathcal{P}(L_0, \varphi_1(L_1)))$$

as follows. If $\eta \in \pi_0(\mathcal{P}(L_0, L_1))$ is represented by $\gamma : [0, 1] \rightarrow M$ then $\varphi_*\eta$ is defined to be the connected component of the path $t \mapsto \varphi_t(\gamma(t))$ in $\mathcal{P}(L_0, \varphi_1(L_1))$.

The isotopy φ induces a canonical isomorphism

$$(4) \quad c_\varphi : HF(L_0, L_1; \eta) \longrightarrow HF(L_0, \varphi_1(L_1); \varphi_*\eta)$$

which comes from a chain level continuation map defined using moving boundary conditions (see e.g. [Oh1]). The isomorphism c_φ depends only on the homotopy class (with fixed end points) of the isotopy φ .

The definition of the isomorphism c_φ involves some subtleties due to our use of the universal Novikov ring \mathcal{A} as base ring: given that the symplectic area of the strips with moving boundaries can vary inside a one parametric moduli space it follows that the naive definition of the morphism c_φ - so that each strip is counted with a weight given by its symplectic area - does not provide a chain map. We explain here in more detail the construction of the map c_φ .

Let $\varphi = \{\varphi_t^H\}$ be a Hamiltonian isotomorphism generated by H . Denote $L'_1 = \varphi_1^H(L_1)$ and assume that L'_1 is also transverse to L and that the Floer complexes $C_1 = CF(L_0, L_1; \eta; 0; \mathbf{J})$ and $C_2 = CF(L_0, L'_1; \varphi_*\eta; 0; \mathbf{J})$ are well-defined.

Put $\psi_t = (\varphi_t^H)^{-1}$. We define the functional $\Theta_H : \mathcal{P}_{\varphi_*\eta}(L_0, L_1) \rightarrow \mathbb{R}$,

$$\Theta_H(\gamma) = \int_0^1 H(\psi_t(\gamma(t)))dt - \int_0^1 H(\gamma_0(t))dt .$$

Let $\beta : \mathbb{R} \rightarrow [0, 1]$ be a smooth function so that $\beta(s) = 0$ for $s \leq 0$, $\beta(s) = 1$ for $s \geq 1$ and β is strictly increasing on $(0, 1)$. Let x be a generator of C_1 . We write

$$(5) \quad \tilde{c}_\varphi(x) = \sum_y \left(\sum_u T^{\omega(v_u) - \Theta_H(y)} \right) y$$

with y going over the generators of C_2 and u going over all the elements of a zero dimensional moduli space of solutions to Floer's homogeneous equation $\bar{\partial}_{\mathbf{J}}u = 0$ that start at x and end at y and verify the boundary conditions

$$u(s, 0) \subset L_0, \quad u(s, 1) \subset \varphi_{\beta(s)}(L_1) .$$

Here the element $v_u : \mathbb{R} \times [0, 1] \rightarrow M$ is defined by the formula $v_u(s, t) = \psi_{t\beta(s)}u(s, t)$ so that $v_u(s, t)$ is a strip with boundary conditions on L_0 and L_1 . It is easy to check that with this definition \tilde{c}_φ is a chain map. Note that the quantity $|\omega(v_u) - \omega(u)|$ is bounded by the variation of H so that \tilde{c}_φ is well defined over \mathcal{A} . Further, the map c_φ induced in homology by \tilde{c}_φ , depends only on the homotopy class with fixed end points of φ . Similar constructions can be used to adapt the rest of the usual Floer theoretic machinery to this moving boundary situation. They show in particular that c_φ induces an isomorphism in homology.

Remark 2.2.3. This argument also applies without modification to cases when M is not compact (but e.g. tame), if we have some control which insures that all solutions u of finite energy as above have their image inside a fixed compact set $K \subset M$.

2.2.3. The pearl complex and Lagrangian quantum homology. Next we briefly describe the version of Lagrangian quantum homology that will be use later in the paper. Let $L \subset M$ be a monotone Lagrangian with minimal Maslov number N_L . Denote by $\Lambda = K[t^{-1}, t]$ the ring of Laurent polynomials in t , graded so that $|t| = -N_L$. (In case L is weakly exact, i.e. $\omega(A) = 0$ for every $A \in \pi_2(M, L)$ we put $\Lambda = K$.) The chain complex used to define the Lagrangian quantum homology $QH(L)$ is denoted by $\mathcal{C}(\mathcal{D})$ and called the pearl complex. It is associated to a triple of auxiliary structures $\mathcal{D} = (f, (\cdot, \cdot), J)$ where $f : L \rightarrow \mathbb{R}$ is a Morse function on L , (\cdot, \cdot) is a Riemannian metric on L and J is an ω -compatible almost complex structure on M . With these structures fixed we have

$$\mathcal{C}(\mathcal{D}) = K\langle \text{Crit}(f) \rangle \otimes \Lambda.$$

This complex is \mathbb{Z} -graded with grading combined from both factors. The grading on the left factor is defined by Morse indices of the critical points. The differential d on this

complex is defined by counting so called pearly trajectories (which are combined from negative gradient flow lines of f with J -holomorphic disks). The homology $H_*(\mathcal{C}(\mathcal{D}), d)$ is independent of \mathcal{D} (up to canonical isomorphisms) and is denoted by $QH_*(L)$. Note that this homology is \mathbb{Z} -graded. We refer the reader to [BC2, BC4, BC3] for the precise construction of this homology.

In what follows we will actually need also to enrich the coefficients of $QH(L)$ to the Novikov ring \mathcal{A} . This is done as follows. Denote by

$$A_L = \min\{\omega(A) \mid A \in \pi_2(M, L), \omega(A) > 0\}$$

the minimal positive area of a disk with boundary on L . We use the convention that $\min \emptyset = \infty$. The Novikov ring \mathcal{A} becomes an algebra over Λ via the ring morphism induced by $\Lambda \ni t \mapsto T^{A_L} \in \mathcal{A}$. (If L is weakly exact we have $\Lambda = K$ and we view \mathcal{A} as an algebra over Λ in the usual way.) Consider now

$$\mathcal{C}(\mathcal{D}; \mathcal{A}) = \mathcal{C}(\mathcal{D}) \otimes_{\Lambda} \mathcal{A}, \quad d_{\mathcal{A}} = d \otimes_{\Lambda} \text{id}.$$

The homology of this complex will be denoted by $QH(L; \mathcal{A})$. Note that in contrast to $\mathcal{C}(\mathcal{D})$ and $QH(L)$ their analogues over \mathcal{A} , $\mathcal{C}(\mathcal{D}; \mathcal{A})$ and $QH(L; \mathcal{A})$ are not graded.

To avoid confusion between Λ and \mathcal{A} we will sometimes write $QH(L; \Lambda)$ for $QH(L)$.

2.2.4. The PSS isomorphism. Let $L \subset M$ be a monotone Lagrangian. Denote by $\eta_0 \in \pi_0(\mathcal{P}(L, L))$ the connected component of a constant path on L . In contrast to the case of two general Lagrangians, the Floer homology of the pair (L, L) is all concentrated in the component η_0 , i.e. $i_{\eta_0} : HF(L, L; \eta_0) \longrightarrow HF(L, L)$ is an isomorphism.

The PSS (Piunikin-Salamon-Schwarz) isomorphism is a comparison between the Lagrangian quantum homology and the self Floer homology of L . More precisely, there is a canonical isomorphism

$$PSS : QH(L; \mathcal{A}) \longrightarrow HF(L, L)$$

coming from a chain morphism $\widetilde{PSS}_{\eta_0} : \mathcal{C}(\mathcal{D}; \mathcal{A}) \longrightarrow CF(L, L; \eta_0; H, \mathbf{J})$. The construction of \widetilde{PSS}_{η_0} is very similar to the one described in [Alb, BC4, BC3] over the ring Λ . The only needed modification when working with \mathcal{A} is to incorporate the total areas of the connecting trajectories that appear in the morphism \widetilde{PSS}_{η_0} .

The map $PSS_{\eta_0} : QH(L; \mathcal{A}) \longrightarrow HF(L, L; \eta_0)$ induced in homology by \widetilde{PSS}_{η_0} is an isomorphism. The isomorphism PSS is now defined as $i_{\eta_0} \circ PSS_{\eta_0}$.

There also exists a version of the PSS morphism which is defined using moving boundary conditions. Specifically, assume that φ is a Hamiltonian isotopy and let $\varphi_1(L) = L'$. Then we have an isomorphism

$$\widetilde{PSS} : QH(L; \mathcal{A}) \longrightarrow HF(L, L') .$$

Its definition is straightforward in view of the standard case recalled above and §2.2.2.

2.2.5. *Products and other structures.* The Lagrangian and ambient quantum homologies as well as the Floer homologies are all related via several compatible algebraic structures endowed with ring and module operations. We will typically denote these operations by the same notation $*$ since they are algebraically compatible. We refer the reader to [BC2, BC4, BC3] for more details.

The quantum homologies $QH(L; \Lambda)$ and $QH(L; \mathcal{A})$ can be endowed with a quantum product $*$ which turns them into associative rings with unity (note that they are in general not commutative). We denote the unity by $[L] \in QH_n(L; \Lambda)$ (in analogy to the fundamental class in classical homology).

For a uniformly monotone pair Lagrangians (L_1, L_2) the Floer homology $HF(L_1, L_2)$ is a left module over $QH(L_1; \mathcal{A})$ and a right module over $QH(L_2; \mathcal{A})$. We denote these module operations by $\alpha_1 * x$ and $x * \alpha_2$, for $x \in HF(L_1, L_2)$, $\alpha_1 \in QH(L_1; \mathcal{A})$, $\alpha_2 \in QH(L_2; \mathcal{A})$. The two module structures are mutually compatible in the sense that associativity holds: $(\alpha_1 * x) * \alpha_2 = \alpha_1 * (x * \alpha_2)$.

For a uniformly monotone triple of Lagrangians L_1, L_2, L_3 there is an associative product

$$HF(L_1, L_2) \otimes_{\mathcal{A}} HF(L_2, L_3) \longrightarrow HF(L_1, L_3), \quad x \otimes y \longmapsto x * y$$

called the triangle or Donaldson product. When $L = L_1 = L_2 = L_3$ this product can be identified via the PSS isomorphism with the quantum product on $QH(L; \mathcal{A})$.

There is a duality isomorphism relating $HF(L_1, L_2)$ and $\text{hom}_{\mathcal{A}}(HF(L_2, L_1), \mathcal{A})$. In case $L = L_1 = L_2$ is exact this duality reduces to Poincaré duality.

Finally, the Floer homology $HF(L_1, L_2)$ is also a module over the ambient quantum homology $QH(M)$. We denote this operation by $*$ too.

2.2.6. *Floer homology over \mathbb{Z}_2 .* Assume that, in addition to the monotonicity conditions discussed in §2.1, every Lagrangian L considered in the Floer theoretic constructions satisfies:

$$(6) \quad \text{image} \left(H_1(L; \mathbb{Z}) \xrightarrow{i_*} H_1(M; \mathbb{Z}) \right) \text{ is torsion,}$$

where i_* is induced by the inclusion $L \subset M$. In this case an observation due to Oh [Oh1] shows that all the structures defined in the previous paragraphs are defined (at the chain level) with coefficients in the “polynomial” ring $\mathcal{A}^0 = \left\{ \sum_{k=0}^n a_k T^{\lambda_k} \mid a_k \in K, n \in \mathbb{Z} \right\}$ (i.e. those elements in \mathcal{A} formed by *finite* sums). There is an obvious ring map $\mathcal{A}^0 \rightarrow K$ obtained by sending $T \rightarrow 1$ and this allows to change the coefficients in all the structures described by specializing to $T = 1$. All the results mentioned in the case of \mathcal{A} remain valid.

In this paper we will use only these two coefficient rings for Floer-theoretic constructions: \mathcal{A} and $K = \mathbb{Z}_2$. Whenever \mathbb{Z}_2 is used, the assumption (6) is supposed to be in force without additional mention.

To shorten notation, \mathcal{K} will denote below one of these two rings. We will use the same notation as the one in the sections above but with \mathcal{K} replacing \mathcal{A} .

2.3. The Fukaya category, derived and not. It is beyond the scope of this section to recall the actual definition and construction of the Fukaya category. Fortunately, this is not really needed as we can refer to Seidel [Sei3] for a careful description of the construction. However, we would like to indicate some of the points that are significant from our point of view. Below we will follow closely Seidel's notation.

We first fix some notation that will be used throughout the paper. For a $d \in K$ and $\rho \in [0, \infty)$, consider the family $\mathcal{L}_d(M)$ formed by the closed, connected Lagrangian submanifolds $L \subset M$ that are monotone with monotonicity constant ρ and with $d_L = d$. (Thus the Lagrangians in $\mathcal{L}_d(M)$ are uniformly monotone.) Denote by $\mathcal{L}_d^0(M) \subset \mathcal{L}_d(M)$ the subset of those Lagrangians that satisfy the homological restriction (6). In all the Floer theoretic constructions in this section we take $\mathcal{K} = \mathbb{Z}_2$. Denote by $\mathcal{L}_d^*(M)$ the subset formed by the non-narrow Lagrangians. We recall from [BC4] that a monotone Lagrangian is non-narrow if its quantum homology does not vanish.

A simplified version of the Fukaya category is the Donaldson category: its objects are the elements of $\mathcal{L}_d^*(M)$ and the morphisms are $\text{Mor}(L_1, L_2) = HF(L_1, L_2)$. Composition is given by the triangle product. The existence of the identity morphism in $\text{Mor}(L, L)$ for each $L \in \mathcal{L}_d^*(M)$ is insured by the fact that L is not narrow.

The main drawback of the Donaldson category can be seen by analogy with usual topology. It is well known that if $X \xrightarrow{f} Y$ is a map of CW-complexes, then the H_*f does not determine the homotopy type of the (homotopy) cofiber of f . We recall that this homotopy type coincides with that of $Y \cup_f CX$, where CX is the cone over X . For instance, the Hopf map $S^3 \rightarrow S^2$ has as cofiber $\mathbb{C}P^2$ but is homologically trivial. In topology it is said that a sequence $X \rightarrow Y \rightarrow Z$ with $Z \simeq Y \cup_f CX$ is a homotopy cofibration sequence, or that Z is obtained from Y by a cone-attachment (over X). In a more general categorical setting, the analogues of the cofibration sequences from the homotopy world are called exact triangles and the proper setting to study cone-decompositions is that of triangulated categories [Wei]. In this language, the problem with the Donaldson category is that it is not triangulated and there is no natural way to complete it to a triangulated category without taking into account more structure. The Fukaya category solves this problem in what first appears to be a simple way: its objects are still the elements of $\mathcal{L}_d^*(M)$ but one intends to now take as morphisms the Floer complexes $CF(L_1, L_2)$

instead of the Floer homology. This change however leads to considerable difficulties. First, Floer complexes actually depend on additional data, almost complex structures, Hamiltonian perturbations etc. Secondly, the composition of morphisms is now defined as a chain-level triangle product and as such is associative only up to homotopy (and similarly for the unit). The first problem can be dealt with by fixing some coherent system of additional data, defining the relevant category associated to it and then remarking that the dependence on the additional data can be controlled by a version of the usual continuation technique. The second issue on the other hand shows that this process does not actually lead to a category but rather to an A_∞ -category - called the Fukaya category of M , $\mathcal{Fuk}^d(M)$. An A_∞ -category is roughly a structure where morphisms compose associatively only up to homotopy, the respective homotopies are themselves included in the structure and related by higher homotopies and so forth (and there are additional identities concerning units). While algebraically A_∞ -categories are complicated objects, the advantage is that the A_∞ -category defined as above admits a reasonably natural completion to a triangulated A_∞ -category. The resulting completion is denoted by $\mathcal{Fuk}^d(M)^\wedge$ (this is not a standard notation). In fact, as explained in [Sei3] there are multiple such completions but all are equivalent from the point of view of the present paper. The precise version that we use here (see Remark 3.21 in [Sei3]) is obtained by first using the Yoneda embedding to view $\mathcal{Fuk}^d(M)$ as a functor category over itself with values into chain complexes and then making use of the usual cone construction at the level of chain complexes to build a triangulated closure of the image of the embedding. In all cases, the advantage of completing at the A_∞ -level is that A_∞ -categories are sufficiently rich in structure so as to recover cofibration sequences like the one associated above to the Hopf map. However, the actual construction of the triangulated completion is rather non-geometric and is not immediate. Finally, the derived Fukaya category, denoted by $D\mathcal{Fuk}^d(M)$, has the same objects as $\mathcal{Fuk}^d(M)^\wedge$ but its morphisms are obtained by applying the homology functor to the morphisms in $\mathcal{Fuk}^d(M)^\wedge$. This category is naturally triangulated, the exact triangles being the image of those in $\mathcal{Fuk}^d(M)^\wedge$ and it contains the Donaldson category.

Remark 2.3.1. There is a further possible completion of $\mathcal{Fuk}^d(M)$ with respect to idempotents (or split factors). This is very useful for computations but we will not make use of it in this paper.

2.4. Cone decompositions over a triangulated category. In this subsection we will discuss a construction valid in any triangulated category. The purpose of the construction is to parametrize the various ways to decompose an object by iterated exact triangles. The morphisms in this category have a certain formal resemblance with the morphisms

in the cobordism category that will be introduced in §3.3. This analogy will be rendered precise later in the paper when we describe the functor $\widetilde{\mathcal{F}}$.

Let \mathcal{C} be a triangulated category. We recall [Wei] that this is an additive category together with a translation automorphism $T : \mathcal{C} \rightarrow \mathcal{C}$ and a class of triangles called *exact triangles*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

that satisfy a number of axioms due to Verdier and to Puppe (see e.g. [Wei]).

A cone decomposition of length k of an object $A \in \mathcal{C}$ is a sequence of exact triangles:

$$T^{-1}X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Y_{i+1} \xrightarrow{w_i} X_i$$

with $1 \leq i \leq k$, $Y_{k+1} = A$, $Y_1 = 0$. (Note that $Y_2 = X_1$.) Thus A is obtained in k steps from $Y_1 = 0$. To such a cone decomposition we associate the family $l(A) = (X_1, X_2, \dots, X_k)$ and we call it the *linearization* of the cone decomposition. This definition is an abstractization of the familiar iterated cone construction in case \mathcal{C} is the homotopy category of chain complexes. In that case T is the shift functor $TX = X[-1]$ and the cone decomposition simply means that each chain complex Y_{i+1} is obtained from Y_i as the mapping cone of a morphism coming from some chain complex, in other words $Y_{i+1} = \text{cone}(X_i[1] \xrightarrow{u_i} Y_i)$ for every i , and $Y_1 = 0$, $Y_{k+1} = A$.

We will now define a category $T^S\mathcal{C}$. The construction of this category starts with the *stabilization category* of \mathcal{C} , $\Sigma\mathcal{C}$: $\Sigma\mathcal{C}$ has the same objects as \mathcal{C} and the morphisms in $\Sigma\mathcal{C}$ from a to $b \in \mathcal{O}b(\mathcal{C})$ are morphisms in \mathcal{C} of the form $a \rightarrow T^s b$ for some integer s . Next, the free monoidal isomorphism category $F^*\Sigma\mathcal{C}$ over $\Sigma\mathcal{C}$ has as objects finite families (x_1, \dots, x_k) where the x_i 's are objects in \mathcal{C} . The monoidal addition, denoted by $+$, is concatenation. The morphisms are corresponding families of *isomorphisms* in $\Sigma\mathcal{C}$ (thus this category differs from the free monoidal category over $\Sigma\mathcal{C}$ because that category has as morphisms families of morphisms and not only isomorphisms).

The category $T^S\mathcal{C}$, called the *category of (stable) triangle (or cone) resolutions over \mathcal{C}* is obtained from $F^*\Sigma\mathcal{C}$ by enriching the morphisms with the elements constructed as follows. Given $x \in \mathcal{O}b(\mathcal{C})$ and $(y_1, \dots, y_q) \in \mathcal{O}b(F^*\Sigma\mathcal{C})$ a morphism $\Psi : x \rightarrow (y_1, \dots, y_q)$ is a triple (ϕ, a, η) , where $a \in \mathcal{O}b(\mathcal{C})$, $\phi : x \rightarrow T^s a$ is an isomorphism for some index s and η is a cone decomposition of the object a with linearization $(T^{s_1} y_1, T^{s_2} y_2, \dots, T^{s_{q-1}} y_{q-1}, y_q)$ for some family of indices s_1, \dots, s_{q-1} . Below we will also sometimes use a shift index s_q attached to the last element y_q with the understanding that $s_q = 0$. Thus, not only a admits a cone decomposition of length q but such a decomposition is part of the data defining the morphism Ψ .

We now define the morphisms between to general objects in $\mathcal{O}b(F^*\Sigma\mathcal{C})$. A morphism

$$\Phi \in \text{Mor}_{T^S\mathcal{C}}((x_1, \dots, x_m), (y_1, \dots, y_n))$$

is a sum $\Phi = \Psi_1 + \dots + \Psi_m$ where $\Psi_j \in \text{Mor}_{T^S\mathcal{C}}(x_j, (y_{\alpha(j)}, \dots, y_{\alpha(j)+\nu(j)}))$, and $\alpha(1) = 1$, $\alpha(j+1) = \alpha(j) + \nu(j) + 1$, $\alpha(m) + \nu(m) = n$.

- Remark 2.4.1.*
- i. For $x \in \mathcal{C}$ the union of all the morphisms $\text{Mor}_{T^S\mathcal{C}}(x, -)$ parametrizes all the possible cone decompositions of x . Of course, the same cone-decomposition appears infinitely many times by changing the x_i 's into $T^l x_i$'s and adjusting accordingly all the indexes s, s_i .
 - ii. The presence of the indexes s, s_i might appear as an unnecessary distraction here and be slightly disconcerting. However, their presence is explained by the fact that in the geometric example that is of interest in this paper the morphisms shift degrees (in Floer homology). Moreover, is it easy to see that it is natural for geometric reasons to allow for the index s to appear in the map ϕ and, on the other hand, not include any shift in front of y_q (or, equivalently, take $s_q = 0$).

The composition of the morphisms in $T^S\mathcal{C}$ is not quite obvious. We make it explicit next. To describe the composition, consider first the case of two morphisms Φ', Φ ,

$$(7) \quad \begin{aligned} \Phi' : x &\longrightarrow (y_1, \dots, y_k), & \Phi' &= (\phi', a', \eta'), \\ \Phi : (y_1, \dots, y_{h-1}, y_h, y_{h+1}, \dots, y_k) &\longrightarrow (y_1, \dots, y_{h-1}, z_1, \dots, z_l, y_{h+1}, \dots, y_k), \end{aligned}$$

where

$$\Phi = \text{id} + \dots + \text{id} + \Psi_h + \text{id} + \dots + \text{id}, \quad \Psi_h : y_h \longrightarrow (z_1, \dots, z_l), \quad \Psi_h = (\phi, a, \eta).$$

We will now define $\Phi'' = \Phi \circ \Phi'$. We will assume for simplicity that the families of shifting degrees indices for both Φ' and Φ are 0 (the general case is a straightforward generalization of the argument below). From the morphism Φ' we get an isomorphism $\phi' : x \rightarrow a'$ and a cone decomposition of a' with linearization y_1, \dots, y_k , i.e. objects $a'_1 = 0, a'_2 = y_1, a'_3, \dots, a'_k, a'_{k+1} = a'$ and exact triangles:

$$(8) \quad T^{-1}y_i \longrightarrow a'_i \longrightarrow a'_{i+1} \longrightarrow y_i, \quad i = 1, \dots, k.$$

For $i = h-1, h, h+1$ we have:

$$(9) \quad \begin{aligned} T^{-1}y_{h-1} &\longrightarrow a'_{h-1} \longrightarrow a'_h \longrightarrow y_{h-1} \\ T^{-1}y_h &\longrightarrow a'_h \longrightarrow a'_{h+1} \longrightarrow y_h \\ T^{-1}y_{h+1} &\longrightarrow a'_{h+1} \longrightarrow a'_{h+2} \longrightarrow y_{h+1}. \end{aligned}$$

Similarly, from the morphism Φ we obtain an isomorphism $\phi : y_h \rightarrow a$, a sequence of objects $a_1 = 0, a_2 = z_1, a_3, \dots, a_l, a_{l+1} = a$ and exact triangles:

$$(10) \quad T^{-1}z_j \longrightarrow a_j \longrightarrow a_{j+1} \longrightarrow z_j, \quad j = 1, \dots, l.$$

We will construct below new objects $a'_{h,1}, \dots, a'_{h,l+1}$ with $a'_{h,1} = a'_h, a'_{h,l+1} = a'_{h+1}$ and exact triangles:

$$(11) \quad T^{-1}z_q \longrightarrow a'_{h,q} \longrightarrow a'_{h,q+1} \longrightarrow z_q, \quad q = 1, \dots, l.$$

With this at hand the composition $\Phi'' = \Phi \circ \Phi' : x \longrightarrow (y_1, \dots, y_{h-1}, z_1, \dots, z_l, y_{h+1}, \dots, y_k)$ is defined by taking the cone decomposition η' (i.e. (8)) and replacing the line $i = h$ in it (i.e. the middle one in (9)) by the list of triangles from (11).

We now turn to the construction of the objects $a'_{h,q}$ and the triangles (11). Given $1 \leq q \leq l$, let $\beta_q : a_q \rightarrow a$ be the morphism obtained by successive composition of the middle arrows of (10) for $j = q, \dots, l$. Consider now the composition $\alpha_q = u'_h \circ (T^{-1}\phi^{-1})\beta_q$ of the following three morphisms:

$$\alpha_q : T^{-1}a_q \xrightarrow{\beta_q} T^{-1}a \xrightarrow{T^{-1}\phi^{-1}} T^{-1}y_h \xrightarrow{u'_h} a'_h,$$

where the last arrow u'_l here is the first arrow in the middle line of (9).

By the axioms of “triangulated category” the morphism α_q can be completed into an exact triangle, i.e. there exists an object $a'_{h,q} \in \mathcal{O}b(\mathcal{C})$ and an exact triangle:

$$(12) \quad T^{-1}a_q \xrightarrow{\alpha_q} a'_h \longrightarrow a'_{h,q} \longrightarrow a_q.$$

By the octahedral axiom and standard results on triangulated categories (see e.g. [Wei]) the triangles in (12) (for q and $q+1$) and those in (10) (for $j = q$ and after a shift) fit into the following diagram:

$$(13) \quad \begin{array}{ccccccc} T^{-2}z_q & \longrightarrow & 0 & \longrightarrow & T^{-1}z_q & \longrightarrow & T^{-1}z_q \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^{-1}a_q & \xrightarrow{\alpha_q} & a'_h & \longrightarrow & a'_{h,q} & \longrightarrow & a_q \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^{-1}a_{q+1} & \xrightarrow{\alpha_{q+1}} & a'_h & \longrightarrow & a'_{h,q+1} & \longrightarrow & a_{q+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^{-1}z_q & \longrightarrow & 0 & \longrightarrow & z_q & \longrightarrow & z_q \end{array}$$

in which all rows and columns are exact triangles. (In fact, the diagram is determined by the upper left square, from which the upper two triangles and left two triangles are extended. The existence of the third column follows from octahedral axioms applied a few

times.) Moreover, all squares in the diagram, except of the lower right, are commutative and the lower right one commutes up to sign.

Note that the third column is precisely the triangle that we needed in (11) in order to complete the construction of the composition in (7).

It remains to define the composition $\Phi \circ \Phi'$ of more general morphisms than (7). The case when the domain of Φ' consists of a tuple of objects in \mathcal{C} and Φ is as in (7) is an obvious generalization of the preceding construction. Next, the case when the rest of the components of Φ are not necessarily id (but rather general cone decompositions too) is done by reducing to the case discussed above by successive compositions. Namely, assume that $\Phi' : x \rightarrow (y_1, \dots, y_k)$ and $\Phi = \Psi_1 + \dots + \Psi_k$ with $\Psi_j : y_j \rightarrow w_j$, where $x, w_j \in \mathcal{O}b(F^*\Sigma\mathcal{C})$. We define

$$\Phi \circ \Phi' = (\Psi_1 + \text{id} + \dots + \text{id}) \circ \dots \circ (\text{id} + \dots + \text{id} + \Psi_k) \circ \Phi',$$

noting that each step of this composition is of the type already defined.

This completes the definition of composition of morphisms in the category $T^S\mathcal{C}$. It is not hard to see that the this composition of morphisms is associative. Note also that, by construction, $T^S\mathcal{C}$ is a monoidal category.

To conclude this discussion we remark that there is a projection functor

$$(14) \quad \mathcal{P} : T^S\mathcal{C} \longrightarrow \Sigma\mathcal{C}$$

that is defined by $\mathcal{P}(x_1, \dots, x_k) = x_k$ and whose value on morphisms is induced by associating to $\Phi \in \text{Mor}_{T^S\mathcal{C}}(x, (x_1, \dots, x_k))$, $\Phi = (\phi, a, \eta)$, the composition:

$$\mathcal{P}(\Phi) : x \xrightarrow{\phi} T^s a \xrightarrow{p} T^s x_k$$

with $p : a \rightarrow x_k$ defined by the last exact triangle in the cone decomposition η of a ,

$$T^{-1}x_k \longrightarrow a_k \longrightarrow a \xrightarrow{p} x_k .$$

A straightforward verification shows the \mathcal{P} is indeed a functor.

Remark 2.4.2. Let $\mathcal{H} : \mathcal{C} \rightarrow \mathcal{A}$ be a covariant cohomological functor with values in the abelian category \mathcal{A} (see [Wei]). Let $\Phi : x \rightarrow (x_1, x_2)$ be a morphism in $T^S\mathcal{C}$ consisting of a cone decomposition of x of length 2. Then the cohomology $\mathcal{H}(x)$ is related to $\mathcal{H}(x_1)$, $\mathcal{H}(x_2)$, by the following long exact sequence:

$$\dots \longrightarrow \mathcal{H}(T^{s_2-1+i}x_2) \longrightarrow \mathcal{H}(T^{s_1+i}x_1) \longrightarrow \mathcal{H}(T^{i-s}x) \longrightarrow \mathcal{H}(T^{s_2+i}x_2) \longrightarrow \dots .$$

Here s, s_1, s_2 , are the shift-indices coming from the morphism Φ .

More generally, a morphism $\Phi \in \text{Mor}_{T^S\mathcal{C}}(x, (x_1, \dots, x_k))$ gives rise to a spectral sequence relating the cohomology $\mathcal{H}(T^*x_i)$ and $\mathcal{H}(T^*x)$ and whose E_2 -term can be computed in terms of $\mathcal{H}(T^*x_i)$.

3. COBORDISM: DEFINITIONS AND GEOMETRIC ASPECTS

In what follows we fix a symplectic manifold M endowed with a symplectic structure ω . The plane \mathbb{R}^2 as well as domains in \mathbb{R}^2 will be endowed with the symplectic structure $\omega_{\mathbb{R}^2} = dx \wedge dy$, $(x, y) \in \mathbb{R}^2$. We endow the product $\mathbb{R}^2 \times M$ with the symplectic form $\omega_{\mathbb{R}^2} \oplus \omega$. We denote by $\pi : \mathbb{R}^2 \times M \rightarrow \mathbb{R}^2$ the projection. For a subset $V \subset \mathbb{R}^2 \times M$ and $S \subset \mathbb{R}^2$ we write $V|_S = V \cap \pi^{-1}(S)$.

3.1. Basic definitions.

3.1.1. Cobordism.

Definition 3.1.1. Let $(L_i)_{1 \leq i \leq k_-}$ and $(L'_j)_{1 \leq j \leq k_+}$ be two families of closed, Lagrangian submanifolds of M . We say that that these two (ordered) families are Lagrangian cobordant, $(L_i) \simeq (L'_j)$, if there exists a smooth compact cobordism $(V; \coprod_i L_i, \coprod_j L'_j)$ and a Lagrangian embedding $V \subset ([0, 1] \times \mathbb{R}) \times M$ so that for some $\epsilon > 0$ we have:

$$(15) \quad \begin{aligned} V|_{[0, \epsilon) \times \mathbb{R}} &= \coprod_i ([0, \epsilon) \times \{i\}) \times L_i \\ V|_{(1 - \epsilon, 1] \times \mathbb{R}} &= \coprod_j ((1 - \epsilon, 1] \times \{j\}) \times L'_j . \end{aligned}$$

The manifold V is called a Lagrangian cobordism from the Lagrangian family (L'_j) to the family (L_i) . We will denote such a cobordism by $V : (L'_j) \rightsquigarrow (L_i)$ or $(V; (L_i), (L'_j))$.

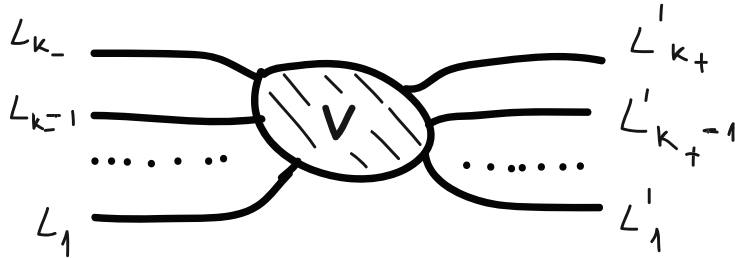


FIGURE 1. A cobordism $V : (L'_j) \rightsquigarrow (L_i)$ projected on \mathbb{R}^2 .

The values of the y coordinate of V (i.e. $1, \dots, k_{\pm}$) is not really important. (This convention is fixed for categorical purposes that will become clear later.) What really matters is the order in which each family of Lagrangians (L_i) and (L'_j) is placed (with respect to the y -coordinate) at the ends of the cobordism. Sometimes we will shift the numbering of the y -coordinates to start with other values of i . It is important to remark that the Lagrangians in the family (L_i) (or (L'_j)) are not assumed to be mutually disjoint

inside M . In fact, one of the roles of the y coordinate at the ends is to “make them disjoint” (inside $0 \times \mathbb{R} \times M$) thus enabling the cobordisms V to be embedded. In this respect our setting is somewhat different than in [Che].

A few variants of this notion will be useful later in the paper. An *elementary* cobordism is a cobordism (which might be connected or not) so that the number of negative ends k_- as well as the number of positive ends k_+ in Definition 3.1.1 both have value at most one. Finally, V is called a *pseudo-isotopy* if it is smoothly trivial which means that there is a diffeomorphism $V \approx ([0, 1] \times \{1\}) \times L$.

A cobordism is called *monotone* if

$$V \subset ([0, 1] \times \mathbb{R}) \times M$$

is a monotone Lagrangian submanifold. Other types of restricted cobordism are, for instance, *oriented* and *spin* - this means that an orientation, respectively a spin structure, is defined on the ends of the cobordism and is preserved by the cobordism itself. Clearly, as in the smooth case, there are many other possible variants of cobordism depending on additional structures.

In practice, particularly when studying one cobordism at a time, it is often more convenient to view cobordisms as embedded in $\mathbb{R}^2 \times M$. Given a cobordism $V \subset ([0, 1] \times \mathbb{R}^2) \times M$ as in Definition 3.1.1 we can extend trivially its negative ends towards $-\infty$ and its positive ends to $+\infty$ thus getting a Lagrangian $\overline{V} \subset \mathbb{R}^2 \times M$. We will in general not distinguish between V and \overline{V} but if this distinction is needed we will call

$$(16) \quad \overline{V} = \left(\prod_i (-\infty, 0] \times \{i\} \times L_i \right) \cup V \cup \left(\prod_j [1, \infty) \times \{j\} \times L'_j \right)$$

The \mathbb{R} -extension of V .

3.1.2. *Lagrangian submanifolds with cylindrical ends.* In what follows it is convenient to make use of a slightly more general notion of Lagrangian submanifold with cylindrical ends that we discuss here.

To simplify notation we will write from now on $\widetilde{M} = \mathbb{R}^2 \times M$ endowed with the split form $\omega_{\mathbb{R}^2} \oplus \omega$. We will also identify in the standard way $\mathbb{R}^2 \cong \mathbb{C}$ endowed with the standard complex structures i .

By a *Lagrangian submanifold with cylindrical ends* we mean a Lagrangian submanifold $\overline{V} \subset \widetilde{M}$ without boundary that has the following properties:

- (1) For every $a < b$ the subset $\overline{V}|_{[a,b] \times \mathbb{R}}$ is compact.

(2) There exists R_+ such that

$$\overline{V}|_{[R_+, \infty) \times \mathbb{R}} = \prod_{i=1}^{k_+} [R_+, \infty) \times \{a_i^+\} \times L_i^+$$

for some $a_1^+ < \dots < a_{k_+}^+$ and some Lagrangian submanifolds $L_1^+, \dots, L_{k_+}^+ \subset M$.

(3) There exists $R_- \leq R_+$ such that

$$\overline{V}|_{(-\infty, R_-] \times \mathbb{R}} = \prod_{i=1}^{k_-} (-\infty, R_-] \times \{a_i^-\} \times L_i^-$$

for some $a_1^- < \dots < a_{k_-}^-$ and some Lagrangian submanifolds $L_1^-, \dots, L_{k_-}^- \subset M$.

We allow k_+ or k_- to be 0 in which case $\overline{V}|_{[R_+, \infty) \times \mathbb{R}}$ or $\overline{V}|_{(-\infty, R_-] \times \mathbb{R}}$ are void.

For every $R \geq R_+$ write $E_R^+(\overline{V}) = \overline{V}|_{[R, \infty) \times \mathbb{R}}$ and call it a positive cylindrical end of \overline{V} . Similarly, we have for $R \leq R_-$ a negative cylindrical end $E_R^-(\overline{V})$.

Obviously if W is a cobordism between (L'_1, \dots, L'_r) and (L_1, \dots, L_s) then its \mathbb{R} -extension \widetilde{W} is a Lagrangian submanifold of \widetilde{M} with cylindrical ends. Vice versa, if \widetilde{W} is a Lagrangian submanifold with cylindrical ends then by an obvious modification of the ends (and a possible symplectomorphism on the \mathbb{R}^2 component) it is easy to obtain a Lagrangian cobordism between the families of Lagrangians corresponding to the positive and negative ends of \widetilde{W} .

In order to simplify terminology, we will say that a Lagrangian with cylindrical ends \overline{V} is cylindrical outside of a compact subset $K \subset \mathbb{R}^2$ if $\overline{V}|_{\mathbb{R}^2 \setminus K}$ consists of horizontal ends, i.e. it is of the form $E_{R_-}^-(\overline{V}) \cup E_{R_+}^+(\overline{V})$.

We will also need the following notion.

Definition 3.1.2. Two Lagrangians with cylindrical ends $\overline{V}, \overline{W} \subset \widetilde{M}$ are said to be cylindrically distinct at infinity if there exists $R > 0$ such that $\pi(E_R^+(\overline{V})) \cap \pi(E_R^+(\overline{W})) = \emptyset$ and $\pi(E_{-R}^-(\overline{V})) \cap \pi(E_{-R}^-(\overline{W})) = \emptyset$.

Finally, let us describe a class of Hamiltonian isotopies that will be useful in the following.

Definition 3.1.3 (Horizontal isotopies). Let $\{\widetilde{V}_t\}_{t \in [0,1]}$ be an isotopy of Lagrangian submanifolds of \widetilde{M} with cylindrical ends. We call this isotopy horizontal if there exists a (not necessarily compactly supported) Hamiltonian isotopy $\{\psi_t\}_{t \in [0,1]}$ of \widetilde{M} with $\psi_0 = \mathbb{1}$ and with the following properties:

- i. $\widetilde{V}_t = \psi_t(\widetilde{V}_0)$ for all $t \in [0, 1]$.
- ii. There exist real numbers $R_- < R_+$ such that for all $t \in [0, 1]$, $x \in E_{R_\pm}^\pm(\widetilde{V}_0)$ we have $\psi_t(x) \in E_{R_\pm}^\pm(\widetilde{V}_0)$.

- iii. There is a constant $K > 0$ so that for all $x \in E_{R^\pm}^\pm(\overline{V}_0)$, $|d\pi_x(X_t(x))| < K$. Here X_t is the (time dependent) vector field of the flow $\{\psi_t\}_{t \in [0,1]}$.

In other words, the Hamiltonian flow ψ_t moves tangentially along the cylindrical ends of \overline{V}_0 and at bounded speed. Of course, the ends of all the Lagrangians \overline{V}_t coincide at infinity. We say that two Lagrangians $\overline{V}, \overline{V}' \subset \widetilde{M}$ with cylindrical ends are *horizontally* isotopic if there exists an isotopy as above $\{\overline{V}_t\}_{t \in [0,1]}$ with $\overline{V}_0 = \overline{V}$ and $\overline{V}_1 = \overline{V}'$. Finally, we will sometime say that an ambient Hamiltonian isotopy $\{\psi_t\}_{t \in [0,1]}$ as above is horizontal with respect to \overline{V}_0 .

3.2. Some constructions. Here are a few examples of constructions of cobordisms.

- a. If $L \subset M$ is a Lagrangian submanifold and $\gamma \in \mathbb{C}$ is any curve so that outside a compact set γ agrees with $\mathbb{R} \times \{y\}$, then $\gamma \times L \in \widetilde{M}$ is an elementary cobordism. If L is monotone, then so is the cobordism $\gamma \times L$, with the same minimal Maslov number and monotonicity constant. More generally, a possibly non-connected curve γ that coincides with $\coprod \mathbb{R} \times \{j\}$ outside a compact set gives rise to cobordisms $L \times \gamma$. In particular, this shows that the Lagrangian family (L, L) is null-bordant.
- b. If the connected Lagrangians $L, L' \subset M$ are Hamiltonian isotopic it is easy to construct an elementary cobordism joining them (notice however that the projection of this cobordism on \mathbb{R}^2 will in general not be a curve).
- c. Let $(V; (L_i), (L'_j))$ be an *immersed* Lagrangian cobordism between two families of *embedded* Lagrangians. This is a cobordism as in Definition 3.1.1 with the exception that $V \rightarrow ([0, 1] \times \mathbb{R}) \times M$ is not a Lagrangian embedding but only a Lagrangian immersion. Such a cobordism can be transformed into an embedded one by first changing the self intersection points of V into generic double points and then resolving these double points by Lagrangian surgery (see for instance [Pol]). It is important to note that by resolving these singularities various properties that the initial V might have verified are in general lost. Monotonicity, for instance, is in general not preserved, nor is orientability. However, if we do not keep track of these additional structures we see that immersed Lagrangian cobordism implies embedded cobordism (as noticed by Chekanov [Che]).
- d. Finally, a less immediate verification shows that the trace of surgery is also a Lagrangian cobordism. In other words, given two transverse Lagrangians L_1, L_2 by applying surgery at each of their intersection points one can obtain (a possibly disconnected) Lagrangian L that is cobordant to the family (L_1, L_2) , the cobordism being given by the composition of the traces of the surgeries. We will elaborate more on this construction in §6.3.

- Remark 3.2.1.*
- i. It is not difficult to see that cobordism is an equivalence relation among Lagrangian families: reflexivity is of course obvious as well as transitivity. For symmetry a little argument is required. Assume V is a cobordism between (L_1, L_2, \dots, L_h) and (L'_1, \dots, L'_k) . The transformation $a : \mathbb{C} \times M \rightarrow \mathbb{C} \times M$ given by $a(z, m) = (-z, m)$ is symplectic and, after adjusting the ends of the cobordism $a(V)$, it provides a cobordism from (L'_k, \dots, L'_1) to (L_h, \dots, L_1) . This cobordism can be easily adjusted at the ends to an immersed cobordism between (L'_1, \dots, L'_k) and (L_1, L_2, \dots, L_h) . By point c. above this can be transformed into an embedded Lagrangian cobordism. (Note that this construction fails for monotone cobordisms, hence being monotone cobordant does not seem to be an equivalence relation.)
 - ii. Given two Lagrangian families $\mathcal{L} = (L_1, \dots, L_h)$ and $\mathcal{L}' = (L'_1, \dots, L'_k)$ define their sum $\mathcal{L} + \mathcal{L}' = (L_1, \dots, L_h, L'_1, \dots, L'_k)$. In view of the properties described above it is easy to see that this operation defines a group structure on the set of equivalence classes of Lagrangian families of M . By applying appropriate surgeries it is easy to see that this group is commutative. (In contrast, there is no apriori reason why $\mathcal{L} + \mathcal{L}'$ should be monotone cobordant to $\mathcal{L}' + \mathcal{L}$.)
 - iii. It is easy to see that elementary cobordism is also an equivalence relation among the Lagrangians of M (surgery is not needed for this argument).
 - iv. Special elementary cobordism of any of the three following types - monotone, oriented, or spin - is an equivalence relation. Again reflexivity is obvious and symmetry follows as in Remark 3.2.1 i. without any need to perform surgeries. Transitivity is obvious too in the orientable and spin cases. In the monotone case, it follows from the Van Kampen theorem for relative $\pi_2(-, -)$'s viewed as cross-modules (see [BH]) that gluing two monotone cobordisms with the same monotonicity constant along a connected monotone end produces a monotone cobordism. However, as already mentioned earlier, non-elementary monotone cobordism is not necessarily an equivalence relation.

3.3. The category $\mathcal{C}ob^d(M)$. The purpose of this subsection is to set up Lagrangian cobordism as a category.

We first introduce an auxiliary category $\widetilde{\mathcal{C}ob}^d(M)$, $d \in K$. Its objects are families (L_1, L_2, \dots, L_r) with $r \geq 1$, $L_i \in \mathcal{L}_d(M)$. (Recall that $\mathcal{L}_d(M)$ stands for the class of uniformly monotone Lagrangians L with $d_L = d$, and when $d \neq 0$ with the same monotonicity constant ρ which is omitted from the notation.)

To describe the morphisms in this category we proceed in two steps. First, for any two horizontal isotopy classes of cobordisms $[V]$ and $[U]$ with $V : (L'_j) \rightsquigarrow (L_i)$ (as in

Definition 3.1.1) and $U : (K'_s) \rightsquigarrow (K_r)$ we define the sum $[V] + [U]$ to be the isotopy class of a cobordism family $W : (L'_j) + (K'_s) \rightsquigarrow (L_i) + (K_r)$ so that $W = V \amalg \tilde{U}$ with $\tilde{U} : (K'_s) \rightsquigarrow (K_r)$ a cobordism horizontally isotopic to U and so that \tilde{U} is disjoint from V (to insure embeddedness we can not simply take V and U in the disjoint union.) Notice that the sum $[V] + [U]$ is not commutative.

The morphisms in $\widetilde{\mathcal{Cob}}^d(M)$ are now defined as follows. A morphism

$$[V] \in \text{Mor}((L'_j)_{1 \leq j \leq S}, (L_i)_{1 \leq i \leq T})$$

is a horizontal isotopy class that can be written as a sum $[V] = [V_1] + \cdots + [V_S]$ with each $V_j \in \mathcal{L}_d(\mathbb{C} \times M)$ a cobordism from the Lagrangian family formed by the *single* Lagrangian L'_j and a subfamily $(L_{r(j)}, \dots, L_{r(j)+s(j)})$ of the (L_i) 's, and do that $r(j)+s(j)+1 = r(j+1)$. In other words, V decomposes as a union of V_i 's each with a single positive end but with possibly many negative ones. We will often denote such a morphism by $V : (L'_j) \longrightarrow (L_i)$.

The composition of morphisms is induced by concatenation followed by a rescaling to reduce the “length” of the cobordism to the interval $[0, 1]$. It is an easy exercise to see that this is well defined precisely because our morphisms are (horizontal) isotopy classes of cobordisms and because morphisms are represented by sums of cobordisms with a single positive end.

We will consider here the void set as a Lagrangian of arbitrary dimension. With the above conventions one would expect that the sum operation described above gives the category $\widetilde{\mathcal{Cob}}^d(M)$ the structure of a strict monoidal category with neutral element \emptyset . However, it is easy to see that $L + \emptyset \neq \emptyset + L$ simply because the two families (L, \emptyset) and (\emptyset, L) are different. We now intend to factor both the objects and the morphisms in this category by equivalence relations that will transform this category in a strict monoidal one. For the objects the equivalence relation is induced by the relations

$$(17) \quad (L, \emptyset) \sim (\emptyset, L) \sim (L).$$

At the level of the morphisms a bit more care is needed. For each $L \in \mathcal{L}_d(M)$ we will define two particular cobordisms $\Phi_L : (\emptyset, L) \rightsquigarrow (L, \emptyset)$ and $\Psi_L : (L, \emptyset) \rightsquigarrow (\emptyset, L)$ as follows. Let $\gamma : [0, 1] \rightarrow [0, 1]$ be an increasing, surjective smooth function, strictly increasing on $(\epsilon, 1 - \epsilon)$ and with $\gamma'(t) = 0$ for $t \in [0, \epsilon] \cup [1 - \epsilon, 1]$. We now let $\Phi(L) = \text{graph}(\gamma) \times L$ and $\Psi(L) = \text{graph}(1 - \gamma) \times L$. The equivalence relation for morphisms is now induced by the following two identifications:

(Eq 1) For every cobordism V we identify $V + \emptyset \sim \emptyset + V \sim V$, where \emptyset is the void cobordism between two void Lagrangians.

(Eq 2) If $V : L \longrightarrow (L_1, \dots, L_i, \emptyset, L_{i+2}, \dots, L_k)$, then we identify $V \sim V' \sim V''$, where $V' = \Phi_{L_{i+2}} \circ V$, $V'' = \Psi_{L_i} \circ V$.

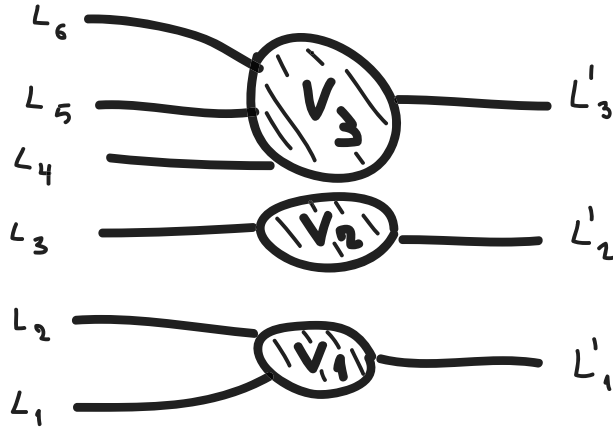


FIGURE 2. A morphism $V : (L'_1, L'_2, L'_3) \longrightarrow (L_1, \dots, L_6)$, $V = V_1 + V_2 + V_3$, projected to \mathbb{R}^2 .

We now construct the category $\mathcal{Cob}^d(M)$. First we consider the full subcategory $\mathcal{S} \subset \widetilde{\mathcal{Cob}}^d(M)$ obtained by restricting the objects only to those families (L_1, \dots, L_k) with L_i non-narrow for all $1 \leq i \leq k$ (this is preferable because our functors ultimately make use of Floer homology and we need the quantum homology of each L_i to be non trivial). Then $\mathcal{Cob}^d(M)$ is obtained by the quotient of the objects of \mathcal{S} by the equivalence relation in (17) and the quotient of the morphisms of \mathcal{S} by the equivalence relation in (Eq 1), (Eq 2).

This category is called the *Lagrangian cobordism category* of M . As mentioned before, it is a strict monoidal category. To recapitulate, its objects are ordered families of non-narrow Lagrangians $\in \mathcal{L}_d(M)$ and its morphisms:

$$[V] : (L'_j) \longrightarrow (L_i)$$

can be represented by cobordisms $V \in \mathcal{L}_d(\mathbb{C} \times M)$ so that all L_i 's are non-void and all L'_j 's are non-void except if there is just a single L'_j which can be void or there is just a single L_i which can be void. Moreover, V can be written as a disjoint union of cobordisms each with a single positive end.

There is a subcategory of $\mathcal{Cob}^d(M)$, that will be denoted by $\mathcal{Cob}_0^d(M)$, whose objects belong to $\mathcal{L}_d^*(M)$ (see §2.3) and whose morphisms are represented by Lagrangian cobordisms V verifying the analogous condition to (6), but in $\mathbb{R}^2 \times M$. This is again a strict monoidal category.

4. CONSTRUCTION OF $\tilde{\mathcal{F}}$

The purpose of this section is to discuss the proof of the following theorem (more details will appear in a more ample paper currently in preparation).

Theorem A. *There exists a monoidal functor*

$$\tilde{\mathcal{F}} : \mathcal{Cob}_0^d(M) \longrightarrow T^S D\mathcal{Fuk}^d(M),$$

which satisfies:

- (1) $\tilde{\mathcal{F}}(L) = L$ for every Lagrangian submanifold $L \in \mathcal{L}_d^*(M)$.
- (2) Every cobordism $V : L \rightsquigarrow (L_1, \dots, L_k)$ is sent by $\tilde{\mathcal{F}}$ to a morphism in $T^S D\mathcal{Fuk}^d(M)$ which is a cone decomposition of L of length k (and with linearization (L_1, \dots, L_k) up to shifts).

Recall from (1) that the functor \mathcal{F} is actually determined by $\tilde{\mathcal{F}}$ by $\mathcal{F} = \mathcal{P} \circ \tilde{\mathcal{F}}$. Nevertheless we will construct \mathcal{F} independently in §4.1 below.

By recalling the construction of the category $T^S(-)$ from §2.4 it is immediate that the existence of $\tilde{\mathcal{F}}$ implies:

Corollary 4.0.1. *For any morphism $[V]$ in $\mathcal{Cob}_0^d(M)$ there are exact triangles in $D\mathcal{Fuk}^d(M)$ that correspond to $\tilde{\mathcal{F}}([V]) \in \text{hom}_{T^S D\mathcal{Fuk}^d(M)}$.*

In slightly different terms:

Corollary 4.0.2. *If $V : L \rightsquigarrow (L_1, \dots, L_k)$ is a monotone cobordism so that $[V]$ is a morphism in $\mathcal{Cob}_0^d(M)$, then L belongs to the subcategory of $D\mathcal{Fuk}^*(M)$ generated by (L_1, \dots, L_k) . In particular, if $V : L_0 \rightsquigarrow L_1$ is an elementary cobordism, then the Lagrangians L_0 and L_1 are isomorphic objects (up to shift) in $D\mathcal{Fuk}^*(M)$.*

The notion of generation of a subcategory here is the one commonly used in the theory of triangulated categories. Namely, if \mathcal{C} is a triangulated category and $S \subset \mathcal{Ob}(\mathcal{C})$ is a collection of objects, then the subcategory generated by S is the minimal full category \mathcal{C}_S that contains S and is closed under isomorphisms and formation of exact triangles (i.e. if A is isomorphic to $B \in \mathcal{C}_S$ then $A \in \mathcal{Ob}(\mathcal{C}_S)$, and if $\varphi : A \rightarrow B$ with $A, B \in \mathcal{Ob}(\mathcal{C}_S)$ then the objects in the completion of φ into an exact triangle also belong to \mathcal{C}_S). In this paper we do not complete with respect to idempotents.

Theorem A and its Corollaries above are formulated for $\mathcal{K} = \mathbb{Z}_2$ and thus assume that condition (6) is satisfied for all Lagrangians and cobordisms involved. Nevertheless, the methods used in the proof also apply when working over \mathcal{A} and thus without the assumption (6), but in this case they lead to less precise results. Still, the following Corollary is true for $\mathcal{K} = \mathcal{A}$ as well as for $\mathcal{K} = \mathbb{Z}_2$ (in this second case it is a simplified version of Corollary 4.0.1).

Corollary 4.0.3. *If $V : L \rightsquigarrow (L_1, \dots, L_k)$ is a monotone cobordism and $N \subset M$ is another Lagrangian so that $L, L_1, \dots, L_k, N \subset \mathcal{L}_d(M)$, $V \in \mathcal{L}_d(\mathbb{R}^2 \times M)$, then there exists*

a sequence of chain complexes K_i and a sequence of chain maps

$$m_i : C(N, L_i) \longrightarrow K_i$$

so that K_{i+1} is the cone over the map m_i (in the category of chain complexes), $K_0 = 0$ and there is a quasi-isomorphism $h : C(N, L) \longrightarrow K_{k+1}$. For $\mathcal{K} = \mathbb{Z}_2$ the chain maps m_i , h , depend only on V up to chain homotopy. For $\mathcal{K} = \mathcal{A}$ each of these maps only depends on V up to chain homotopy and product with some unit in \mathcal{A} .

We recall that \mathcal{K} is not graded so we have omitted any grading/suspension etc in this statement.

Note that Corollary 4.0.3 together with Example d. in §3.2 (expanded in §6.3) imply the existence of exact sequences associated to surgery over \mathcal{K} . An exact sequence of Floer homologies corresponding to Lagrangian surgery (of two Lagrangian submanifolds) has been previously obtained in [FOOO1] by other methods. See also [Sei2].

Before passing to the proof of Theorem A we will discuss the definition of the functor \mathcal{F} as this already indicates what are the basic technical ingredients of the whole machinery. Moreover, the definition of this functor is relatively simple on the algebraic side.

4.1. The functors \mathcal{F} and $\tilde{\mathcal{F}}$. In this subsection we will work over the ring \mathcal{K} that can be either \mathcal{A} or \mathbb{Z}_2 . We will indicate in each relevant statement the differences between these two cases. Each time a statement is claimed over \mathbb{Z}_2 , the condition (6) is assumed for all involved Lagrangians in M and in $\mathbb{R}^2 \times M$.

4.1.1. The functor \mathcal{F} . The construction of \mathcal{F} is relatively easy. However, we emphasize it is only the existence of the functor $\tilde{\mathcal{F}}$ that implies the exact sequences and isomorphisms in the Corollaries 4.0.1 and 4.0.2 above.

At the level of the objects, the functor \mathcal{F} is defined as follows:

$$\mathcal{O}b(\mathcal{C}ob_0^d(M)) \ni (L_1, \dots, L_k) \longrightarrow L_k \in \mathcal{O}b(D\mathcal{F}uk^d(M)) .$$

We now describe the functor \mathcal{F} on morphisms. We will show how to associate to an appropriate cobordism $V : L \rightsquigarrow (L_1, \dots, L_k)$ with a single positive end, a homology class

$$\alpha_V \in HF(L, L_k)[-l] = \text{hom}_{D\mathcal{F}uk^d(M)}(L, T^l L_k),$$

where $[-l]$ is the degree shift by $l \in \mathbb{Z}$ and T is the shift in the triangulated category $D\mathcal{F}uk^d(M)$. We then put $\mathcal{F}([V]) = \alpha_V$. For a more general morphism, $W = V_1 + \dots + V_k$ with each V_i with a single positive end, we put $\mathcal{F}(W) = \mathcal{F}(V_k) = \alpha_{V_k}$. We will not verify here the properties of \mathcal{F} with respect to composition. We only notice here that they follow from those of the functor $\tilde{\mathcal{F}}$.

In the sequel we will make use of Floer homology for pairs of Lagrangian submanifolds with cylindrical ends (recall the definition of Lagrangians with such ends from §3.1.2). This is analogous to usual Floer homology (for closed Lagrangians) with a few modifications that are described in §4.2.2 below.

We will make use of the following result whose proof is postponed.

Theorem 4.1.1. *Let $\overline{W}, \overline{W}' \subset \mathbb{R}^2 \times M$ be two Lagrangian submanifolds with cylindrical ends. Assume that \overline{W} and \overline{W}' are cylindrically distinct at infinity. Then the Floer homology $HF(\overline{W}, \overline{W}')$ is well defined over \mathcal{K} . If $\phi = \{\phi_t\}_{t \in [0,1]}$ is a horizontal isotopy with respect to \overline{W} , then there is an isomorphism $HF(\overline{W}, \overline{W}') \rightarrow HF(\phi(\overline{W}), \overline{W}')$ that only depends on the homotopy class of the path of Hamiltonian diffeomorphisms ϕ_t (with fixed end-points). A similar statement is valid if we act with a horizontal isotopy on \overline{W}' and keep \overline{W} fixed.*

Versions of this theorem apply also to the case when $\overline{W}, \overline{W}'$ are not cylindrically distinct at infinity. See §4.2.2 for the details.

Returning to the construction of \mathcal{F} we will now construct a particular family of cobordisms that will be useful for our considerations. Let $a \geq 0, q, r, s \in \mathbb{R}$. Consider a smooth function $\sigma_{a;q,r,s} : \mathbb{R} \rightarrow \mathbb{R}$, with the following properties:

- i. $\sigma_{a;q,r,s}(t) = q$ for $t \leq -a$, $\sigma_{a;q,r,s}(t) = s$ for $t \geq 3$.
- ii. $\sigma_{a;q,r,s}(t) = r$ for $t \in [-a+1, 2]$.
- iii. $\sigma_{a;q,r,s}$ is strictly monotone on $(-a, -a+1)$ and strictly monotone on $(2, 3)$.

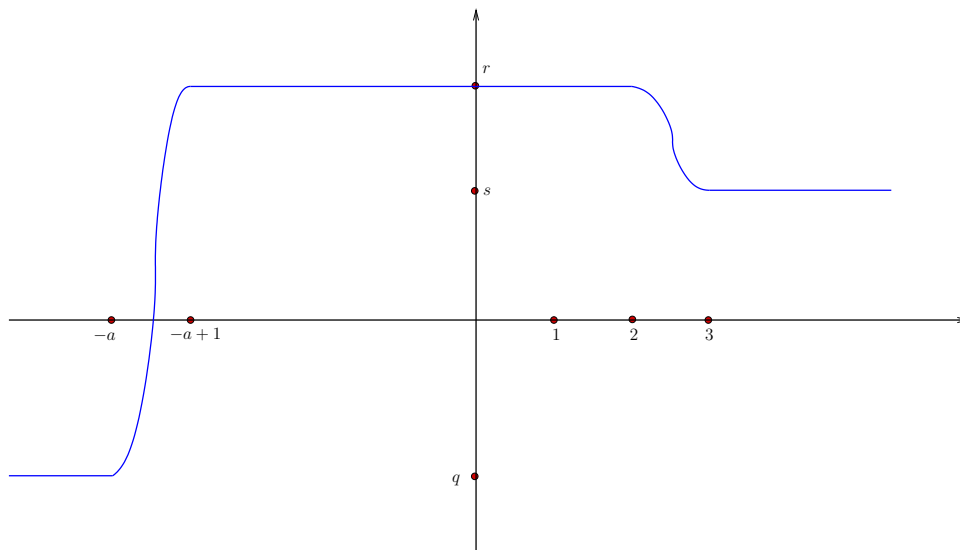


FIGURE 3. The graph of $\sigma_{a;q,r,s}$.

We denote by $\gamma_{a;q,r,s} \subset \mathbb{R}^2$ the graph of $\sigma_{a;q,r,s}$. To simplify the notation we also put

$$\lambda_k = \text{graph}(\sigma_{-2;k-1/2,k+1,1/2}).$$

Let $V : L \rightsquigarrow (L_1, \dots, L_k)$ be a cobordism with one positive end. Let $L' \subset M$ be a small Hamiltonian perturbation of L which is transverse to L as well as to L_1, \dots, L_k . By a possible isotopy of V we may assume that $V \subset ([0, 1] \times (-\infty, k]) \times M$. Consider now the Lagrangian $\lambda_k \times L'$. Clearly $\lambda_k \times L'$ intersects \bar{V} transversely and

$$(\lambda_k \times L') \cap \bar{V} = \{(b, k)\} \times (L' \cap L_k) \cup \{(c, 1)\} \times (L' \cap L),$$

for some $b \in (-2, -1)$, $c \in (2, 3)$.

It follows from Theorem 4.1.1 that the Floer homology $HF(\bar{V}, \lambda_k \times L')$ is well defined and depend only on the horizontal isotopy classes of \bar{V} and $\lambda_k \times L'$. We now need another variant of Theorem 4.1.1 whose proof will also be postponed.

Corollary 4.1.2. *There exist (time dependent) almost complex structures $\tilde{\mathbf{J}} = \{\tilde{J}_t\}_{t \in [0,1]}$ on $\mathbb{R}^2 \times M$ with the following properties:*

- (1) *For every t , \tilde{J}_t is tamed by the symplectic structure $\omega_{\mathbb{R}^2} \oplus \omega$ of $\mathbb{R}^2 \times M$.*
- (2) *For every t , π is (\tilde{J}_t, i) -holomorphic over $\mathbb{R}^2 \setminus ([-1, 2] \times [-K, K])$ for a large positive constant K . Here i is the standard complex structure on $\mathbb{R}^2 \cong \mathbb{C}$.*
- (3) *The Floer complexes $CF(L', L_k; \mathbf{J}^b)$ and $CF(L', L; \mathbf{J}^c)$ are well defined over \mathcal{K} for $\mathbf{J}^b = \tilde{\mathbf{J}}|_{\{(b,k)\} \times M}$ and $\mathbf{J}^c = \tilde{\mathbf{J}}|_{\{(a,1)\} \times M}$.*

Moreover, for a generic such $\tilde{\mathbf{J}}$ the Floer complex $(CF(\lambda_k \times L', \bar{V}; \tilde{\mathbf{J}}), \bar{d})$ is well defined and has the form:

$$CF(\lambda_k \times L', \bar{V}; \tilde{\mathbf{J}}) = CF(L', L_k; \mathbf{J}^b)[l_1] \oplus CF(L', L; \mathbf{J}^c)[l_2]$$

for some $l_1, l_2 \in \mathbb{Z}$ and differential

$$D = \begin{pmatrix} d_1 & -\phi_{V, \lambda_k} \\ 0 & -d_2 \end{pmatrix},$$

where d_1 and d_2 are, up to sign, the Floer differentials of $CF(L', L_k; \mathbf{J}^b)$ and $CF(L', L; \mathbf{J}^c)$ respectively.

Since $D^2 = 0$ it follows that $\phi_{V, \lambda_k} : CF(L', L; \mathbf{J}^c) \rightarrow s^{l_1-l_2} CF(L', L_k; \mathbf{J}^b)$ is a chain morphism. (See figure 4.)

Next recall that to a monotone Lagrangian L we can associate its quantum homology $QH(L; \mathcal{K})$ (see §2.2.3, and also [BC3, BC4, BC2]). This is a ring with a unit $[L] \in QH(L; \mathcal{K})$. Recall also that there is a canonical PSS isomorphism (see §2.2.4)

$$PSS : QH(L; \mathcal{K}) \rightarrow HF(L, L).$$

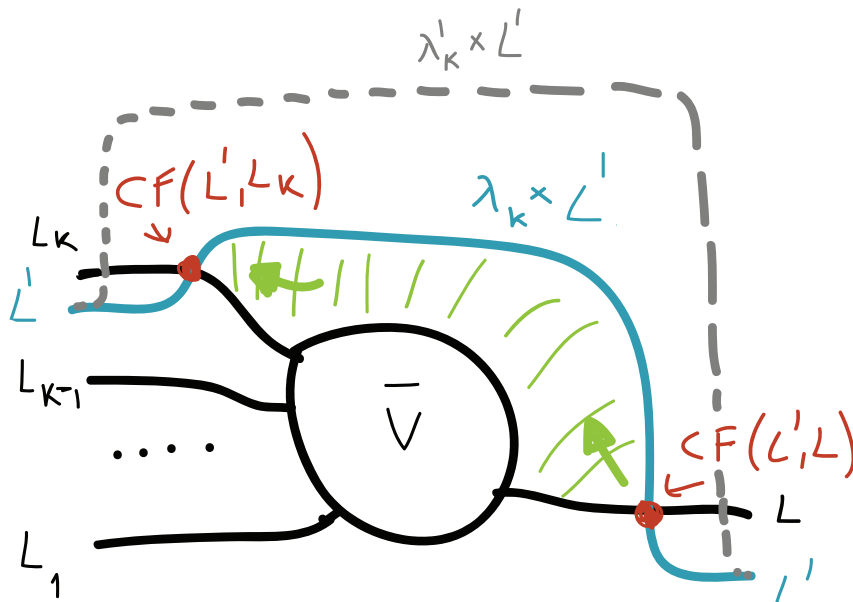


FIGURE 4. The two cobordisms, $\lambda_k \times L'$ and \bar{V} and, in green the $\tilde{\mathcal{J}}$ -holomorphic strips $u : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^2 \times M$ giving ϕ_{V, λ_k} (all after projection on \mathbb{R}^2).

We now define

$$\alpha_{V, \lambda_k} := H(\phi_V) \circ PSS([L]) \in HF(L, L_k),$$

where we have identified here $HF(L, L) = H(CF(L, L; H)) \xrightarrow{c_\varphi} H(CF(L', L; H)) = HF(L', L)$, $HF(L', L_k) \xrightarrow{c_\varphi^{-1}} HF(L, L_k)$ by using the moving boundaries continuation isomorphisms c_φ as described in §2.2.2 with φ a (small) Hamiltonian isotopy with $\varphi_1(L) = L'$; $H(\phi_{V, \lambda_k})$ stands for the map induced in homology by ϕ_{V, λ_k} ; H is an auxiliary generic Hamiltonian. It is not difficult to see that $H(\phi_{V, \lambda_k})$ depends possibly on V and λ_k but not on the other choices made in its construction and the same is true for the class α_{V, λ_k} .

We now restrict to the case $\mathcal{K} = \mathbb{Z}_2$. Put

$$\alpha_V = \alpha_{V, \lambda_k}.$$

We claim that over \mathbb{Z}_2 this defines indeed a functor as desired.

Remark 4.1.3. Although we will not prove this claim here it is useful to understand why the restriction to $\mathcal{K} = \mathbb{Z}_2$ is necessary. The reason is that, for this construction to produce a functor with domain \mathcal{Cob} , the class α_{V, λ_k} , or equivalently the morphism $H(\phi_{V, \lambda_k})$, has to be independent of the choice of λ_k (at least for k large enough) and also has to be invariant with respect to horizontal isotopies of V . Over \mathcal{A} the invariance with respect to λ_k is not verified simply because in the expression of ϕ_{V, λ_k} the areas of the green strips in

Figure 4 are taken into account. Thus, if $\lambda_k \times L'$ is moved by a horizontal isotopy induced from a planar Hamiltonian vector field to $\lambda'_k \times L'$ for some other curve λ'_k - as the one in Figure 4 - the new morphism ϕ_{V, λ'_k} will differ from ϕ_{V, λ_k} because the areas of these strips will change. In fact, the exact difference is that ϕ_{V, λ_k} and $T^a \phi_{V, \lambda'_k}$ are chain homotopic for some real constant a . This is easily seen by using the moving boundary formulas in §2.2.2 (the adaptation to the setting of Lagrangians cylindrical at infinity is discussed in §4.2.2). Clearly, over $\mathcal{K} = \mathbb{Z}_2$ this dependence disappears and the independence on horizontal isotopies of V is also easily established.

Example 4.1.4. If V_ζ is the cobordism between L and itself associated to a loop of Hamiltonian diffeomorphisms $\zeta = \{\zeta_t\}_{t \in S^1}$, then α_{V_ζ} coincides with the Seidel element of η relative to L . In other words, if $S_\zeta \in QH(M; \mathcal{K})$ is the Seidel element associated to the loop ζ then

$$PSS(S_\zeta * [L]) = \alpha_{V_\zeta}$$

where $*$: $QH(M; \mathcal{K}) \otimes QH(L; \mathcal{K}) \rightarrow QH(L; \mathcal{K})$ is the quantum module product [BC3]. In particular, it is easy to find examples when the morphism $\mathcal{F}(V_\zeta)$ is not trivial.

4.1.2. *The functor $\tilde{\mathcal{F}}$.* The construction of $\tilde{\mathcal{F}}$ is very simple at the level of objects:

$$Ob(Cob_0^d(M)) \ni (L_1, \dots, L_k) \xrightarrow{\tilde{\mathcal{F}}} (L_1, \dots, L_k) \in Ob(T^S D\mathcal{F}uk^d(M)) .$$

To describe the functor $\tilde{\mathcal{F}}$ on morphisms we first mention that this will be by definition a monoidal functor so that it is enough to describe $\tilde{\mathcal{F}}(\Phi)$ where

$$\Phi \in \text{Mor}_{Cob_0^d(M)}(L, (L_1, \dots, L_k)) .$$

Let

$$V : L \rightsquigarrow (L_1, \dots, L_k) \text{ with } [V] = \Phi .$$

A part of the difficulty of the construction resides in the fact that we now need to use explicitly the triangulated structure of $D\mathcal{F}uk^d(M)$ and as that is induced from the structure of $\mathcal{F}uk^d(M)^\wedge$ we need to deal with the triangulated structure of this last A_∞ -category.

To simplify notation we will denote $\mathfrak{A} = \mathcal{F}uk^d(M)$. We denote by Ch the category of chain complexes viewed as an A_∞ -category (with operations of order ≥ 3 being trivial). We also recall from [Sei3] the Yoneda embedding:

$$\mathcal{Y} : \mathfrak{A} \rightarrow F(\mathfrak{A}^{opp}, Ch)$$

where $F(A, B)$ denotes the (non-unitary) A_∞ -functor category from the category A to the category B , and A^{opp} is the A_∞ -category opposite to A . The definition of the functor \mathcal{Y} is so that, in our context, $\mathcal{Y}(L) = CF(-, L)$ where $CF(-, L)$ represents the Floer chains

over $\mathcal{K} = \mathbb{Z}_2$ viewed as a functor in the first variable (and actually defined as a directed system because the actual definition of the Floer chain complex also depends on the almost complex structure and on various perturbation data). The Yoneda embedding is faithful and the actual triangulated structure of $\mathcal{F}uk^d(M)^\wedge$ is provided by the triangulated completion of the image of \mathcal{Y} . In turn, the exact triangles in $F(\mathfrak{A}^{opp}, Ch)$ are induced from the exact triangles in Ch . In short, we need to show how to associate to $V : L \rightsquigarrow (L_1, \dots, L_k)$ a sequence η_V of exact triangles in $F(\mathfrak{A}^{opp}, Ch)$:

$$(18) \quad T^{s_i-1}CF(-, L_i) \longrightarrow Z_i \longrightarrow Z_{i+1} \longrightarrow T^{s_i}CF(-, L_i), \quad 1 \leq i \leq k,$$

where $Z_1 = 0$, $s_k = 1$. Moreover, writing $A_V := Z_{k+1}$, we need to provide also an isomorphism $\phi_V : CF(-, L) \longrightarrow T^r A_V$. We then define

$$\tilde{\mathcal{F}}(\Phi) = (\phi_V, A_V, \eta_V).$$

More specifically, we will construct for each Lagrangian $N \in \mathcal{L}_d^*(M)$ the following: a sequence of chain complexes Z_i^N , $1 \leq i \leq k+1$, with $Z_1^N = 0$, and chain morphisms $u_i : CF(N, L_i)[-s_i] \longrightarrow Z_i^N$ so that

$$(19) \quad Z_{i+1}^N = \text{cone}(CF(N, L_i)[-s_i] \xrightarrow{u_i} Z_i^N), \quad \forall 1 \leq i \leq k,$$

as well as a chain homotopy equivalence $\phi_V^N : CF(N, L) \longrightarrow Z_{k+1}^N[-r]$. (So that we take $A_V = CF(N, L)$.)

Of course, the above is not sufficient in order to prove that $\tilde{\mathcal{F}}$ is a functor as one needs to also verify that the exact sequences (19) are functorial in N , that they respect all the higher structures of an A_∞ -category and that $\tilde{\mathcal{F}}$ so defined respects the composition of morphisms. We will omit these verifications here - they are postponed to the more complete version of this work that is in preparation. Still, the basic idea behind these verifications is rather obvious once the construction of the triple (ϕ_V, A_V, η_V) needed in order to define $\tilde{\mathcal{F}}(\Phi)$ is made explicit. This is the construction that we will describe below.

4.1.3. *Construction of the mapping cones (19)*. In this subsection we will work over \mathcal{K} which can be either \mathcal{A} or \mathbb{Z}_2 . The geometric constructions described here imply Corollary 4.0.3. However, we emphasize that for reasons similar to the discussion in Remark 4.1.3, the functoriality properties required for the construction of $\tilde{\mathcal{F}}$ are satisfied only over \mathbb{Z}_2 .

The construction below is another corollary of Theorem 4.1.1. We will use the family of functions and the notation introduced just before the statement of Corollary 4.1.2. First, to simplify the construction we will assume that N is transverse to L as well as to L_i for all $1 \leq i \leq k$. By a possible isotopy of \bar{V} we may assume that

$$\bar{V} \subset \mathbb{R} \times [0, k] \times M, \quad \bar{V}|_{[1, \infty) \times \mathbb{R}} = [1, \infty) \times \{1\} \times L$$

and that for some large enough $a > 2$ we have

$$\overline{V}|_{(-\infty, -a+2] \times \mathbb{R}} = \prod_{i=1}^k (-\infty, -a+2] \times \{i\} \times L_i.$$

We now consider two Lagrangians in $\mathbb{R}^2 \times M$:

$$N^\wedge = \gamma_{a;-1, k+1, 2} \times N, \quad N^\vee = \gamma_{a;-1, -2, 2} \times N.$$

The intersections between these Lagrangians and \overline{V} is:

$$N^\wedge \cap \overline{V} = \bigcup_{i=1}^k \{(q_i, i)\} \times (N \cap L_i), \quad \text{where } q_i \in (-a, -a+1), \sigma_{a;-1, k+1, 2} = i,$$

$$N^\vee \cap \overline{V} = \{(p, 1)\} \times (N \cap L), \quad \text{where } p \in (2, 3), \sigma_{a;-1, -2, 2}(p) = 1.$$

The following is another variant of Theorem 4.1.1.

Corollary 4.1.5. *There exist (time dependent) almost complex structures $\tilde{\mathbf{J}} = \{\tilde{J}_t\}_{t \in [0, 1]}$ on $\mathbb{R}^2 \times M$ with the following properties:*

- (1) *For every t , \tilde{J}_t is tamed by $\omega_{\mathbb{R}^2} \oplus \omega$.*
- (2) *For every t , π is (\tilde{J}_t, i) -holomorphic on $(\mathbb{R}^2 \times M) \setminus ([-a+1, 2] \times [-K, K] \times M)$ (for a large positive constant K). Here i is the standard complex structure on $\mathbb{R}^2 \cong \mathbb{C}$.*
- (3) *The Floer complexes $CF(N, L_i; \mathbf{J}^i)$, $i = 1, \dots, k$, $CF(N, L; \mathbf{J}^0)$, $CF(N^\wedge, \overline{V}; \tilde{\mathbf{J}})$, $CF(N^\vee, \overline{V}; \tilde{\mathbf{J}})$ are all well defined, where $\mathbf{J}^i = \tilde{\mathbf{J}}|_{\{(q_i, i)\} \times M}$, $\mathbf{J}^0 = \tilde{\mathbf{J}}|_{\{(p, 1)\} \times M}$.*

Moreover, $CF(N^\vee, \overline{V}) = CF(N, L)$ and there is a chain homotopy-equivalence (unique up to homotopy) $\overline{\phi}_V^N : CF(N^\vee, \overline{V}) \rightarrow CF(N^\wedge, \overline{V})$ implied by the fact that N^\vee and N^\wedge are isotopic. The complex $CF(N^\wedge, \overline{V})$ has the form:

$$CF(N^\wedge, \overline{V}) = \left(CF(N, L_1[-s_1]) \oplus CF(N, L_2[-s_2]) \oplus \dots \oplus CF(N, L_k[-s_k]), D \right)$$

with the differential given by an upper triangular matrix $D = (D_{ij})$ whose diagonal entries D_{ii} are up to sign the differentials of the complex $CF(N, L_i)$, and the indexes $s_i \in \mathbb{Z}$ are independent of N . (See figure 5.)

Assuming this corollary we will now construct a cone decomposition of $CF(N^\wedge, \overline{V})$ with linearization $(CF(N, L_1), \dots, CF(N, L_k))$. Define (graded) vector spaces $Z_1^N = 0$,

$$Z_q^N = \bigoplus_{i=1}^{q-1} CF(N, L_i)[-s_i], \quad \forall 2 \leq q \leq k+1.$$

We endow each Z_q^N with an operator $\delta_q : Z_q^N \rightarrow Z_q^N$ which is the upper left $q \times q$ sub-matrix of the differential D (which appears in Corollary 4.1.5). Define also maps

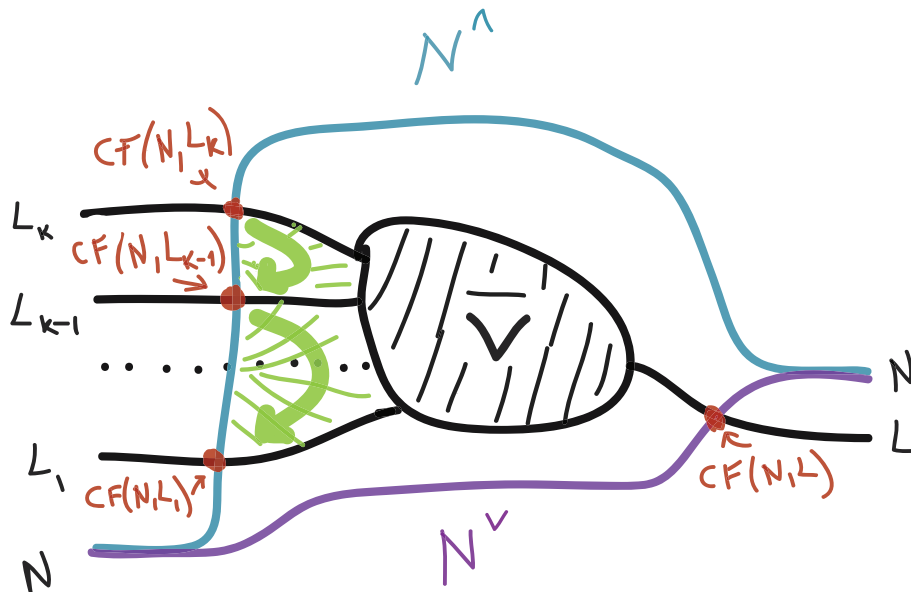


FIGURE 5. The cobordisms V , N^\wedge and N^\vee together with, in green, some of the \tilde{J} -holomorphic strips relevant for the iterated cone structure (everything projected to \mathbb{R}^2).

$u_q : CF(N, L_q)[-s_q] \longrightarrow Z_q^N$, by

$$u_q = \sum_{i=1}^{q-1} D_{i,q}.$$

Since D is upper triangular and $D^2 = 0$ it easily follows that:

- (1) each δ_q is a differential.
- (2) each u_q is a chain map.
- (3) $(Z_{q+1}^N, \delta_{q+1}) = \text{cone}\left((CF(N, L_q)[-s_q], D_{qq}) \xrightarrow{u_q} (Z_q^N, \delta_q)\right)$ for every q .

The chain map $\bar{\phi}_V^N$ can be used to produce ϕ_V^N after adjusting the relevant degree shifts.

Note that in this construction there are some signs that we have omitted (in the matrix D as well as in u_q and δ_q). Since we work with the ground field \mathbb{Z}_2 we just ignore them.

4.2. Some technical ingredients. The purpose of this subsection is to summarize the main ingredients needed to prove the technical Theorem 4.1.1 and its Corollaries 4.1.2 and 4.1.5.

There are essentially three ingredients that are important in these proofs: a compactness argument, a definition of Floer complexes and homology for Lagrangian with cylindrical ends, and finally a method to use plane curve combinatorics to deduce algebraic properties of the differential in such Floer complexes. We will briefly discuss each of them below. Variants of these constructions appear in slightly different settings in the literature (see for

instance the works of Seidel [Sei3], Abouzaid [Abo], Auroux [Aur], as well as older work of Oh [Oh4]). At the same time, each of these ingredients is sufficiently non-standard so that we feel it useful to discuss all three of them here. Besides this, standard techniques together with the methods in [BC2],[BC4] are sufficient to deal with transversality issues.

4.2.1. *Compactness.* Given that cobordisms are viewed as Lagrangians with cylindrical ends and thus non-compact, it is clear that compactness for pseudo-holomorphic curves with boundaries on such Lagrangians is the first main technical issue that one has to deal with.

In this subsection we will discuss the main idea that insures compactness in this setting. Variants of this idea will appear in all the constructions contained later in the section.

For this discussion we will fix two Lagrangians with cylindrical ends \overline{W} and \overline{W}' as in Theorem 4.1.1 - see also Figure 6. In contrast to (16) we do not assume that they are

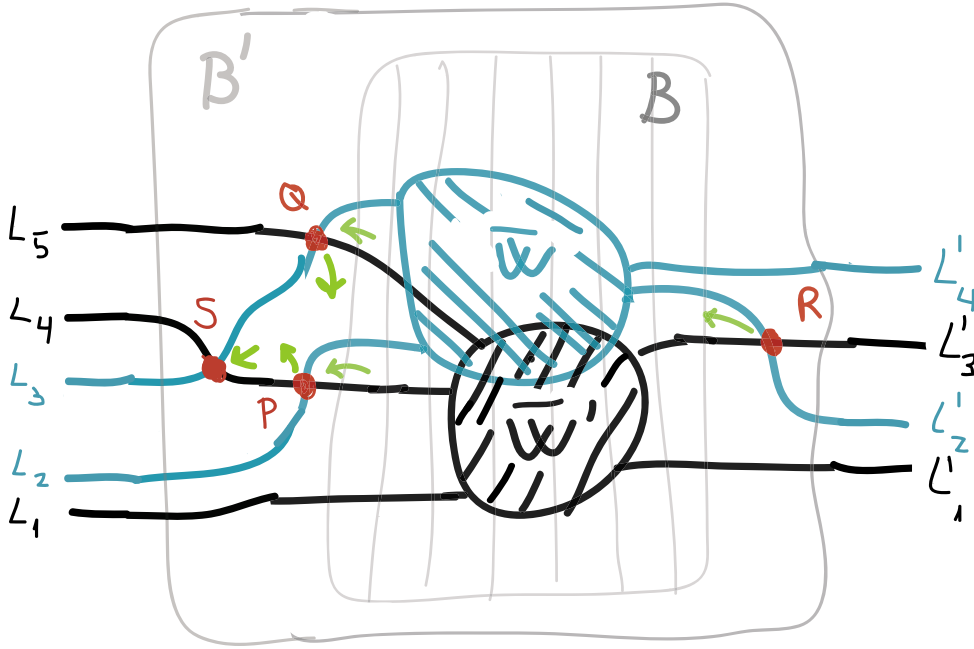


FIGURE 6. Two Lagrangians with cylindrical ends \overline{W} and \overline{W}' projected on the plane with the box B outside of which π is (\tilde{J}, i) -holomorphic and with \tilde{J} -holomorphic strips starting and entering intersection points. Outside the box B' the ends are flat and do no longer intersect.

cylindrically horizontal outside of $[0, 1] \times \mathbb{R} \times M$, but rather that they are cylindrically horizontal outside a compact subset $K \subset \mathbb{R}^2$ as described in §3.1.2. We also fix a compact region in the plane $B \subset \mathbb{C} \cong \mathbb{R}^2$ and we will only consider almost complex structures \tilde{J} so

that π is (\tilde{J}, i) -holomorphic outside $B \times M$. Moreover, outside B each of the cobordisms coincide for the negative ends with products $\gamma_i^- \times L_i^-$ between certain planar curves γ_i^- and Lagrangians $L_i \subset M$ and similarly for the positive ends, they are products $\gamma_j^+ \times L_j^+$ with Lagrangians $L_j^+ \subset M$ and γ_j^+ curves in \mathbb{C} .

Moreover, we assume that the negative planar curves of \overline{W} and those of \overline{W}' intersect transversely, and similarly for the positive planar curves of the two cobordisms. Further, two curves that correspond to positive (respectively, negative) ends of \overline{W} do not intersect outside B and similarly for \overline{W}' . Further, we will also assume that the Lagrangians in M corresponding to the positive ends of \overline{W} and those corresponding to the positive ends of \overline{W}' are two-by-two transverse in M and similarly for the negative ends. We will assume also that there is a compact set $B' \subset \mathbb{C}$ with $B \subset B'$ and so that outside B' the curves γ_i^+ and γ_j^- coincide with horizontal lines, and similarly for the curves of \overline{W}' .

The basic argument here appeared already in Chekanov's work [Che] and is as follows. Assume that $u : \Sigma \rightarrow \mathbb{C} \times M$ is a \tilde{J} -holomorphic curve where Σ is either the disk D^2 , the strip $\mathbb{R} \times [0, 1]$ or the sphere S^2 . In case Σ is the disk we assume that u maps the boundary $\partial\Sigma$ either to \overline{W} or to \overline{W}' , and if Σ is the strip, we assume $u(\mathbb{R} \times \{0\}) \subset \overline{W}$, $u(\mathbb{R} \times \{1\}) \subset \overline{W}'$.

Lemma 4.2.1. *Assume that the symplectic energy of u is finite. Then either $\pi \circ u$ is constant or $\pi \circ u(\Sigma) \subset B'$.*

Proof. The first remark is that $\pi \circ u(\Sigma)$ is bounded. Indeed, this is clear for $\Sigma = D^2, S^2$. If $\Sigma = \mathbb{R} \times [0, 1]$ then due to the finite energy condition we get that $u(\Sigma)$ converges at $\pm\infty$ to some point in $\overline{W} \cap \overline{W}'$. But as $\pi(\overline{W} \cap \overline{W}') \subset B'$ we get that $\pi \circ u(\Sigma)$ is bounded in this case too.

Now assume that $\pi \circ u(\Sigma) \not\subset B'$. Notice that $\mathbb{C} \setminus (B' \cup \pi(\overline{W}) \cup \pi(\overline{W}'))$ is a union of unbounded domains in \mathbb{C} . As $Im(\pi \circ u)$ is bounded it follows that $\pi \circ u$ is constant. Indeed, otherwise an application of the open mapping theorem to the holomorphic map $\pi \circ u$ implies that the image of $\pi \circ u|_{\text{Int}\Sigma}$ contains an unbounded region. \square

Remark 4.2.2. It is a simple exercise to show that the conclusion of the Lemma 4.2.1 remains valid even if u is not \tilde{J} -holomorphic but rather it verifies a perturbed Cauchy-Riemann equation of the form $\bar{\partial}_{\tilde{J}}u - \tilde{J}X_H(z, u) = 0$ where $H_z : \tilde{M} \rightarrow \mathbb{R}$, $z \in \Sigma$ is a smooth family of Hamiltonians with a compact support contained in $B' \times M$.

4.2.2. *Floer homology for Lagrangians with cylindrical ends.* Here we explain the necessary modifications needed for the constructions and structures from §2.2 to work for

Lagrangian cobordisms (rather than just closed Lagrangian submanifolds). In particular, the precise type of Floer homology in Theorem 4.1.1 and the Floer complexes in the Corollaries 4.1.2 and 4.1.5 appear as a special cases of the construction below.

Let \overline{W} and \overline{W}' be two uniformly monotone Lagrangians with cylindrical ends. We will *not* assume for now that they are cylindrically distinct at ∞ .

We intend to define the Floer complex $CF(\overline{W}, \overline{W}'; \eta; (H, f); \tilde{\mathbf{J}})$ with coefficients in \mathcal{K} with \mathcal{K} either \mathcal{A} or \mathbb{Z}_2 (see §2.2) and we now describe the data involved in this definition.

A. The almost complex structure $\tilde{\mathbf{J}} = \{\tilde{J}_t\}_{t \in [0,1]}$. For a compact subset $B \subset \mathbb{R}^2$ denote by $\tilde{\mathcal{J}}_B$ the (families of) almost complex structures $\{\tilde{J}_t\}_{t \in [0,1]}$ on $(\tilde{M}, \tilde{\omega}) = (\mathbb{C} \times M, \omega_0 \oplus \omega)$ with the following properties:

- (1) For every t , \tilde{J}_t is an $\tilde{\omega}$ -tamed almost complex structure on \tilde{M} .
- (2) For every t , the projection π is (\tilde{J}_t, i) -holomorphic on $(\mathbb{R}^2 \setminus B) \times M$.

If $B = \emptyset$ we simply write $\tilde{\mathcal{J}}$.

B. The component $\eta \in \pi_0(\mathcal{P}(\overline{W}, \overline{W}'))$ is fixed as in §2.2.1.

C. The perturbation (H, f) . To describe these perturbations we first fix the notation for the ends of \overline{W} (see §3.1.2). Thus for R_+ and R_- sufficiently big we assume

$$\overline{W}|_{[R_+, \infty) \times \mathbb{R}} = \prod_{i=1}^{k_+} [R_+, \infty) \times \{a_i^+\} \times L_i^+$$

for some $a_1^+ < \dots < a_{k_+}^+$ and

$$\overline{W}|_{(-\infty, R_-] \times \mathbb{R}} = \prod_{i=1}^{k_-} (-\infty, R_-] \times \{a_i^-\} \times L_i^-$$

for some $a_1^- < \dots < a_{k_-}^-$.

The couple (H, f) consists of two Hamiltonians $H : [0, 1] \times \tilde{M} \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with the following properties:

- (1) The support of H is compact.
- (2) The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ verifies:
 1. The support of f is contained in the union of the sets

$$U_i^+ = [R_+ + 1, \infty) \times [a_i^+ - \epsilon_i^+, a_i^+ + \epsilon_i^+]$$

and

$$U_i^- = (-\infty, R_- - 1] \times [a_i^- - \epsilon_i^-, a_i^- + \epsilon_i^-]$$

where the positive constants ϵ_i^\pm are small enough (and the numbers R_+ and R_- are big enough) so that the all the sets U_i^\pm above are pairwise disjoint.

2. The restriction of f on each set $V_i^+ = [R_+ + 2, \infty) \times [a_i^+ - \epsilon_i^+/2, a_i^+ + \epsilon_i^+/2]$ and $V_i^- = (-\infty, R_+ - 2] \times [a_i^- - \epsilon_i^-/2, a_i^- + \epsilon_i^-/2]$ is of the form

$$f(x, y) = \alpha_i^\pm x + \beta_i^\pm$$

with $\alpha_i^\pm \in \mathbb{R}$ sufficiently small so that the planar Hamiltonian diffeomorphism ϕ_t^f associated to f leaves the sets $[R_+ + 2, \infty) \times \{a_i^+\}$ and $(-\infty, R_+ - 2] \times \{a_i^-\}$ inside the respective V_i^\pm for $0 \leq t \leq 1$.

3. We assume R_+ and R_- sufficiently big so that \overline{W}' is cylindrical on $(-\infty, R_-] \times \mathbb{R}$ as well as on $[R_+, \infty) \times \mathbb{R}$ and we assume that ϵ_i^\pm is sufficiently small so that

$$(20) \quad (U_i^+ \setminus ([R_+ + 1, \infty) \times \{a_i^+\})) \cap \overline{W}' = \emptyset$$

for all indexes i and similarly

$$(21) \quad (U_i^- \setminus ((-\infty, R_- - 1] \times \{a_i^-\})) \cap \overline{W}' = \emptyset.$$

We denote the space of pairs (H, f) as above by $\mathcal{H}(\overline{W}, \overline{W}')$. The role of the function f is to perturb the Lagrangian \overline{W} so as to render it cylindrically distinct at infinity from \overline{W}' by using the Hamiltonian flow associated to the composition $e = f \circ \pi$ with $\pi : \widetilde{M} = \mathbb{R}^2 \times M \rightarrow \mathbb{R}^2$ the projection. This is precisely the meaning of the requirements in equations (20), (21): the Hamiltonian flow ϕ_t^e associated to e has the property that $\phi_t^e(\overline{W})$ and \overline{W}' are cylindrically distinct at infinity for all $t \in (0, 1]$ whether or not \overline{W} and \overline{W}' are cylindrically distinct at infinity to start with. Clearly, if \overline{W} and \overline{W}' are not cylindrically distinct at infinity, then the constants α_i^\pm associated to those ends of \overline{W} that coincide with some ends of \overline{W}' verify $\alpha_i^\pm \neq 0$. This implies that in this case the space $\mathcal{H}(\overline{W}, \overline{W}')$ has more than a single connected component. It is easy to see that each such component is convex, hence contractible. Moreover, these components only depend on f and not on H so that we will denote the path component of $\mathcal{H}(\overline{W}, \overline{W}')$ associated to a pair (H, f) by $[f]$.

Finally, we define the complex $CF(\overline{W}, \overline{W}'; \eta; (H, f); \tilde{\mathbf{J}})$ where η is as at point B . above, $(H, f) \in \mathcal{H}(\overline{W}, \overline{W}')$ generic and $\tilde{\mathbf{J}} \in \tilde{\mathcal{J}}_B$ for some compact set B is also generic.

We put:

$$(22) \quad CF(\overline{W}, \overline{W}'; \eta; (H, f); \tilde{\mathbf{J}}) := CF(\phi_1^{f \circ \pi}(\overline{W}), \overline{W}'; \eta'; H; \tilde{\mathbf{J}}),$$

where η' is the path component that corresponds to η under the isotopy $\phi_t^{f \circ \pi}$.

Of course, we still have to justify the right term in equation (22). In view of the fact that H is compactly supported and due to our choice of $\tilde{\mathbf{J}}$ it is immediate to see that the (standard) construction of the Floer complex - recalled in §2.2.1 - carries over to

this setting. This is true because compactness for the finite energy solutions of Floer's equation

$$(23) \quad \bar{\partial}_{\tilde{\mathbf{J}}} u + \nabla H(t, u) = 0$$

for $u : [0, 1] \times \mathbb{R} \rightarrow \widetilde{M}$ subject to the boundary conditions $u(\{0\} \times \mathbb{R}) \subset \phi_1^e(\overline{W})$ and $u(\{1\} \times \mathbb{R}) \subset \overline{W}'$ follows from an immediate adaptation of Lemma 4.2.1 as indicated in Remark 4.2.2. Thus the Floer complex $CF(\phi_1^{f \circ \pi}(\overline{W}), \overline{W}'; \eta'; H; \tilde{\mathbf{J}})$ is well-defined.

As in §2.2.1 we omit the component η in case we take into account all Hamiltonian chords, irrelevant of the component of $\mathcal{P}(\overline{W}, \overline{W}')$. Clearly, in case \overline{W} and \overline{W}' are cylindrically distinct at infinity, then one can also take $f = 0$ in the construction above.

The purpose of the rather complicated definition above is to provide a notion of Floer homology that is sufficiently robust so that the standard constructions familiar from the compact case admit analogues in the “cylindrical at infinity” setting.

The key statement is the following:

Proposition 4.2.3. *The homology of the complex $CF(\overline{W}, \overline{W}'; (H, f); \tilde{\mathbf{J}})$ is independent of H , $\tilde{\mathbf{J}}$ and only depends on the path connected component, $[f] \in \pi_0(\mathcal{H}(\overline{W}, \overline{W}'))$ up to canonical isomorphism. This homology is invariant with respect to horizontal isotopies of \overline{W} and \overline{W}' .*

We will denote the homology of this complex by $HF(\overline{W}, \overline{W}'; [f])$. In case \overline{W} and \overline{W}' are distinct at infinity, then, as indicated before, $\pi_0(\mathcal{H}(\overline{W}, \overline{W}'))$ reduces to a single point and f may be taken $f = 0$. In this case we denote the homology simply by $HF(\overline{W}, \overline{W}')$. Moreover, if $\overline{W}, \overline{W}'$ are distinct at infinity and transverse, then for generic $\tilde{\mathbf{J}} \in \tilde{\mathcal{J}}_B$ (with B sufficiently big) the complex $CF(\overline{W}, \overline{W}'; (0, 0); \tilde{\mathbf{J}})$ is well-defined and we denote it by $CF(\overline{W}, \overline{W}'; \tilde{\mathbf{J}})$. This is the notation used in Theorem 4.1.1 and in the Corollaries 4.1.2 and 4.1.5.

Proof. We now sketch the argument for the proof of Proposition 4.2.3. First, the standard invariance arguments for Floer homology easily adapt to this setting, again by using the compactness argument in Lemma 4.2.1, to show independence with respect to choices of H and $\tilde{\mathbf{J}}$. The only less immediate invariance statements concern the independence of f - inside the same connected component of $\mathcal{H}(\overline{W}, \overline{W}')$ - and with respect to horizontal homotopies.

The invariance in both these cases follows from the standard construction of Floer Lagrangian comparison maps in the case of moving Lagrangian boundary conditions - as described in §2.2.2 combined with yet another application of the compactness Lemma 4.2.1.

We exemplify the argument in the case of independence with respect to f . Thus assume that f and f' are so that $(H, f), (H, f') \in \mathcal{H}(\overline{W}, \overline{W}')$ and $[f] = [f']$. We also pick a compact set $B \subset \mathbb{R}^2$ as well as a generic $\tilde{\mathbf{J}} \in \tilde{\mathcal{J}}_B$. Let $\nu : \mathbb{R} \rightarrow [0, 1]$ be an increasing C^∞ function so that $\nu(\tau) = 0$ for $\tau \leq 0$ and $\nu(\tau) = 1$ for $\tau \geq 1$. Define $f_\tau = \nu(\tau)f + (1 - \nu(\tau))f'$, $f_\tau : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\tau \in \mathbb{R}$. Let $e_\tau = f_\tau \circ \pi$. Denote by $\overline{W}_\tau = \phi_1^{e_\tau}(\overline{W})$. Therefore, $\overline{W}_\tau = \phi_1^{f' \circ \pi}(\overline{W})$ for $\tau \leq 0$ and $\overline{W}_\tau = \phi_1^{f \circ \pi}(\overline{W})$ for $\tau \geq 1$. We now define a morphism:

$$\psi : CF(\phi_1^{f' \circ \pi}(\overline{W}), \overline{W}'; H; \tilde{\mathbf{J}}) \rightarrow CF(\phi_1^{f \circ \pi}(\overline{W}), \overline{W}'; H; \tilde{\mathbf{J}})$$

by a sum like in Equation (5) running over the elements of zero dimensional moduli spaces consisting of finite energy solutions to Floer's equation (23) subject to the boundary conditions

$$(24) \quad u(0, s) \in \overline{W}_s, \quad (1, s) \in \overline{W}' \quad \forall s \in \mathbb{R}.$$

The only difficulty in checking that this morphism is well-defined and verifies the expected properties in standard Floer theory (i.e. it induces a canonical isomorphism in homology as in §2.2.2) reside in insuring that the moduli spaces of finite energy Floer trajectories with moving boundary conditions as above verify the usual compactness properties. But this follows immediately by noticing that, because $[f] = [f']$ we have that \overline{W}_τ and \overline{W}' are cylindrically distinct at infinity for all $\tau \in \mathbb{R}$. This implies that Lemma 4.2.1 can still be applied and it shows that the image of a finite energy solution of equation (23) subject to (24) is either constant or it has its image contained in a compact set $K \subset \widetilde{M}$ that contains the support of H and whose projection on \mathbb{R}^2 contains B as well as the rectangle $[R_- - 3, R_+ + 3] \times [a, b]$ where $a < a_i^\pm - \epsilon_i^\pm$ and $b > a_i^\pm - \epsilon_i^\pm$ for all i .

The argument showing invariance with respect to horizontal isotopies is similar.

Remark 4.2.4. Given a pair of Lagrangians with cylindrical ends that are cylindrically distinct at infinity the fact that the fibration $\pi : \mathbb{C} \times M \rightarrow \mathbb{C}$ is trivial provides a variety of possible horizontal isotopies. Each of them leads to some relation among appropriate Floer complexes. An example appears in Figure 5 where the horizontal isotopy between N^\wedge and N^\vee produces a chain homotopy equivalence $CF(N^\vee, \overline{V}) \rightarrow CF(N^\wedge, \overline{V})$.

□

4.2.3. Non-existence of certain holomorphic strips. The arguments here are needed to justify the particular form of the Floer complexes as described for instance in Corollaries 4.1.2 and 4.1.5, in particular the upper triangular form of the matrix of the differential D . To describe them we return to the setting in §4.2.1: in other words $\overline{W}, \overline{W}'$ are Lagrangians in \widetilde{M} with cylindrical ends and $\tilde{\mathbf{J}} \in \tilde{\mathcal{J}}_B$ for some compact set $B \subset \mathbb{R}^2$. We also

recall the particular choices for the ends of \overline{W} and \overline{W}' from §4.2.1. Namely, outside B each of these two cobordisms coincides - for the negative ends - with products $\gamma_i^- \times L_i^-$ between certain planar curves γ_i^- and Lagrangians $L_i \subset M$ and similarly for the positive ends, they are products $\gamma_j^+ \times L_j^+$ with Lagrangians $L_j^+ \subset M$ and γ_j^+ curves in \mathbb{C} . We also assume the transversality conditions mentioned in §4.2.1.

The actual argument reduces to the following remark. Assume that $u : \mathbb{R} \times [0, 1] \rightarrow M$ is a \tilde{J} -holomorphic strip of finite energy with $u(\mathbb{R} \times \{0\}) \subset \overline{W}$, $u(\mathbb{R} \times \{1\}) \subset \overline{W}'$ and such that $\pi \circ u$ is not constant. Assume also that $a = \lim_{s \rightarrow -\infty} u(s, t)$ verifies $a \in (\overline{W} \cap \overline{W}') \setminus B$. This means that a is of the form $a = (p, x)$, with p an intersection point of an end of \overline{W} with an end of \overline{W}' and x is an intersection point of two of the curves, say γ_i and γ_j' . Moreover, in the neighborhood of x , the map $v = \pi \circ u$ is a \mathbb{C} -valued holomorphic map with transverse (embedded) Lagrangian boundary conditions. The intersection pattern of γ_i and γ_j' together with an application of the open mapping theorem limit the possible choices for the point x . More precisely, after an orientation preserving change of coordinates in the neighborhood \mathcal{U} of x we may assume that γ_i coincides with the real axis and γ_j' with the imaginary axis. Then $\mathcal{U} \setminus (\gamma_i \cup \gamma_j')$ is divided into four quadrants. Then the image of v can possibly lie only in two of the four quadrants, namely in the first or the third. Moreover, if the connected component of one of these admissible quadrants inside $\mathbb{C} \setminus (\pi(\overline{W}) \cup \pi(\overline{W}'))$ is unbounded then v cannot have its image inside that quadrant. See Figure 7. Similarly, in Figure 6 the only possible choices for such an x are the points R, Q, P but not S .

The same type of arguments also apply to the positive end of the strip u , i.e. $b = \lim_{s \rightarrow +\infty} u(s, t)$. In case $b \in (\overline{W} \cap \overline{W}') \setminus \pi^{-1}(B)$ then $b = (q, y)$ where again q is an intersection of an end of \overline{W} and one of \overline{W}' and y is an intersection point of two of the curves γ_i or γ_j' . Then the only admissible quadrants are the second and fourth (see Figure 7). And again, if one of these quadrants is inside an unbounded component of the complement of $\pi(\overline{W}) \cup \pi(\overline{W}')$ then this quadrant is ruled out. In Figure 6 for instance, this means that the choices for y are S, Q, P but not R . A little more detailed analysis shows that a strip starting at Q can end at S but not at P or Q itself (except, in this last case, if v is the constant strip at Q). The reason is that, by the open mapping theorem, a strip starting from Q has to have S either as end or inside its image. But if it is inside the image this image is not bounded which is not possible. Thus S is the end of the strip. This also explains Figure 5 where each intersection point of N^\wedge and \overline{V} can not be the exit point of holomorphic strips ending at any of the intersection points in $N^\wedge \cap \overline{V}$ that have a strictly bigger vertical coordinate. This shows that in Corollary 4.1.5 the differential D is upper triangular and it also follows that the diagonal elements have the form claimed.

5.1. Quantum homology for Lagrangians with cylindrical ends. We first discuss the definition of quantum homology in this context and then will see how the PSS-type comparison morphisms between quantum homology and Floer homology (recalled in §2.2.4) adapt to this setting.

5.1.1. *Quantum Homology.* Let $\overline{W} \subset \widetilde{M}$ be a monotone Lagrangian with cylindrical ends and let S be a union of some of its ends. In other words, assume the ends of \overline{W} are

$$E_{R_-}^-(\overline{W}) = \prod_{j=1}^{k_-} (-\infty, R_-] \times \{a_j^-\} \times L_j^-, \quad E_{R_+}^+(\overline{W}) = \prod_{i=1}^{k_+} [R_+, \infty) \times \{a_i^+\} \times L_i^+$$

then

$$S = \bigcup_{j \in J_-} \{a_j^-\} \times L_j^- \cup \bigcup_{i \in J_+} \{a_i^+\} \times L_i^+$$

where $J_- \subset \{1, \dots, k_-\}$ and $J_+ \subset \{1, \dots, k_+\}$.

The quantum homology $QH(\overline{W}, S)$ is defined as follows. Fix $\epsilon > 0$ and put $W = \overline{W}|_{[R_- - \epsilon, R_+ + \epsilon] \times \mathbb{R}}$, so that W is a compact manifold with boundary

$$\partial W = \left(\prod_{j=1}^{k_-} \{(R_- - \epsilon, a_j^-\} \} \times L_j^- \right) \amalg \left(\prod_{i=1}^{k_+} \{(R_+ + \epsilon, a_i^+\} \} \times L_i^+ \right)$$

Let S' be the part of the boundary of W that corresponds to S :

$$S' = \left(\prod_{j \in J_-} \{(R_- - \epsilon, a_j^-\} \} \times L_j^- \right) \cup \left(\prod_{i \in J_+} \{(R_+ + \epsilon, a_i^+\} \} \times L_i^+ \right)$$

Choose a Morse function $\tilde{f} : W \rightarrow \mathbb{R}$ together with a Riemannian metric (\cdot, \cdot) and an almost complex structure \tilde{J} on \widetilde{M} . We require the function \tilde{f} to be so that its negative gradient $-\nabla \tilde{f}$ is transverse to ∂W and moreover it points outside of W along S' and inside W along $\partial W \setminus S'$. We also require \tilde{J} to be so that the projection π is holomorphic outside a compact set $K \subset [R_- - \epsilon/2, R_+ + \epsilon/2] \times \mathbb{R} \times M$. Denote by $\mathcal{D}_S = (\tilde{f}, (\cdot, \cdot), \tilde{J})$ our data.

Proposition 5.1.1. *If the data \mathcal{D}_S is generic, then the pearl complex $\mathcal{C}(\mathcal{D}_S)$ is well-defined by the same construction as the one recalled in §2.2.3. The resulting quantum homology does not depend, up to canonical isomorphism, on the choice of data \mathcal{D}_S nor on the choice of ϵ and R_+, R_- above. We denote the resulting homology by $QH(\overline{W}, S)$.*

Similarly to the conventions in §2.2.3 we will denote by $QH(\overline{W}, S; \mathcal{A})$ the homology of the complex $\mathcal{C}(\mathcal{D}_S) \otimes_{\Lambda} \mathcal{A}$.

Sketch of the proof of Proposition 5.1.1. Recall that the relevant pearly trajectories are composed of flow lines of $-\nabla\tilde{f}$ and \tilde{J} -holomorphic disks. By Lemma 4.2.1 and our restriction on \tilde{J} , there are no pseudo-holomorphic disks with boundary on \overline{W} with non-constant projection to \mathbb{R}^2 that reach the complement of K . In view of the fact that $-\nabla\tilde{f}$ is transverse to ∂W we deduce that all pearly trajectories that originate and end at critical points of \tilde{f} can not reach the boundary of W . This immediately implies that the complex $\mathcal{C}(\mathcal{D}_S)$ is well defined and indeed a chain complex. The same argument also applies to show the rest of the statement. \square

The notation $QH(\overline{W}, S)$ is coherent with the standard notation in Morse (or singular) homology. Moreover, the standard manipulations with Morse functions relative to boundaries adapt to the ‘‘pearly’’ setting and induce exact sequences that are quantum homology versions of the respective Morse theoretic sequences. The following lemma will be useful later in the paper.

Lemma 5.1.2. *Assume \overline{W} is as in Proposition 5.1.1. Pick a union of the ends of \overline{W} and denote it by A . Take also some other union B of the ends of \overline{W} so that $A \cap B = \emptyset$. There is a long exact sequence:*

$$(25) \quad \rightarrow QH_*(A) \rightarrow QH_*(\overline{W}, B) \rightarrow QH_*(\overline{W}, A \cup B) \rightarrow QH_{*-1}(A) \rightarrow \dots$$

A similar exact sequence also exists with coefficients in \mathcal{A} .

Proof. We put $S = A \cup B$ and we intend to construct a particular function \tilde{f} as the one appearing in the definition of $QH(\overline{W}, S)$ but with a number of additional properties. We use below the same notation as the one fixed before the statement of Proposition 5.1.1. In particular, $J_+ \subset \{1, \dots, k_+\}$, $J_- \subset \{1, \dots, k_-\}$ are so that

$$S' = \left(\prod_{j \in J_-} \{(R_- - \epsilon, a_j^-)\} \times L_j^- \right) \cup \left(\prod_{i \in J_+} \{(R_+ + \epsilon, a_i^+)\} \times L_i^+ \right)$$

is the part of the boundary of W corresponding to S . We also denote by $J'_+ = \{1, \dots, k_+\} \setminus J_+$ and $J'_- = \{1, \dots, k_-\} \setminus J_-$.

Let $\tilde{f} : W \rightarrow \mathbb{R}$ be a Morse function with the following properties.

$$(26) \quad \begin{aligned} \tilde{f}(x, a_i^+, p) &= f_i^+(p) + \sigma_i^+(x), & \sigma_i^+ : [R_+ + \epsilon/4, R_+ + \epsilon] &\rightarrow \mathbb{R}, & p \in M, & j = 1, \dots, k_+, \\ \tilde{f}(x, a_j^-, p) &= f_j^-(p) + \sigma_j^-(x), & \sigma_j^- : [-R_- - \epsilon, -R_- - \epsilon/4] &\rightarrow \mathbb{R}, & p \in M, & j = 1, \dots, k_-, \end{aligned}$$

where $f_i^+ : L_i^+ \rightarrow \mathbb{R}$, $f_j^- : L_j^- \rightarrow \mathbb{R}$ are Morse functions. The functions σ_i^+ , σ_j^- are also Morse, each with a *single* critical point and are required to satisfy the following conditions:

- (1) $\sigma_i^+(x)$ is a non-constant linear function for $x \in [R_+ + 3\epsilon/4, R_+ + \epsilon]$. Moreover, in this interval σ_i^+ is decreasing if $i \in J_+$ and increasing if $i \in J'_+$. Further, σ_i^+ has a single critical point at $R_+ + \epsilon/2$ and this is of index 1 if $i \in J_+$ and of index 0 if $i \in J'_+$.
- (2) $\sigma_j^-(x)$ is a non-constant linear function for $x \in [-R_- - \epsilon, -R_- - 3\epsilon/4]$. Moreover, in this interval σ_j^- is increasing if $j \in J_-$ and decreasing if $j \in J'_-$; σ_j^- has a single critical point at $R_- - \epsilon/2$ and this is of index 1 if $j \in J_-$ and of index 0 if $j \in J'_-$.

A function \tilde{f} with these properties will be called *adapted to the exit region S*.

We now pick a Riemannian metric (\cdot, \cdot) on W which splits as $g^\pm \oplus dx^2$ on $W \cap \pi^{-1}([R_+ + \epsilon/4, R_+ + \epsilon] \times \mathbb{R})$ and $W \cap \pi^{-1}([-R_- - \epsilon, -R_- - \epsilon/4] \times \mathbb{R})$ for some Riemannian metrics g^\pm on the manifolds $\coprod_i L_i^+$ and $\coprod_j L_j^-$. We call such a metric *adapted to the ends of \widetilde{W}* . Finally we also pick (a time independent) almost complex structure \tilde{J} on \widetilde{M} such that π is (\tilde{J}, i) -holomorphic outside a compact set contained in $\widetilde{M} \setminus \pi^{-1}([R_- - \epsilon/4, R_+ + \epsilon/4] \times \mathbb{R})$.

Let now I_-, I_+ be index sets so that

$$A' = \left(\prod_{j \in I_-} \{(R_- - \epsilon, a_j^-)\} \times L_j^- \right) \cup \left(\prod_{i \in I_+} \{(R_+ + \epsilon, a_i^+)\} \times L_i^+ \right)$$

corresponds to A and let $U(A')$ be a tubular neighborhood of A' in W given by

$$U(A') = \left(\prod_{j \in I_-} [R_- - \epsilon, R_- - 5\epsilon/8] \times \{a_j^-\} \times L_j^- \right) \cup \left(\prod_{i \in I_+} [R_+ + 5\epsilon/8, R_+ + \epsilon] \times \{a_i^+\} \times L_i^+ \right).$$

We now let $V = W \setminus U(A')$ and also denote

$$A'' = \left(\prod_{j \in I_-} \{(R_- - \epsilon/2, a_j^-)\} \times L_j^- \right) \cup \left(\prod_{i \in I_+} \{(R_+ + \epsilon/2, a_i^+)\} \times L_i^+ \right).$$

We assume the various choices made are generic so that the pearl complexes $\mathcal{C}(W, \tilde{f}, \tilde{J})$, $\mathcal{C}(A'', \tilde{f}|_{A''}, \tilde{J})$ and $\mathcal{C}(V, \tilde{f}|_V, \tilde{J})$ are well defined. These three complexes are related by an obvious short exact sequence:

$$0 \rightarrow \mathcal{C}(V, \tilde{f}|_V, \tilde{J}) \rightarrow \mathcal{C}(W, \tilde{f}, \tilde{J}) \rightarrow \mathcal{C}(A'', \tilde{f}|_{A''}, \tilde{J}) \rightarrow 0.$$

The claim now follows by noticing that $\mathcal{C}(A'', \tilde{f}|_{A''}, \tilde{J})$ is isomorphic to a pearl complex associated to A with a shift in degree by one, $H(\mathcal{C}(V, \tilde{f}|_V, \tilde{J})) = QH(V'', B) = QH(\overline{W}, B)$ and, by definition, $H(\mathcal{C}(W, \tilde{f}, \tilde{J})) = QH(\overline{W}, A \cup B)$.

□

Remark 5.1.3. We will mainly apply the construction above to Lagrangians \overline{V} that are the \mathbb{R} -extensions of Lagrangian cobordisms V . In this case we denote $QH(\overline{V}, S)$ by $QH(V, S)$ and similarly when working over \mathcal{A} .

5.1.2. *The PSS isomorphism for Lagrangians with cylindrical ends.* Let $\overline{W} \subset \widetilde{M}$ be a Lagrangian with cylindrical ends and assume that S is a union of some of its ends as in §5.1.1. The choice of S determines a path component $c_S \in \pi_0(\mathcal{H}(\overline{W}, \overline{W}))$ in the following way. Consider a perturbation function f , as at point C in §4.2.2, so that:

- (1) for each positive end i of \overline{W} , the constant α_i^+ is negative if the end is in S and is positive if the end i is not in S .
- (2) for each negative end j , the constant α_j^- is positive if the end is in S and is negative if the end j is not in S

and put $c_S := [f]$.

The purpose of this subsection is to discuss the proof of the following result.

Proposition 5.1.4. *There exists a PSS-type isomorphism over \mathcal{A}*

$$\overline{PSS}_S : HF(\overline{W}, \overline{W}; c_S) \cong QH(\overline{W}, S; \mathcal{A}) .$$

Proof. With the notations in the proof of Lemma 5.1.2, let $\tilde{f} : W \rightarrow \mathbb{R}$ be adapted to the exit region S . Extend the function \tilde{f} to the whole of \overline{W} by using the formulas in (26) and extending the functions $\sigma_i^+(x)$ linearly beyond $R_+ + \epsilon$ and also extending linearly the functions $\sigma_j^-(x)$ linearly below $R_- - \epsilon$.

Fix a Darboux-Weinstein neighborhood \mathcal{U} of \overline{W} in \widetilde{M} which is symplectomorphic to a neighborhood of the zero-section in $T^*\overline{W}$. Due to the cylindrical ends of \overline{W} we can choose \mathcal{U} so that $\pi(\mathcal{U}) \cap ((-\infty, -R_-] \times \mathbb{R})$ contains the strips $\cup_i (-\infty, R_-] \times (a_i^- - \delta, a_i^- + \delta)$ for some $\delta > 0$ and similarly for $\pi(\mathcal{U}) \cap ([R_+, \infty) \times \mathbb{R})$.

After multiplying \tilde{f} by a small positive constant we may assume that \tilde{f} has a small differential $d\tilde{f}$ so that the graph of $d\tilde{f}$ fits inside of \mathcal{U} . Recall that \tilde{f} has a linear horizontal component along the ends. Extend the function \tilde{f} first to a function on \mathcal{U} using the identification of \mathcal{U} with a neighborhood of $\overline{V} \subset T^*\overline{W}$ (making it constant along each cotangent fibre), and then to the rest of \widetilde{M} so that the resulting function $H_{\tilde{f}}$ vanishes outside a closed neighborhood \mathcal{U}' of \mathcal{U} .

Pick a generic autonomous almost complex structure $\tilde{J} \in \tilde{\mathcal{J}}_B$ with B a compact set sufficiently large so that \overline{W} is cylindrical outside B . We will also assume R_+ and $|R_-|$ sufficiently big so that $[R_+, \infty) \times \mathbb{R}$ as well as $(-\infty, R_-] \times \mathbb{R}$ are both outside B .

The linearity of the function \tilde{f} at infinity immediately shows that $\overline{W}_1 := \phi_1^{H_{\tilde{f}}}(\overline{W})$ and \overline{W} are cylindrically distinct at infinity, that for a generic choice of \tilde{f} the Floer complex $CF(\overline{W}_1, \overline{W}; \tilde{J})$ is well defined and that, by Proposition 4.2.3, its homology is canonically identified with $HF(\overline{W}, \overline{W}; c_S)$ (see §4.2.2 and in particular (22)).

We will also need below another function $\tilde{f}' : W \rightarrow \mathbb{R}$ with the same properties from Lemma 5.1.2 as \tilde{f} except that the value of ϵ used to construct \tilde{f}' is fixed to be $\epsilon' = \epsilon/2$.

We also fix a metric (\cdot, \cdot) on W that is adapted to the ends of \widetilde{W} (in the sense indicated in the proof of Lemma 5.1.2) and so that the pearl complex $\mathcal{C}(\mathcal{D}_S)$ is defined for $\mathcal{D}_S = (\widetilde{f}', (\cdot, \cdot), \widetilde{J})$. We will work in this proof only over \mathcal{A} so that the homology computed by $\mathcal{C}(\mathcal{D}_S)$ is $QH(\overline{W}, S; \mathcal{A})$.

We now intend to consider the moving boundaries PSS - chain morphism - see §2.2.4:

$$\widetilde{PSS} : \mathcal{C}(\mathcal{D}_S) \rightarrow CF(\overline{W}_1, \overline{W}; \widetilde{J}) .$$

In fact, the only issue that is specific to our cylindrical at infinity setting is again whether the necessary compactness is satisfied by the moduli spaces used to define this map. If this is the case, the rest of the construction takes place like in the compact setting. In particular, we also obtain that this morphism induces an isomorphism in homology.

Thus, our focus will now be to describe the relevant moduli spaces and indicate the reason why compactness holds.

Let $x \in \text{Crit}(\widetilde{f}')$ and let $a \in \overline{W}_1 \cap \overline{W}$ be an intersection point. Consider a C^∞ function $\beta : \mathbb{R} \rightarrow [0, 1]$ so that $\beta(s) = 0$ for $s \leq 0$, $\beta(s) = 1$ for $s \geq 1$ and β is strictly increasing on $(0, 1)$. Put $\overline{W}_s = \phi_{\beta(s)}^{H_{\widetilde{f}'}}$.

We consider the moduli space $\mathcal{M}(x, a; \widetilde{J})$ consisting of pairs (v, u) where v is a string of pearls on \overline{W} formed by flow lines of $-\nabla \widetilde{f}'$ (the first one originating at x) alternating with \widetilde{J} -holomorphic disks in \widetilde{M} with boundary on \overline{W} (see [BC4, BC3]) so that the last flow line in the string v ends at a point $b \in \overline{W}$. This point b is the starting point of a solution $u : [0, 1] \times \mathbb{R} \rightarrow \widetilde{M}$, of the Cauchy-Riemann equation $\overline{\partial}_{\widetilde{J}} u = 0$ subject to the following moving boundary condition:

$$(27) \quad u(0, s) \in \overline{W}_s, \quad u(1, s) \in \overline{W}.$$

By “starting point” we mean that $\lim_{s \rightarrow -\infty} u(-, s) = b$. We also have $\lim_{s \rightarrow \infty} \overline{u}(-, s) = a$.

It is easy to see that the needed compactness properties for the definition of \widetilde{PSS} as well as that of its (homological) inverse and all the other relevant properties are an immediate consequences of the following result.

Lemma 5.1.5. *With the notation above*

$$\text{Im}(\pi \circ u) \subset B \cup (([R_- - \epsilon/2, R_+ + \epsilon/2] \times \mathbb{R}) \cap \mathcal{U}') .$$

Proof of Lemma 5.1.5. Put $P = \{R_+ + \epsilon/2\} \times a_i^+$ and notice that this is a point of intersection of $l_s = \pi(\overline{W}_s)$ and $l = \pi(\overline{W})$ for all s , the intersection is transverse for $s > 0$. This is because P is a critical point for the function σ_i^+ . Let now $u' = \pi \circ u$ and let $z \in \{0\} \times \mathbb{R} \cup \{1\} \times \mathbb{R}$. The key remark for proving the lemma is that it is not possible that $u'(z) = P$. The reason for this is again the open mapping theorem: if $u'(z) = P$

then the image of u would contain a small open quadrant around P as in Figure 8 which is not possible. A similar statement holds also for the negative ends of \overline{W} .

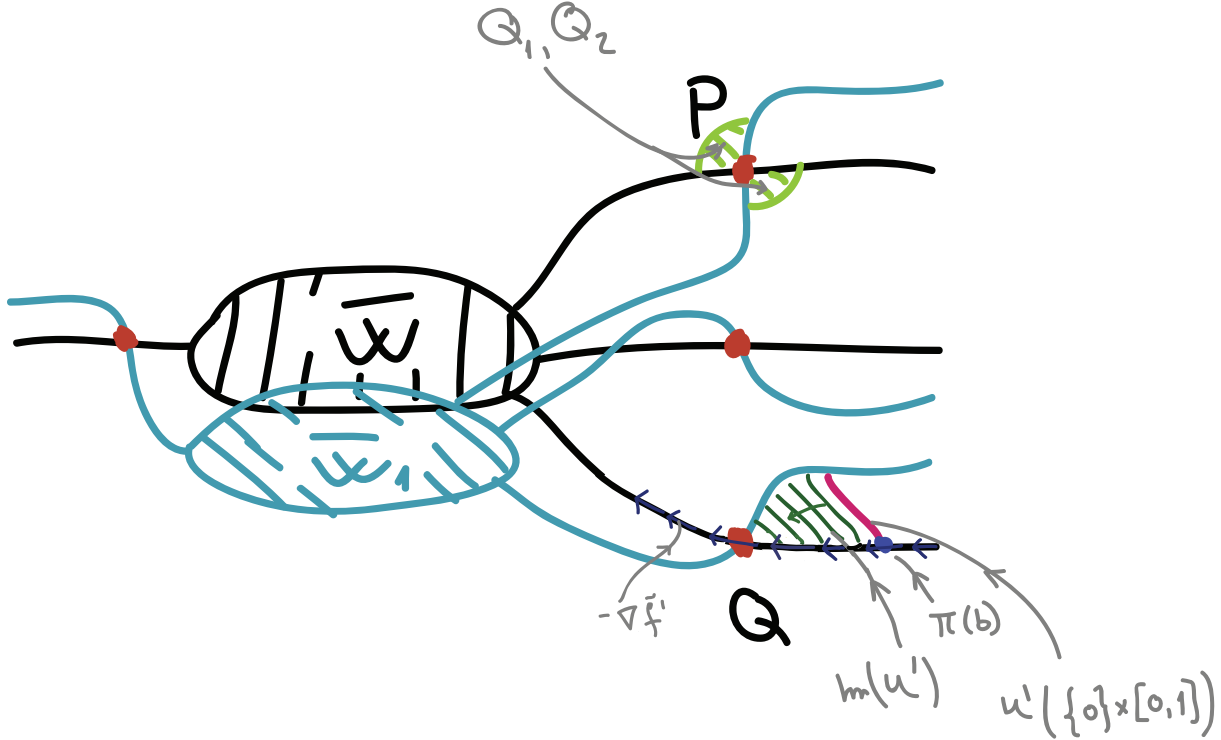


FIGURE 8. The cobordisms \overline{W}_1 and \overline{W} . The quadrants Q_1, Q_2 around P . Also appears the image of u' , the points $Q = \pi_a$ and $\pi(b)$ as well as the direction of the flow $-\nabla \tilde{f}'$ when projected on \mathbb{R}^2 .

By using again the open mapping theorem we deduce that if the image of u does not verify the claim, then the whole image of u' is contained in $((-\infty, R_- - \epsilon/2] \cup [R_+ + \epsilon/2, \infty)) \times \mathbb{R}$. This means that there is some point Q of the form $Q = \{R_+ + \epsilon/2\} \times a_{i_0}^+$ or $Q = \{R_- - \epsilon/2\} \times a_{j_0}^-$ so that $\pi(a) = Q$. To simplify the discussion assume that we are in the first case, the second one is treated in a perfectly similar fashion. The fact that $\pi(a) = Q$ implies that the strip u' “arrives” at Q and this is easily seen to imply that $\text{ind}_{\sigma_{i_0}^+}(Q) = 0$. Moreover, $\pi(b)$ can be written as $\pi(b) = (b', a_{i_0}^+)$ with $b' \geq R_+ + \epsilon/2$. At this point we use the particular form of the function \tilde{f}' : as the function $\sigma_{i_0}^+$ used in the construction of \tilde{f}' is increasing on the interval $[R_+ + 3\epsilon/8, +\infty)$ (because $\epsilon' = \epsilon/2$) and, as the metric (\cdot, \cdot) is adapted to the ends of \overline{W} , we deduce that there can not be any flow lines of $-\nabla(\tilde{f}')$ that come from the interior of the region $\overline{W} \cap \pi^{-1}([R_- - \epsilon/2, R_+ + \epsilon/2])$ and reach the point b . Clearly, by Lemma 4.2.1, there can not be any \tilde{J} -holomorphic disk

with boundary on \widetilde{W} reaching b either. Taken together, these two facts contradict our assumption on the image of u and this concludes the proof of the lemma. \square

The proof of Proposition 5.1.4 follows now by standard arguments. \square

5.2. Elementary cobordisms.

Theorem 5.2.1. *Let $V \in \text{Mor}_{\text{Cob}^d(M)}(L, L')$, then V is a quantum h -cobordism in the sense that $QH(V, L) = 0 = QH(V, L')$ and moreover $QH(L)$ and $QH(L')$ are isomorphic (via an isomorphism that depends on $[V]$).*

Remark 5.2.2. In case L, L' and V are all exact we have $QH_*(V, L) \cong H(V, L) \otimes \Lambda$ (where Λ is the Novikov ring) so by Theorem 5.2.1 we obtain $H_*(V, L) = 0$. Thus in this case, quantum h -cobordism implies usual h -cobordism.

Proof. By Proposition 5.1.4 the Floer homology associated to the two Lagrangian cobordisms, \bar{V} and \bar{V}' , in Figure 9 verifies:

$$(28) \quad HF(\bar{V}', \bar{V}) \cong HF(\bar{V}, \bar{V}; c_L) \cong QH(V, L; \mathcal{A}),$$

where $c_L \in \pi_0(\mathcal{H}(\bar{V}, \bar{V}))$ is defined as at the beginning of §5.1.2.



FIGURE 9. The elementary cobordism \bar{V} , its (non-isotopic) deformation \bar{V}' together with one horizontally isotopic deformation of \bar{V}' , \bar{V}'' . We have $QH(V, L; \mathcal{A}) \cong HF(\bar{V}', \bar{V}) \cong HF(\bar{V}'', \bar{V})$.

It is clear that, as in the picture we may find \bar{V}'' horizontally isotopic to \bar{V}' and disjoint from \bar{V} . Thus, $HF(\bar{V}', \bar{V}) \cong HF(\bar{V}'', \bar{V}) = 0$. But now, from Lemma 5.1.2, we also have the long exact sequence:

$$\rightarrow QH(L; \mathcal{A}) \rightarrow QH(V; \mathcal{A}) \rightarrow QH(V, L; \mathcal{A}) \rightarrow$$

as well as a similar exact sequence over Λ . From the exact sequence over \mathcal{A} we deduce $QH(L; \mathcal{A}) \rightarrow QH(V; \mathcal{A})$ is an isomorphism. It is easy to see that $QH(-; \mathcal{A}) = QH(-) \otimes_{\Lambda} \mathcal{A}$ and the map $QH(-) \rightarrow QH(-; \mathcal{A})$ is injective (because \mathcal{A} is a field extension of Λ). Thus we obtain $QH(V, L) = 0$ and therefore $QH(L) \rightarrow QH(V)$ is also an isomorphism. For further use, this arrow can be viewed, as in the Morse case, as induced by inclusion. Clearly, a similar argument is valid for $QH(L') \rightarrow QH(V)$. \square

5.3. Two ended splitting. Here we discuss properties of morphisms

$$V \in \text{Mor}_{\text{Cob}^k(M)}(L, (L_1, L_2)) .$$

Our main result is valid under a strong additional assumption.

Theorem 5.3.1. *Assume that L_1 and L_2 are not narrow (over Λ) and that $QH(L)$ is a field (in other words, each element in $QH(L)$ admits an inverse with respect to the quantum multiplication, and moreover the unity $[L] \in QH(L)$ is different from 0). Then the inclusion $QH(L) \rightarrow QH(V)$ is injective. Moreover, for each k we have the inequality:*

$$(29) \quad rk(QH_k(L)) \leq |rk(QH_k(L_1)) - rk(QH_k(L_2))| .$$

Proof. The first part of the argument is based on the existence of the diagram:

$$(30) \quad \begin{array}{ccccccc} QH_*(V, L) & \xrightarrow{j_1} & QH_*(V, L_1 \cup L) & \xrightarrow{s_1} & QH_{*-1}(L_1) & \xrightarrow{l_1} & QH_{*-1}(V, L) \\ \downarrow j_2 & & \downarrow \eta_1 & & \downarrow k_1 & & \\ QH_*(V, L_2 \cup L) & \xrightarrow{\eta_2} & QH_{*-1}(L) & \xrightarrow{i_2} & QH_{*-1}(V, L_2) & & \\ \downarrow s_2 & & \downarrow i_1 & & \downarrow r_2 & & \\ QH_{*-1}(L_2) & \xrightarrow{k_2} & QH_{*-1}(V, L_1) & \xrightarrow{r_1} & QH_{*-1}(V, L_1 \cup L_2) & & \\ \downarrow l_2 & & & & & & \\ QH_{*-1}(V, L) & & & & & & \end{array}$$

where the columns and rows are exact. Here $i_1, i_2, j_1, j_2, k_1, k_2, l_1, l_2, r_1, r_2$ are induced by inclusions and η_1, η_2 and s_1, s_2 are connecting morphisms in the long exact sequences associated to these inclusions. A further important remark is that η_1 is dual to i_2 , η_2 is dual to i_1 , s_1 is dual to k_1 and s_2 is dual to k_2 - the duality here is similar to Poincaré duality (for pearl homology it appears in §4.4 of [BC4]). The existence of this commutative diagram is shown in a way similar to the proof of Lemma 5.1.2. Note that Diagram (30) exists, together with the dualities indicated above, also with coefficients in \mathcal{A} .

The next step is to notice the commutativity of the diagram

$$(31) \quad \begin{array}{ccc} QH(L; \mathcal{A}) & \xrightarrow{i_1} & QH(V, L_1; \mathcal{A}) \\ PSS \downarrow & & \downarrow PSS' \\ HF(L, L) & \xrightarrow{\phi_V} & HF(L, L_2) \end{array}$$

up to multiplication by T^a for some $a \in \mathbb{R}$. Here ϕ_V is the morphism from Corollary 4.1.2 and PSS and PSS' are both isomorphisms as explained below. Recall from Remark 4.1.3 that over \mathcal{A} the map ϕ_V is in fact unique only up to multiplication by a real power of T . This explains why we can expect the commutativity of (31) only up to multiplication by T^α .

The morphism PSS is just the Piunikin-Salamon-Schwarz-type isomorphism $QH(L; \mathcal{A}) \rightarrow HF(L, L)$ as recalled in §2.2.4. The morphism PSS' is given by the composition:

$$(32) \quad QH(V, L_1; \mathcal{A}) \xrightarrow{\overline{PSS}_{L_1}} HF(\bar{V}, \bar{V}; c_{L_1}) \xrightarrow{\eta} HF(\bar{V}', \bar{V}) \xrightarrow{\xi} HF(L, L_2) .$$

Here the first morphism \overline{PSS}_{L_1} is the PSS-type isomorphism discussed in Proposition 5.1.4. The second isomorphism, η , follows from the definition of $HF(-, -)$ in §4.2.2 and

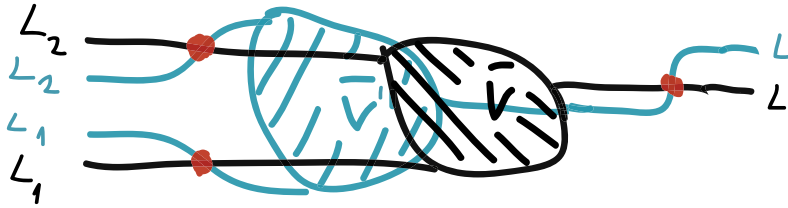


FIGURE 10. The cobordism \bar{V}' obtained by a Hamiltonian deformation associated to a small function $f : \bar{V} \rightarrow \mathbb{R}$ adapted to the exit region L_1 . We have $QH(V, L_1; \mathcal{A}) \cong HF(\bar{V}', \bar{V})$.

Proposition 4.2.3. The third isomorphism, ξ , is itself a composition of two isomorphisms

$$HF(\bar{V}', \bar{V}) \xrightarrow{\xi'} HF(\bar{V}'', \bar{V}) \xrightarrow{\xi''} HF(L, L_2) .$$

Here ξ' is provided (again via Proposition 4.2.3) by the fact that \bar{V}' is horizontally isotopic to the cobordism \bar{V}'' in Figure 11. As for ξ'' , it is an identification $\xi'' : HF(\bar{V}'', \bar{V}) = HF(L, L_2)$ that follows from the fact that $\pi(\bar{V}'')$ and $\pi(\bar{V})$ intersect in a single point in the region where both \bar{V}'' and \bar{V} are just products between curves in the plane and, respectively, L and L_2 .

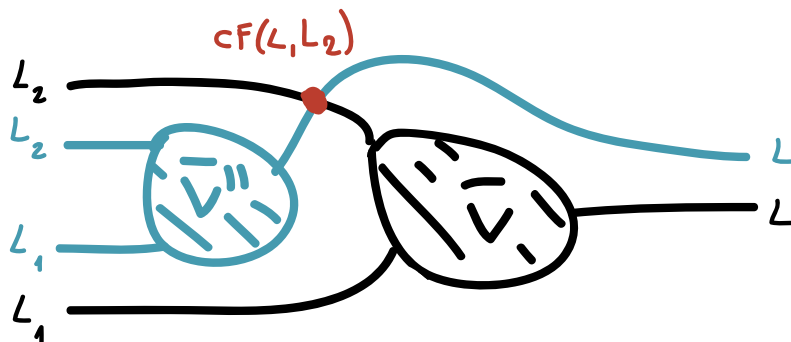


FIGURE 11. The cobordism \bar{V}'' is isotopic to \bar{V}' . We have $HF(\bar{V}', \bar{V}) \cong HF(\bar{V}'', \bar{V}) = HF(L, L_2)$.

The next step is to justify the commutativity of Diagram 31. This verification is based on a slight strengthening of the argument used to justify the isomorphism PSS' . The purpose is to identify geometrically the maps ϕ_V and i_1 and to relate them to the construction of PSS' . The geometric part of this argument consists in composing the two isotopic cobordisms \bar{V}' and \bar{V}'' from the Figures 10 and 11 with a cobordism of the form $\gamma \times L$ as in the Figure 12. To be more precise, assume, without loss of generality, that the cylindrical positive end of both \bar{V}' and \bar{V}'' coincide with $[1, +\infty) \times \{2\} \times L$. Assume also that the positive end of \bar{V} coincides with $[1, +\infty) \times \{1\} \times L$. Now take the curve γ to be the graph of a function $g : [1, +\infty) \rightarrow \mathbb{R}$ so that g is smooth, $g(t) = 2$ for $t \in [1, 2] \cup [4, +\infty)$, g attains its minimum at the point 3 with minimal value $g(3) = -1$ and 3 is the single critical point of g in the interval $(2, 4)$. The curve γ intersects (transversely) the curve $y = 1$ in two points $P = (p, 1)$ and $Q = (q, 1)$ with $p < q$. Finally, we put $\bar{W}' = (\bar{V}' \cap \pi^{-1}((-\infty, 1] \times \mathbb{R})) \cup \gamma \times L$ and similarly $\bar{W}'' = (\bar{V}'' \cap \pi^{-1}((-\infty, 1] \times \mathbb{R})) \cup \gamma \times L$. Certainly, \bar{W}' is horizontally isotopic to \bar{W}'' (and both are horizontally isotopic with \bar{V}' and \bar{V}''). We will use the fact that the isotopy from \bar{W}' to \bar{W}'' may be assumed constant on $\pi^{-1}([1, +\infty) \times \mathbb{R})$.

We use the two cobordisms \bar{V} and \bar{W}' to deduce the commutativity of the following diagram:

$$(33) \quad \begin{array}{ccc} QH(L; \mathcal{A}) & \xrightarrow{i_1} & QH(V, L_1; \mathcal{A}) \\ PSS \downarrow & & \downarrow PSS'' \\ HF(L, L) & \xrightarrow{j} & HF(\bar{W}', \bar{V}) \end{array}$$

where j is the map induced in homology by the inclusion of the subcomplex of $CF(\bar{W}', \bar{V})$ generated by the intersection points of \bar{W}' and \bar{V} that project onto Q ; PSS'' is a composition like $\eta \circ \overline{PSS}_{L_1}$ from Equation (32) only with \bar{W}' instead of \bar{V}' .

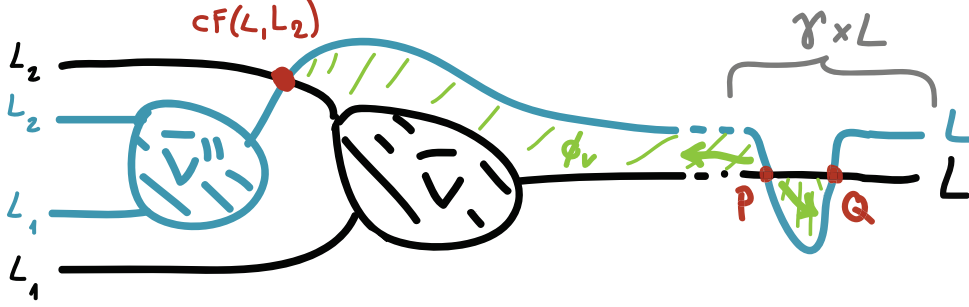


FIGURE 12. The cobordism \bar{W}'' obtained by the extension of \bar{V}'' by $\gamma \times L$ and its intersections with \bar{V} . We have $HF(\bar{V}', \bar{V}) \cong CF(\bar{W}'', \bar{V}) = (CF(L, L_2) \oplus CF(L, L)[1] \oplus CF(L, L), D)$. In green the two non-internal components of D : ϕ_V (to the left) and $id_{CF(L, L)}[-1]$ (to the right).

We now use the cobordisms \bar{W}'' and \bar{V} . The fact that the horizontal isotopy from \bar{W}' to \bar{W}'' may be assumed constant $\pi^{-1}([1, +\infty) \times \mathbb{R})$ implies the commutativity of the triangle below up to multiplication with a term of the form T^a :

$$(34) \quad \begin{array}{ccc} HF(L, L) & \xrightarrow{j} & HF(\bar{W}', \bar{V}) \\ & \searrow \phi_V & \downarrow \xi' \\ & & HF(\bar{W}'', \bar{V}) = HF(L, L_2) . \end{array}$$

Indeed, with the correct choice of perturbations and almost complex structure, the Floer complex $CF(\bar{W}'', \bar{V})$ is of the form $(CF(L, L_2) \oplus CF(L, L)[1] \oplus CF(L, L), D)$ where the differential D is just the internal differential on both $CF(L, L_2)$ and $CF(L, L)$ and on $CF(L, L)[1]$ (which is represented geometrically by the intersection points of \bar{W}'' and \bar{V} that project on P) it has the form $D = d_L[1] + \phi_V - id_{CF(L, L)}[-1]$ where d_L is the differential on $CF(L, L)$. The choice of isotopy shows that j corresponds to the inclusion

$$CF(L, L) \rightarrow CF(L, L_2) \oplus CF(L, L)[1] \oplus CF(L, L)$$

and this implies the commutativity of Diagram (34) up to multiplication by T^a .

To summarize what was shown till now, we proved that Diagram (31) commutes and that PSS and PSS' are isomorphisms. The next important point is that the morphism ϕ_V is a $QH(L; \mathcal{A})$ -module morphism. This is easy to show by a direct argument (of course,

by using a specific choice of function λ_k as in Corollary 4.1.2 and acting on the left with $QH(L')$. As $QH(L)$ is a field this means that either ϕ_V is null or it is an injection. Thus, the same is true for i_1 and it is easy to see that a similar argument can be applied to the morphism i_2 from Diagram (30). The exactness of (30) together with the duality between the i_j 's and the η_r 's implies that one of the i_j 's has to vanish and the other is injective. We will assume that i_1 is injective and that i_2 vanishes. It is immediate to see that injectivity of i_1 with coefficients in \mathcal{A} immediately implies that the corresponding morphism $i_1^\Lambda : QH(L) \rightarrow QH(V, L_1)$ is also injective. Moreover, the vanishing of i_2 with coefficients in \mathcal{A} also implies the vanishing of i_2 over Λ . To shorten notation we will not indicate the coefficients in the notation for these morphisms i_1, i_2 , etc as long as there is no risk of confusion.

The first claim of the Theorem now follows easily. Indeed i_1 (now taken over Λ) factors:

$$QH(L) \rightarrow QH(V) \rightarrow QH(V, L_1)$$

and thus $QH(L) \rightarrow QH(V)$ is injective. The rank inequality (29) follows immediately if we can show that if $I_i = \text{Im}(QH(L_i) \rightarrow QH(V))$ and $I_0 = \text{Im}(QH(L) \rightarrow QH(V))$, then $I_1 \oplus I_0 \subset I_2$ and $QH(L_1) \rightarrow QH(V)$ is injective.

To do this we go back to the Diagram (30) and we start by noticing that the vanishing of i_2 implies that k_1 vanishes. This is seen as follows. First, by an argument similar to that applied to i_1 and i_2 we see that, over \mathcal{A} , k_1 is a $QH(L_1; \mathcal{A})$ -module map. Thus it suffices to show that $k_1([L_1]) = 0$ ($[L_1]$ is the fundamental class and is the unit in $QH(L_1; \mathcal{A})$). Secondly, by using explicitly the form of the pearl complexes associated to a function $f : V \rightarrow \mathbb{R}$ adapted to the exit region $L \cup L_1$ it is easy to see that $i_2([L]) = k_1([L_1])$ and thus $k_1([L_1]) = 0$. This means that k_1 vanishes over \mathcal{A} . But this implies that it also vanishes over Λ . Now k_1 and s_1 are dual so the vanishing of k_1 implies that of s_1 which means that $l_1 : QH(L_1) \rightarrow QH(V, L)$ is injective. But this implies that $QH(L_1) \rightarrow QH(V)$ is injective.

We now show that $I_0, I_1 \subset I_2$. This follows from the exact sequence:

$$\rightarrow QH(L_2) \rightarrow QH(V) \rightarrow QH(V, L_2) \rightarrow$$

combined with the fact that both maps $k_1 : QH(L_1) \rightarrow QH(V) \rightarrow QH(V, L_2)$ and $i_2 : QH(L) \rightarrow QH(V) \rightarrow QH(V, L_2)$ vanish.

The last step is to show that $I_0 \cap I_1 = \{0\}$. This follows from the exact sequence

$$\rightarrow QH(L_1) \rightarrow QH(V) \rightarrow QH(V, L_1) \rightarrow$$

together with the fact that the map $i_1 : QH(L) \rightarrow QH(V) \rightarrow QH(V, L_1)$ is injective. \square

- Remark 5.3.2.*
- i. The inequality (29) shows that often a Lagrangian with $QH(L)$ a field can not be split in two non-narrow parts by a Lagrangian cobordism. For instance, it is easy to see that the inequality (29) can never be verified in dimension $n = 2$ if L_0 and L_1 are non-narrow.
 - ii. Concerning the condition that $QH(L)$ be a field we mention that, when working under enough assumptions so that the base field can be taken to be \mathbb{Q} , a 2-torus has its quantum homology a field as soon as its discriminant - as defined in [BC1] - is not a perfect square.

6. EXAMPLES

This section describes some explicit constructions of Lagrangian cobordisms. They expand on the constructions sketched in §3.2 and are all based on the Lagrangian surgery construction as described for instance by Polterovich in [Pol].

6.1. Topologically non-trivial monotone cobordisms. Assume that L and L' are two Lagrangians $L, L' \in \mathcal{L}_k(M)$ and suppose that, additionally, L and L' are transverse and that they intersect in a single point. Fix a real number $1 < l < 2$ and consider the curve $C^+(l) = \{z+l \in \mathbb{C} : |z| = 1, \operatorname{Re}(z) \geq 0\} \cup \{z \in \mathbb{C} : |\operatorname{Im}(z)| = r, 0 \leq \operatorname{Re}(z) \leq l\}$ - this is a semicircle of radius 1, centered at $(l, 0)$ and extended horizontally till it reaches the imaginary axis. Let $C(l)$ be a C^∞ curve in \mathbb{C} obtained by smoothing $C^+(l)$. We now define two Lagrangian cobordisms in $M \times T^*([0, 2])$: $\bar{L} = L \times C(l)$ and $\bar{L}' = L' \times [0, 2] \times \{0\}$. Obviously, \bar{L} and \bar{L}' intersect transversely in a single point P . We now consider the Lagrangian V obtained from \bar{L} and \bar{L}' by surgery at P (there are actually two choices of surgeries here but for this example this choice is not relevant). It is easy to see that V is monotone with the same monotonicity constant as L and L' and with $d_V = d_L = d_{L'}$.

Notice that V is a monotone, connected, cobordism that is non-trivial topologically. In our language $V : L' \rightsquigarrow (L, L', L)$.

Of somewhat more interest is the cobordism constructed by doubling V . We consider $\bar{V} = \xi(V)$ where $\xi : M \times \mathbb{C} \rightarrow M \times \mathbb{C}$, $\xi(x, z) = (x, -z)$. As ξ is a symplectic diffeomorphism we see that \bar{V} is also a Lagrangian cobordism. The two Lagrangians V and \bar{V} can be glued along their intersection that equals $L \times \{0\} \times \{1\} \cup L' \times \{0\} \times \{0\} \cup L \times \{0\} \times \{-1\}$ - see also Figure 13 and we denote by W the resulting Lagrangian.

Notice that the group $H_2(M \times T^*[-2, 2], W)$ has two additional generators besides those coming from V and \bar{V} . They are denoted by a and b . The first one, a , corresponds to the cycle in the plane given by $\{(s, \{0\}) \in \mathbb{C} : -1 - l \leq s \leq 1 + l\} \cup \{z \in (C(l) \cup -C(l)) : \operatorname{Im}(z) \geq 0\}$ and b corresponds to $\{(s, \{0\}) \in \mathbb{C} : -1 - l \leq s \leq 1 + l\} \cup \{z \in (C(l) \cup -C(l)) :$

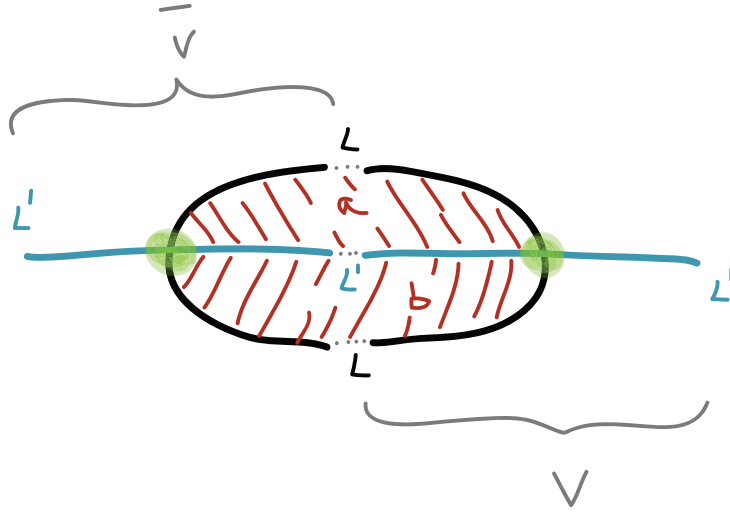


FIGURE 13. The cobordism V its mirror, \bar{V} and the double $W = V \cup \bar{V}$. In green the points where the surgery is operated. The two new generators, a and b , of $H_2(M \times T^*[-2, 2], W)$ are in red.

$Im(z) \leq 0\}$. Endowed with the correct orientation it is easy to see that these two generators are of Maslov index 2 and of positive symplectic area. Moreover, by adjusting appropriately the constant l (and, if needed, also changing the radius of the semi-circle $C^+(l)$) we may insure that a verifies $\omega_0(a) = \rho\mu(a)$ where ρ is the monotonicity constant of V and \bar{V} . After possibly a small Lagrangian isotopy supported on the branch of $C(l)$ with $Im(C(l)) < 0$ we can achieve the relation $\omega_0(b) = \rho\mu(b)$ and thus the resulting Lagrangian cobordism W' is monotone.

Clearly, $W' : L' \rightsquigarrow L'$ and W' is monotone, connected but topologically non-trivial.

Example 6.1.1. The simplest example of Lagrangians verifying the conditions required for this construction are the longitude and the latitude on the torus, $L = S^1 \times \{1\} \subset S^1 \times S^1$, $L' = \{1\} \times S^1 \subset S^1 \times S^1$. One can construct many other examples by taking products of this cobordism with other suitable Lagrangians.

6.2. The trace of surgery as Lagrangian cobordism. The purpose here is to show that the trace of a Lagrangian surgery gives rise to a Lagrangian cobordism. As we shall see, this is a bit less obvious than the reader might first expect because Lagrangian cobordism is less flexible than Lagrangian isotopy.

We start with the local picture and fix the following two Lagrangians: $L_1 = \mathbb{R}^n \subset \mathbb{C}^n$ and $L_2 = i\mathbb{R}^n \subset \mathbb{C}^n$.

We define a particular curve $H \subset \mathbb{C}$, $H(t) = a(t) + ib(t)$, $t \in \mathbb{R}$, with the following properties (see also Figure 14):

- i. H is smooth.
- ii. $(a(t), b(t)) = (t, 0)$ for $t \in (-\infty, -1]$.
- iii. $(a(t), b(t)) = (0, t)$ for $t \in [1, +\infty)$.
- iv. $a'(t), b'(t) > 0$ for $t \in (-1, 1)$.

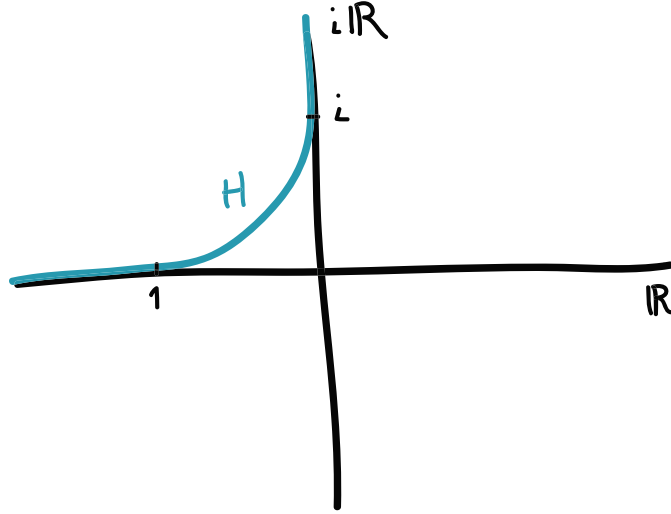


FIGURE 14. The curve $H \subset \mathbb{C}$.

Consider $L = H \cdot S^{n-1} \subset \mathbb{C}^n$ or more explicitly

$$L = \left\{ ((a(t) + ib(t))x_1, \dots, (a(t) + ib(t))x_n) \mid t \in \mathbb{R}, \sum x_i^2 = 1 \right\} \subset \mathbb{C}^n .$$

Lemma 6.2.1. *The submanifold $L \subset \mathbb{C}^n$ as defined above is Lagrangian and there is a Lagrangian cobordism $L \rightsquigarrow (L_1, L_2)$.*

By a slight abuse of notation (because we omit the handle from the notation) we will denote $L = L_1 \# L_2$.

Proof. A straightforward calculation shows that $L \subset \mathbb{C}^n$ is Lagrangian (see e.g. [Pol]).

To construct the desired cobordism we now define

$$\widehat{H} = H \cdot S^n \subset \mathbb{C}^{n+1} .$$

Or more explicitly

$$\widehat{H} = \left\{ ((a(t) + ib(t))x_1, \dots, (a(t) + ib(t))x_{n+1}) \mid t \in \mathbb{R}, \sum x_i^2 = 1 \right\} .$$

A similar computation as before shows that \widehat{H} is also Lagrangian.

We consider the projection $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$, $\pi(z_1, \dots, z_{n+1}) = z_{n+1}$ and we denote by $\widehat{\pi}$ its restriction to \widehat{H} :

$$\widehat{\pi}((a(t) + ib(t))x_1, \dots, (a(t) + ib(t))x_{n+1}) = (a(t) + ib(t))x_{n+1} .$$

Define $W = \widehat{\pi}^{-1}(S_+)$ where $S_+ = \{(x, y) \in \mathbb{R}^2 : y \geq x\}$, see Figure 15. (As usual, we identify $\mathbb{R} = \{(x, y)\}$ with \mathbb{C} under $(x, y) \rightarrow x + iy$.)

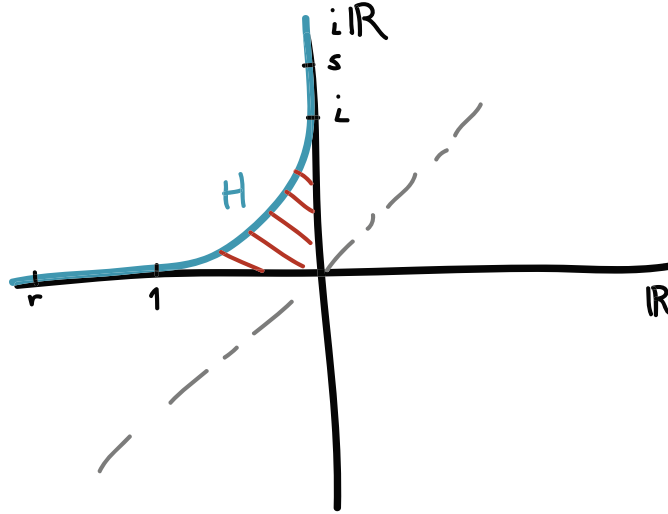


FIGURE 15. The projection of W is the red region together with the two semi-axes $(-\infty, 0] \subset \mathbb{R}$ and $i[0, +\infty) \subset i\mathbb{R}$ and the curve H .

Fix some $r < 0$ and notice that if

$$\widehat{\pi}((a(t) + ib(t))(x_1, \dots, x_{n+1})) = (r, 0) ,$$

then $b(t) = 0$ so that $t \leq -1$ and $a(t) = t$. Moreover, $tx_{n+1} = r$ so that $\sum_{i=1}^n t^2 x_i^2 = t^2 - r^2$. Thus, for $r \leq -1$, we have $\widehat{\pi}^{-1}(r, 0) = L_1 \times (r, 0) \subset \mathbb{C}^n \times \mathbb{C}$. Similarly, for $s \geq 1$, $\widehat{\pi}^{-1}(0, s) = L_2 \times (0, s) \subset \mathbb{C}^n \times \mathbb{C}$. Also notice that $L = \widehat{\pi}^{-1}(0)$.

We now only look to $W_0 = W \cap \pi^{-1}([-2, 0] \times [0, 2])$. It is not difficult to see that $\partial W_0 = L_1 \times \{(-2, 0)\} \cup L_2 \times \{(0, 2)\} \cup L \times \{0, 0\}$. We would like to be able to say that W_0 is a cobordism $W_0 : L \rightsquigarrow (L_1, L_2)$. For this however we still need to show that the L -boundary component of W_0 can be continued to be cylindrical. We now describe explicitly this adjustment (the argument here is in fact quite general). Let $V_L \subset \mathbb{C}^n \times \mathbb{C}$ be the Lagrangian given by $V_L = L \times \{(x, y) \in \mathbb{C} : x = -y\}$.

Put $L^0 = L \times \{(0, 0)\}$. Note that $V_L \cap \pi^{-1}((0, 0)) = \widehat{H} \cap \pi^{-1}((0, 0)) = L^0$. Fix a small neighborhood $U(L^0) \subset \widehat{H}$ of $L^0 \subset \widehat{H}$ and a Darboux-Weinstein neighborhood $\mathcal{N} \subset \mathbb{C}^{n+1}$ of $U(L^0)$ and identify symplectically \mathcal{N} with a tubular neighborhood of $U(L^0)$

in $T^*U(L^0)$. Write $p : \mathcal{N} \rightarrow U(L^0)$ for the projection corresponding via this identification to the projection in the cotangent bundle $T^*U(L^0) \rightarrow U(L^0)$.

Note that at each point of L^0 , V_L projects 1-1 on the tangent space of \widehat{H} (via p). Thus reducing $U(L^0)$ if necessary we can write $V_L \cap \mathcal{N}$ as the graph of a 1-form α on $U(L^0)$ that vanishes on L^0 . Since V_L is Lagrangian the form α is closed. As $U(L^0)$ can be chosen so that it contracts to L^0 , we have $H^1(U(L^0), L^0) = 0$ hence α is exact. Let $f : U(L^0) \rightarrow \mathbb{R}$ be so that $\alpha = df$. Using a partition of unity construct $g : W \cup U(L^0) \rightarrow \mathbb{R}$ so that it agrees with f on $U(L^0) \setminus W$ and vanishes outside a neighborhood of $U(L^0)$. Then the Lagrangian W' obtained by isotoping W_0 by the time-one Hamiltonian diffeomorphism induced by X_g provides the cobordism desired between L and (L_1, L_2) - see also Figure 16. \square

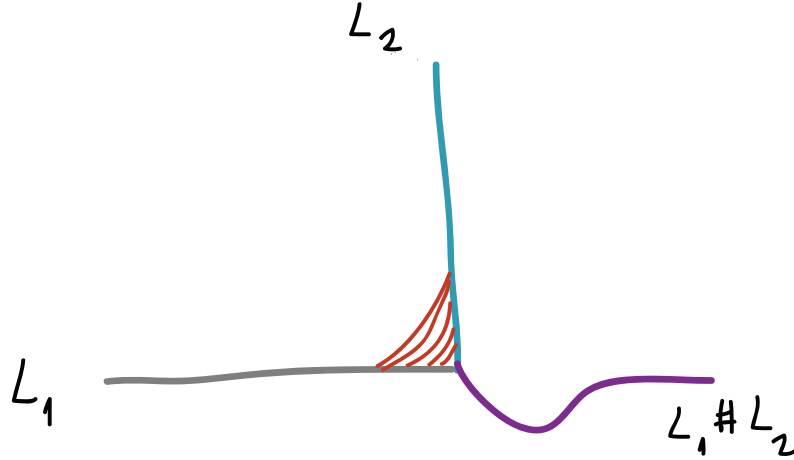


FIGURE 16. The trace of the surgery after projection on the plane.

Remark 6.2.2. i. For further use, we notice that the homotopy type of the total space of the cobordism $W' : L \rightsquigarrow L_1 \# L_2$ constructed above coincides with that of the set $L_1 \cup L_2$.

ii. The construction described here also provides a cobordism $W'' : (L_2, L_1) \rightarrow L_2 \# L_1$ by simply using instead of W the region $\widehat{\pi}^{-1}(S_-)$ with

$$S_- = \{(x, y) \in \mathbb{R}^2 : y \leq x\} .$$

Going from the local argument above to a global one is easy. Suppose that we have two Lagrangians L' and L'' that intersect transversely, possibly in more than a single point. At *each* intersection point we fix symplectic coordinates mapping (locally) L' to $\mathbb{R}^n \subset \mathbb{C}^n$ and mapping (again locally) L'' to $i\mathbb{R}^n \subset \mathbb{C}^n$. We then apply the construction above at

each of these intersection points. This produces a new Lagrangian submanifold $L' \# L''$ as well as a cobordism $L' \# L'' \rightsquigarrow (L', L'')$ (we use $\#$ in the notation as $L' \# L''$ is topologically not a connected sum if there are several intersection points). The homotopy type of V coincides with that of the set $L' \cup L'' \subset M$.

Remark 6.2.3. One can easily generalize the previous construction to a configuration of Lagrangian submanifolds (L_1, \dots, L_r) and the total surgery L of the Lagrangians in the configuration. The result will be a Lagrangian cobordism $V : L \rightsquigarrow (L_1, \dots, L_r)$ with one positive end and r negative ends. Of course, monotonicity will in general not be preserved in this case. However, if the intersection diagram of the configuration is a tree, and if (L_1, \dots, L_r) are uniformly monotone it seems that the Lagrangian L and the cobordism V will be monotone too. An interesting example is when (L_1, \dots, L_r) is a configuration of Lagrangian spheres corresponding to a simple singularity. The relation between singularity theory and Fukaya categories has been extensively studied in recent years (see e.g. [Sei3]). Thus the constructions above (together with Corollaries 4.0.3, 4.0.1, 4.0.2) suggests that the cobordism category is relevant in this study.

6.3. Cobordant Lagrangians that are not isotopic. In this subsection we will make use of the constructions described in §6.2 to construct an example of non smoothly isotopic, monotone cobordant, connected Lagrangians. A variety of other examples can be constructed following the same ideas.

We will start our construction in the ambient manifold $M = \mathbb{C}$. We consider two circles $A = \{z \in \mathbb{C} : |z + 1/2| = 1\}$ and $B = \{x \in \mathbb{C} : |z - 1/2| = 1\}$. We denote by $D(A)$ and $D(B)$ respectively the two disks bounded by A and B . We also consider two smooth curves in the plane \mathbb{C} , $\gamma_1 : [-1, 1] \rightarrow \mathbb{C}$ and $\gamma_2 : [-1, 1] \rightarrow \mathbb{C}$ so that - see Figure 17:

- i. $\gamma_1(t) = t$ for $t \in [-1, -1/2]$
- ii. $\gamma_1(t) = 1 + (1 - t)i$ for $t \in [1/2, 1]$
- iii. $\gamma_1(t)$ is strictly increasing for $t \in (-1/2, 1/2 - \epsilon)$ and strictly decreasing for $t \in (1/2 - \epsilon, 1/2)$.
- iv. $\gamma_2(t) = -\gamma_1(t)$ for all $t \in [-1, 1]$.

We now consider the Lagrangians $A' = \gamma_2 \times A \subset \mathbb{C} \times M$ and $B' = \gamma_1 \times B \subset \mathbb{C} \times M$. By performing surgery - as explained in §6.2 - at both intersection points $A \cap B$ we can extend the union of the two Lagrangians $A' \cup B'$ towards the positive end as well as towards the negative end as in the Figure 17 thus obtaining a cobordism $V : A \# B \rightsquigarrow B \# A$.

Put $L = A \# B$ and $L' = B \# A$. With our choice of handles it is easy to see that L and L' look as in Figure 18. Moreover, if the surgeries used in both intersection

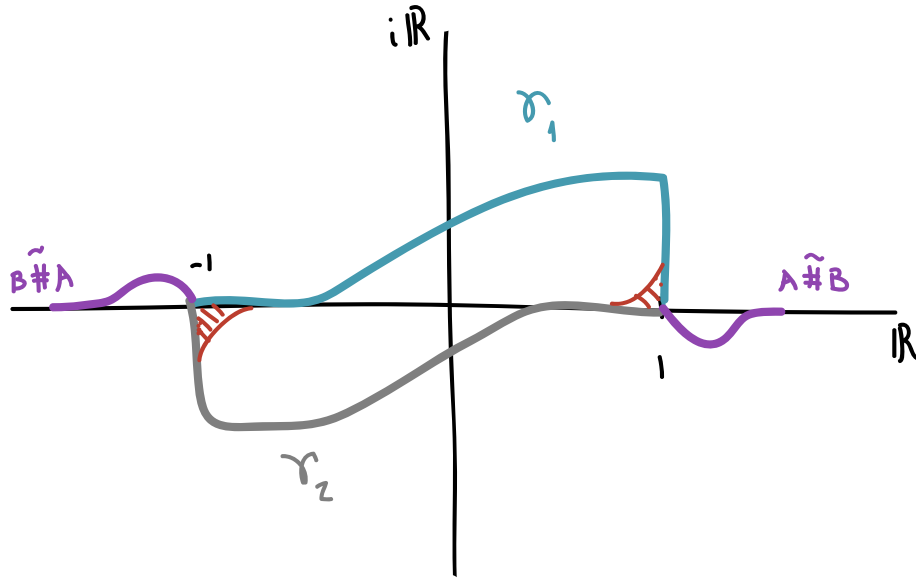


FIGURE 17. The projection of V on \mathbb{C} ; in red the surgery regions; the curves γ_1 (in blue) and γ_2 in gray.

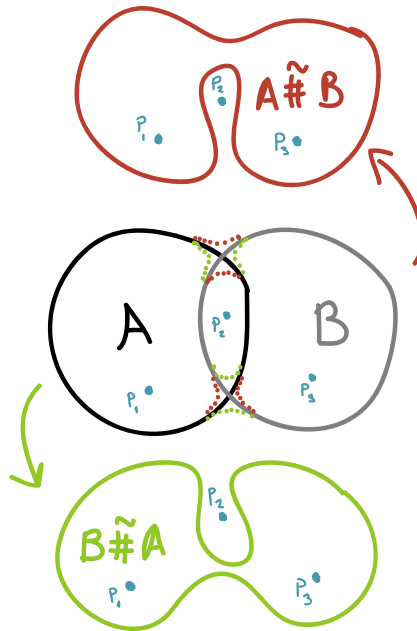


FIGURE 18. The two circles A and B as well as $A\#\tilde{B}$ and $\tilde{B}\#A$. The three puncture points are indicated in blue.

points of A and B use the same handle H , then the area inside both circles is precisely $D(A) + D(B) - \text{Area}(D(A) \cap D(B))$ (the two handles can also be picked differently and

this can modify the areas bounded by these two circles, thus producing a - non-monotone - cobordism relating non-Hamiltonian isotopic, connected Lagrangians).

It is easy to see that V as constructed before is not orientable. Moreover, V is also not monotone. However there is an easy way to transform the cobordism into a monotone one and we describe it now.

Instead of performing all the construction above in $M = \mathbb{C}$ we can as well do it in $M' = \mathbb{C} \setminus \{P_1, P_2, P_3\}$ where the three points P_i are such that $P_1 \in D(A) \setminus D(B)$, $P_2 \in D(A) \cap D(B)$ and $P_3 \in D(B) \setminus D(A)$ as in Figure 18. We will explicitly check monotonicity below. We notice for now that in M' , L and L' are not even smoothly isotopic.

To verify that V is monotone in $\mathbb{C} \times M'$ we write $V = V_+ \cup V_-$, where $V_+ = V \cap \pi^{-1}([0, +\infty) \times \mathbb{R})$ and $V_- = V \cap \pi^{-1}((-\infty, 0] \times \mathbb{R})$. Put $\widetilde{M}'_+ = M' \times ([0, +\infty) \times \mathbb{R})$, $\widetilde{M}'_- = M' \times ((-\infty, 0] \times \mathbb{R})$. Moreover, $V_+ \cap V_- = A \times \{P\} \cup B \times \{Q\}$, where $P = \gamma_2 \cap i\mathbb{R}$ and $Q = \gamma_1 \cap i\mathbb{R}$. Each of V_+ and V_- are homotopy equivalent to $A \cup B$. In particular, $H_2(\widetilde{M}'_+, V_+) = 0$ and $H_2(\widetilde{M}'_-, V_-) = 0$. This implies that $H_2(\mathbb{C} \times M', V) = \mathbb{Z} \oplus \mathbb{Z}$. There are two generators for this group, u and v , each associated to one of the intersection points of A and B . Each of them is represented by a flat lift of the region R bounded by the union of the two curves γ_1 and γ_2 and the two planar projection of the handles at the ends. If we take these generators to be of positive areas we see that they have Maslov class 3: the horizontal loop is of Maslov class 2 and there is also a vertical half loop of Maslov +1. Moreover, by choosing the handles used for the intersection points appropriately we may arrange that these two generators have equal areas. Thus V is monotone of minimal Maslov class equal to 3.

Remark 6.3.1. It is possible to compactify M' to a surface of high genus, while still keeping V monotone. This can be done by enlarging the punctures around the points P_i and adding appropriate handles.

REFERENCES

- [Abo] M. Abouzaid. Homogeneous coordinate rings and mirror symmetry for toric varieties. *Geom. Topol.*, 10:1097–1157 (electronic), 2006.
- [Alb] P. Albers. A Lagrangian Piunikhin-Salamon-Schwarz morphism and two comparison homomorphisms in Floer homology. *Int. Math. Res. Not. IMRN*, (4):Art. ID rnm134, 56, 2008.
- [ALP] M. Audin, F. Lalonde, and L. Polterovich. Symplectic rigidity: Lagrangian submanifolds. In M. Audin and J. Lafontaine, editors, *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 271–321, Basel, 1994. Birkhäuser Verlag.
- [Arn1] V. Arnol'd. Lagrange and Legendre cobordisms. I. *Funktsional. Anal. i Prilozhen.*, 14(3):1–13, 96, 1980.
- [Arn2] V. Arnol'd. Lagrange and Legendre cobordisms. II. *Funktsional. Anal. i Prilozhen.*, 14(4):8–17, 95, 1980.

- [Aud] M. Audin. Quelques calculs en cobordisme lagrangien. *Ann. Inst. Fourier (Grenoble)*, 35(3):159–194, 1985.
- [Aur] D. Auroux. Fukaya categories of symmetric products and bordered Heegaard-Floer homology. *J. Gökova Geom. Topol. GGT*, 4:1–54, 2010.
- [BC1] P. Biran and O. Cornea. Lagrangian topology and enumerative geometry. Preprint (2010). Can be found at <http://arxiv.org/pdf/1011.2271>.
- [BC2] P. Biran and O. Cornea. Quantum structures for Lagrangian submanifolds. Preprint (2007). Can be found at <http://arxiv.org/pdf/0708.4221>.
- [BC3] P. Biran and O. Cornea. A Lagrangian quantum homology. In *New perspectives and challenges in symplectic field theory*, volume 49 of *CRM Proc. Lecture Notes*, pages 1–44. Amer. Math. Soc., Providence, RI, 2009.
- [BC4] P. Biran and O. Cornea. Rigidity and uniruling for Lagrangian submanifolds. *Geom. Topol.*, 13(5):2881–2989, 2009.
- [BH] R. Brown and P. Higgins. On the connection between the second relative homotopy groups of some related spaces. *Proc. London Math. Soc. (3)*, 36(2):193–212, 1978.
- [Che] Y. Chekanov. Lagrangian embeddings and Lagrangian cobordism. In *Topics in singularity theory*, volume 180 of *Amer. Math. Soc. Transl. Ser. 2*, pages 13–23. Amer. Math. Soc., Providence, RI, 1997.
- [Eli] Y. Eliashberg. Cobordisme des solutions de relations différentielles. In *South Rhone seminar on geometry, I (Lyon, 1983)*, Travaux en Cours, pages 17–31. Hermann, Paris, 1984.
- [FHS] A. Floer, H. Hofer, and D. Salamon. Transversality in elliptic Morse theory for the symplectic action. *Duke Math. J.*, 80(1):251–292, 1995.
- [FOOO1] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian intersection Floer theory - anomaly and obstruction, chapter 10. Preprint, can be found at <http://www.math.kyoto-u.ac.jp/~fukaya/Chapter10071117.pdf>.
- [FOOO2] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.
- [FOOO3] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.
- [Oh1] Y.-G. Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. *Comm. Pure Appl. Math.*, 46(7):949–993, 1993.
- [Oh2] Y.-G. Oh. Addendum to: "Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I." [Comm. Pure Appl. Math. 46 (1993), no. 7, 949–993]. *Comm. Pure Appl. Math.*, 48(11):1299–1302, 1995.
- [Oh3] Y.-G. Oh. Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings. *Internat. Math. Res. Notices*, 1996(7):305–346, 1996.
- [Oh4] Y.-G. Oh. Floer homology and its continuity for non-compact Lagrangian submanifolds. *Turkish J. Math.*, 25(1):103–124, 2001.
- [Pol] L. Polterovich. The surgery of Lagrange submanifolds. *Geom. Funct. Anal.*, 1(2):198–210, 1991.
- [Sei1] P. Seidel. Graded Lagrangian submanifolds. *Bull. Soc. Math. France*, 128(1):103–149, 2000.

- [Sei2] P. Seidel. A long exact sequence for symplectic floer cohomology. *Topology*, 42(5):1003–1063, 2003.
- [Sei3] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [Wei] A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

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