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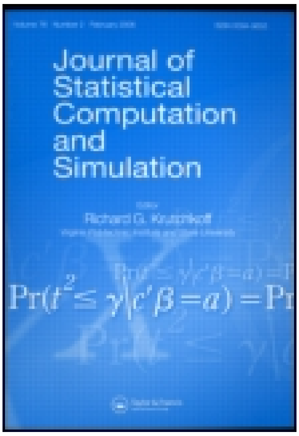

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Inference for a leptokurtic symmetric family of distributions represented by the difference of two gamma variates

Maciej Augustyniak [a] & Louis G. Doray [a]

[a] Département de Mathématiques et de Statistique, Université de Montréal, C.P. 6128, Succursale Centre-ville, Montréal, Québec, Canada, H3C 3J7

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Inference for a leptokurtic symmetric family of distributions represented by the difference of two gamma variates

Maciej Augustyniak* and Louis G. Doray

Département de Mathématiques et de Statistique, Université de Montréal, C.P. 6128, Succursale Centre-ville, Montréal, Québec, Canada H3C 3J7

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We introduce a family of leptokurtic symmetric distributions represented by the difference of two gamma variates. Properties of this family are discussed. The Laplace, sums of Laplace and normal distributions all arise as special cases of this family. We propose a two-step method for fitting data to this family. First, we perform a test of symmetry, and second, we estimate the parameters by minimizing the quadratic distance between the real parts of the empirical and theoretical characteristic functions. The quadratic distance estimator obtained is consistent, robust and asymptotically normally distributed. We develop a statistical test for goodness of fit and introduce a test of normality of the data. A simulation study is provided to illustrate the theory.

Keywords: double gamma difference; gamma distribution; leptokurtic distribution; symmetric distribution; Laplace distribution; empirical characteristic function; quadratic distance; parameter estimation; goodness of fit; normality test; test of symmetry

1. Introduction

We introduce a family of leptokurtic symmetric distributions by presenting its characteristic function. Consider X and Y to be independent and identically distributed random variables from a gamma distribution with parameters of shape $1/\lambda$ and scale $\sqrt{\lambda\theta}$ (i.e. $X, Y \sim \Gamma(1/\lambda, \sqrt{\lambda\theta})$), where λ and θ are defined on the positive real line. The new family is represented by the random variable Z, where $Z = X - Y$ with characteristic function

$$\phi_Z(t) = \phi_X(t)\phi_{-Y}(t) = \left(\frac{1}{1 - \mathrm{i}t\sqrt{\lambda\theta}}\right)^{1/\lambda}\left(\frac{1}{1 + \mathrm{i}t\sqrt{\lambda\theta}}\right)^{1/\lambda} = \left(\frac{1}{1 + t^2\lambda\theta}\right)^{1/\lambda}.$$

In the limit, as $\lambda \to 0$, we have $\phi_Z(t) \to \mathrm{e}^{-t^2\theta}$, which is the characteristic function of a normal random variable centred at 0 and with variance 2θ. Hence, we define this family of symmetric

*Corresponding author. Email: augusty@dms.umontreal.ca

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distributions by its characteristic function

$$\phi(t) = \begin{cases} \left(\dfrac{1}{1+t^2\lambda\theta}\right)^{1/\lambda} & \text{for } \lambda > 0, \theta \geq 0, \\ \mathrm{e}^{-t^2\theta} & \text{for } \lambda = 0, \theta \geq 0. \end{cases} \tag{1}$$

We will use the notation DGD(λ, θ) for the double gamma difference distribution with parameters (λ, θ) and characteristic function given by Equation (1). For $\lambda < 0$ or $\theta < 0, \phi(t)$ is not a characteristic function, because $|\phi(t)|$ is not bounded by 1. Refer to Lukacs [1] for more details on the properties of characteristic functions.

When $\lambda = 1$, the family becomes the difference of two independent exponentially distributed variates with mean $\sqrt{\theta}$. Kotz *et al.* [2] proved that a Laplace random variable centred at the origin can be represented as the difference between two independent exponentially distributed variates; the characteristic function of a classical Laplace random variable centred at 0 with scale parameter s is

$$\frac{1}{1+t^2s^2}.$$

Hence, the classical Laplace distribution is a special case of the DGD family with parameters $(\lambda = 1, \theta = s^2)$. When $\lambda = 1/n, n \in \mathbb{N}$, the difference of two independent gamma variates can be seen as the sum of n differences of two independent exponential variates, which is simply the sum of n independent Laplace variates. In the limit, when $n \to \infty$ (i.e. $\lambda \to 0$), our result is consistent with the central limit theorem, as the sum of n independent Laplace variates converges to the normal distribution. We now list some properties of this family.

Property 1: Odd and even moments The moment-generating function (mgf) of the DGD family can be easily computed by using the mgf of a gamma distribution. Since this mgf always exists in a neighbourhood of 0, all the moments of the DGD family are finite. Moreover, the characteristic function of the DGD family is real and even and, consequently, it is a family of symmetric density functions centred at 0. Thus, the odd moments are 0; the positive even moments can be calculated from the formula in Proposition 1, the proof of which is given in the appendix.

Proposition 1 *Let Z be a DGD*(λ, θ) *random variable with characteristic function* $\phi(t)$ *as defined by Equation* (1). *Then*,

$$\mathrm{E}[Z^{2k}] = \frac{\theta^k (2k)!}{k!} \prod_{j=0}^{k-1} (1 + j\lambda), \quad k = 1, 2, \ldots.$$

Property 2: Kurtosis From Proposition 1, we obtain the variance and the kurtosis of a DGD(λ, θ) random variable, which are 2θ and $(3 + 3\lambda)$, respectively. Kurtosis is defined as the fourth central moment divided by the square of the variance and it is a measure of peakedness of the probability distribution and of heaviness of the tails. Since $\lambda \geq 0$, the kurtosis is always greater or equal to 3. Thus, the family is leptokurtic, because the kurtosis is always at least that of the normal distribution.

Property 3: Closure under transformations Let $Z_1, \ldots, Z_n$ be independent and identically distributed DGD$(n\lambda, \theta/n)$ variates and consider $Z = Z_1 + \ldots + Z_n$, then

$$\phi_Z(t) = \phi_{Z_1}(t) \cdots \phi_{Z_n}(t) = [\phi_{Z_1}(t)]^n = \left(\frac{1}{1+t^2\lambda\theta}\right)^{1/\lambda}. \tag{2}$$

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Clearly, Z is a DGD(λ, θ) random variable. Also, if $a \in \mathbb{R}$, then the characteristic function of aZ is $\phi_{aZ}(t)$, where

$$\phi_{aZ}(t) = \phi_Z(at) = \left(\frac{1}{1 + t^2 a^2 \lambda \theta} \right)^{1/\lambda}.$$

This entails that aZ is a DGD$(\lambda, a^2\theta)$ random variable. Thus, the family is closed under scale and convolution operations but not under the translation operation as the centre of symmetry is fixed at the origin. Moreover, from Equation (2), we can recognize that its characteristic function is infinitely divisible.

When the family reduces to Laplace or normal random variables, the density function can be expressed in a closed form. For example, when $\lambda = 1$, we obtain the classical Laplace random variable centred at 0 with density function equal to

$$f(z; \lambda = 1, \theta = s^2) = \frac{1}{2s} \mathrm{e}^{-|z|/s}.$$

When $\lambda = 1/n, n \in \mathbb{N}$, and $\theta = n$, Kotz *et al.* [2] obtained the density function of this sum of n standard classical Laplace variates. However, in the general case, the density function does not have a closed-form expression.

Since the family consists of symmetric leptokurtic distributions, this suggests that data exhibiting the properties of being symmetric around the origin and of having excess kurtosis can be fitted to this family. In the following section, we develop a two-step method for fitting data to the DGD family. The first step comprises model validation and the second step comprises parameter estimation. In Section 3, goodness of fit tests for the simple and composite hypotheses are presented. The test statistics are shown to follow a chi-square distribution asymptotically. In addition, we explain how the parameter λ can be employed to test for distributional assumptions. More precisely, a test of normality of the data is presented. In Section 4, we provide simulation results for the methods developed.

2. Fitting to the DGD family

2.1. *Introduction*

We suggest a two-step method for fitting data to the DGD family. The first step consists of assessing the compatibility between the data and the family. Since the distributions in the family are symmetric and leptokurtic, the data have to exhibit those characteristics. Deviations from symmetry can be evaluated by performing a test of symmetry, while leptokurtosis implies that the model is only suitable for data with tails that are at least as heavy as the normal distribution (i.e. the sample kurtosis should be greater than 3). Once we have confirmed that the family is well suited for the data, we proceed with parameter estimation, which is the second step. Parameter estimation is achieved through a minimum-distance method based on the characteristic function. We choose the parameters which minimize the distance between the real parts of the theoretical characteristic function and the empirical characteristic function. The estimators obtained are consistent, robust and asymptotically normal.

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2.2. *Testing symmetry*

2.2.1. *Introduction*

Let $x_1, \ldots, x_n$ be n independent observations from a continuous random variable X with distribution function F, density f and known centre μ_0. We consider the problem of testing

$$H_0 : F(\mu_0 - x) = 1 - F(\mu_0 + x) \quad \text{against}$$
$$H_a : F(\mu_0 - x) \neq 1 - F(\mu_0 + x).$$

Thus, we are interested in testing whether the density f is symmetric about the known median μ_0 or skewed.

Many tests of symmetry have been described in the literature [3,4]. McWilliams [5] and Modarres and Gastwirth [6] used tests based on a runs statistic. Tajuddin [7] and Thas *et al*. [8] used tests based on the Wilcoxon signed-rank statistic. Also, Cheng and Balakrishnan [9] proposed a modified sign test for symmetry.

We suggest using the hybrid test proposed by Modarres and Gastwirth [10] to test the hypothesis of symmetry around a known median. We favour this test due to its high power and simplicity. Thas *et al*. [8] performed extensive simulations comparing the power of different tests of symmetry, which revealed that the hybrid test is more powerful than most alternatives considered.

2.2.2. *Hybrid test*

The hybrid test is defined in two stages. Stage I consists of the sign test at level $\alpha_1 < \alpha$. If H_0 is accepted in stage I, then the percentile-modified two-sample Wilcoxon test is performed in stage II at level $\alpha_2 < \alpha$. The hybrid procedure is an α-level test, where $\alpha = \alpha_1 + (1 - \alpha_1)\alpha_2$. Modarres and Gastwirth [10] suggested that α_1 should be small relative to α_2 and proposed that $\alpha_1 = 0.01$ and $\alpha_2 = 0.0404$ yielding an overall level of $\alpha = 0.05$. Refer to Modarres and Gastwirth [10] for a detailed description of the hybrid procedure.

The first step of our method involves validating the compatibility between the data and the DGD family. It consists of two elements: the sample kurtosis should be greater than 3 and the hybrid test must not reject the hypothesis of symmetry around μ_0. If the data qualify, then we can carry on with the second step, parameter estimation. For the DGD family, μ_0 is conveniently set to 0. However, in the particular case where μ_0 is known and $\mu_0 \neq 0$, μ_0 must be subtracted from the data and the shifted data can be fitted. If μ_0 is unknown, our model must be extended by adding a third parameter for location. This will be discussed in Section 5.

2.3. *Parameter estimation*

2.3.1. *Introduction*

We will estimate the parameters through a minimum-distance method based on the characteristic function. There is an extensive literature involving the characteristic function in parameter estimation. For example, it is a widely used method with stable distributions. References include [11–16]. Moreover, Yu [17] showed how techniques relying on the characteristic function are used in mixtures of normal distributions, in the variance gamma distribution, in stable autoregressive moving average (ARMA) processes and in a diffusion model.

Traditionally, the maximum-likelihood approach is widely favoured due to its generality and asymptotic efficiency (see [18,19] or [20] for examples of application of the method to the beta generalized exponential distribution, the three-parameter gamma distribution and the Weibull

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regression). However, the likelihood function is not always tractable, as is the case with stable laws. When this occurs, the characteristic function might be used. Since the empirical characteristic function retains all the information in the sample, estimation and inference via the empirical characteristic function should work as efficiently as the likelihood-based approaches. Feuerverger and McDunnough [12] showed that the asymptotic variance–covariance matrix of the parameters estimated using a minimum-distance method based on the characteristic function can be made arbitrarily close to the Cramér–Rao bound so that the method can attain arbitrarily high asymptotic efficiency. Moreover, the estimators obtained are consistent, robust and asymptotically normally distributed. Feuerverger and McDunnough [21] noted that the robustness properties for procedures associated with the empirical characteristic function are the result of a bounded influence curve for the estimators. For more details on the influence curve, see [22].

2.3.2. *The empirical characteristic function*

Consider $Z_1, \ldots, Z_n$ to be independent and identically distributed observations from the $\mathrm{DGD}(\lambda, \theta)$. Let us define the empirical and theoretical characteristic functions at a specific point t_0 as $\phi_n(t_0)$ and $\phi(t_0)$, respectively, where

$$\phi_n(t_0) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}t_0 Z_j} = \frac{1}{n}\sum_{j=1}^{n} [\cos(t_0 Z_j) + i\sin(t_0 Z_j)]$$

and

$$\phi(t_0) = \left(\frac{1}{1 + t_0^2\lambda\theta}\right)^{1/\lambda}.$$

Thus, $\phi(t_0)$ only has a real part, and let us denote the real part of $\phi_n(t_0)$ as $\phi_n^{\mathrm{Re}}(t_0)$, where

$$\phi_n^{\mathrm{Re}}(t_0) = \frac{1}{n}\sum_{j=1}^{n} \cos(t_0 Z_j). \tag{3}$$

For any fixed t_0, $\phi_n(t_0)$ is an average of bounded independent and identically distributed random variables having mean $\phi(t_0)$ and finite variance. Therefore, it follows by the strong law of large numbers that $\phi_n(t_0)$ converges almost surely to $\phi(t_0)$. Furthermore, Feuerverger and Mureika [23] proved, for fixed $T < \infty$, the convergence of

$$\sup_{|t| \le T} |\phi_n(t) - \phi(t)| \to 0$$

almost surely as $n \to \infty$ and asserted that $\phi_n^{\mathrm{Re}}(t)$ will become uniformly close to $\phi(t)$ when the underlying distribution is symmetric. This implies that the imaginary part of $\phi_n(t)$, denoted by $\phi_n^{\mathrm{Im}}(t)$, is approximately 0 for large n. Thus, any discrepancies observed between $\phi_n^{\mathrm{Im}}(t)$ and 0 will be due to sampling error and consequently $\phi_n^{\mathrm{Im}}(t)$ will not hold any information about the parameters λ and θ. Since we are only fitting data that are symmetric around the origin, we will only consider the real parts of $\phi_n(t)$ and $\phi(t)$ to estimate the parameters, as the imaginary parts will be uninformative.

2.3.3. *Quadratic distance*

The method used is a form of non-linear weighted least-squares (WLS) estimation. It is similar to the $k - L$ procedure introduced by Feuerverger and McDunnough [12], and it is a special distance

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within the class of quadratic distances introduced by Luong and Thompson [24], where a unified theory for estimation and goodness of fit was developed. More precisely, the technique consists in choosing the parameters which minimize the quadratic distance between the real parts of the theoretical characteristic function and the empirical characteristic function. We note that it is not necessary to include the quadratic distance between the imaginary parts in the minimization process, as this expression does not depend on the parameters since the imaginary part of the theoretical characteristic function is equal to 0 for the DGD family.

Let us choose the points $t_1, \ldots, t_k > 0$ and let us define the column vectors

$$\mathbf{Z}_n = [\phi_n^{\mathrm{Re}}(t_1), \ldots, \phi_n^{\mathrm{Re}}(t_k)]'$$
$$\mathbf{Z}(\lambda, \theta) = [\phi(t_1), \ldots, \phi(t_k)]'.$$

The quadratic distance estimator (QDE) based on the characteristic function, denoted by $(\hat{\lambda}, \hat{\theta})$, is defined as the value of (λ, θ) which minimizes the distance

$$d(\lambda, \theta) = [\mathbf{Z}_n - \mathbf{Z}(\lambda, \theta)]'\mathbf{Q}(\lambda, \theta)[\mathbf{Z}_n - \mathbf{Z}(\lambda, \theta)], \tag{4}$$

where $\mathbf{Q}(\lambda, \theta)$ is a positive definite matrix which may depend on (λ, θ). Luong and Thompson [24] showed that an optimal choice of $\mathbf{Q}(\lambda, \theta)$ in the sense of minimizing the norm of the variance–covariance matrix of the estimated parameters is $\mathbf{Q}(\lambda, \theta) = \mathbf{\Sigma}^{-1}(\lambda, \theta)$, where $\mathbf{\Sigma}(\lambda, \theta)$ is the variance–covariance matrix of $\mathbf{Y}_n(\lambda, \theta) = \sqrt{n}[\mathbf{Z}_n - \mathbf{Z}(\lambda, \theta)]$. With this choice of matrix $\mathbf{Q}(\lambda, \theta)$, the QDE will be denoted by $(\hat{\lambda}^*, \hat{\theta}^*)$.

Let $Y_n(t) = \sqrt{n}[\phi_n^{\mathrm{Re}}(t) - \phi(t)]$, then $\mathbf{\Sigma}(\lambda, \theta) = (\sigma_{ij})$ is the $k \times k$ symmetric matrix with elements

$$\sigma_{ij} = \mathrm{Cov}[Y_n(t_i), Y_n(t_j)] = \frac{1}{2}[\phi(t_i + t_j) + \phi(t_i - t_j)] - \phi(t_i)\phi(t_j).$$

This result follows because $E[\cos(tZ)] = \phi(t)$ and

$$E[\cos(tZ)\cos(sZ)] = E\left[\frac{1}{2}(\cos((t+s)Z) + \cos((t-s)Z))\right] = \frac{1}{2}[\phi(t+s) + \phi(t-s)].$$

Since minimization of $d(\lambda, \theta)$ involves the inverse of the matrix $\mathbf{\Sigma}(\lambda, \theta)$, which depends on the parameters, a simpler procedure would be to replace $\mathbf{\Sigma}(\lambda, \theta)$ by a consistent estimate $\hat{\mathbf{\Sigma}}$ and minimize

$$d'(\lambda, \theta) = [\mathbf{Z}_n - \mathbf{Z}(\lambda, \theta)]'\hat{\mathbf{\Sigma}}^{-1}[\mathbf{Z}_n - \mathbf{Z}(\lambda, \theta)]. \tag{5}$$

Let (λ_0, θ_0) be the true value of (λ, θ) and $\mathbf{\Sigma}(\lambda_0, \theta_0) = \mathbf{\Sigma}$, then, if $\hat{\mathbf{\Sigma}} \xrightarrow{\mathcal{P}} \mathbf{\Sigma}$ (i.e. $\hat{\mathbf{\Sigma}}$ is a consistent estimate of $\mathbf{\Sigma}$), Luong and Doray [25,26] asserted that minimization of Equation (4) with $\mathbf{Q}(\lambda, \theta) = \mathbf{\Sigma}^{-1}(\lambda, \theta)$ and Equation (5) yields asymptotically equivalent estimators. For example, $\mathbf{\Sigma}_n^{\mathrm{Re}}$ defined analogously to $\mathbf{\Sigma}$ in terms of $\phi_n^{\mathrm{Re}}(t)$ is a consistent estimate of $\mathbf{\Sigma}$. More precisely, $\mathbf{\Sigma}_n^{\mathrm{Re}} = (a_{ij})$ is the $k \times k$ matrix with elements

$$a_{ij} = \frac{1}{2}[\phi_n^{\mathrm{Re}}(t_i + t_j) + \phi_n^{\mathrm{Re}}(t_i - t_j)] - \phi_n^{\mathrm{Re}}(t_i)\phi_n^{\mathrm{Re}}(t_j).$$

Luong and Doray [25] suggested an iterative procedure to estimate $(\hat{\lambda}^*, \hat{\theta}^*)$. First, we obtain $(\tilde{\lambda}, \tilde{\theta})$ by choosing $\mathbf{Q}(\lambda, \theta) = \mathbf{I}$, the identity matrix. Despite the fact that $(\tilde{\lambda}, \tilde{\theta})$ is less efficient, it can be used to estimate $\mathbf{\Sigma}$, by letting $\hat{\mathbf{\Sigma}} = \mathbf{\Sigma}(\tilde{\lambda}, \tilde{\theta})$. We can then use $\hat{\mathbf{\Sigma}}$ to obtain the first iteration for $(\hat{\lambda}^*, \hat{\theta}^*)$, and this procedure can be repeated with $\mathbf{\Sigma}$ re-estimated at each step; $(\hat{\lambda}^*, \hat{\theta}^*)$ is defined as the convergent vector value of the procedure.

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2.3.4. *Asymptotic properties of the QDE*

From Equation (3), we observe that $\phi_n^{\mathrm{Re}}(t)$ is an average of bounded processes and it follows, by means of the multivariate central limit theorem, that $\mathbf{Y}_n(\lambda_0, \theta_0) = \mathbf{Y}_n$ converges in law to a multivariate normal distribution with zero mean and covariance structure $\mathbf{\Sigma}$. Thus, we have

$$\mathbf{Y}_n = \sqrt{n}[\mathbf{Z}_n - \mathbf{Z}(\lambda_0, \theta_0)] \xrightarrow{\mathcal{D}} N(0, \mathbf{\Sigma}). \tag{6}$$

Let $(\hat{\lambda}^*, \hat{\theta}^*)$ be the estimator obtained by minimizing Equation (4) with $\mathbf{Q}(\lambda, \theta) = \mathbf{\Sigma}^{-1}(\lambda, \theta)$. Under the conditions that $d(\lambda, \theta)$ attains its minimum at an interior point of $\mathbf{\Theta} = \{\lambda, \theta \in \mathbb{R}; \lambda \geq 0, \theta \geq 0\}$ and that $\mathbf{Z}(\lambda, \theta)$ and $\mathbf{Q}(\lambda, \theta)$ are differentiable, the estimator $(\hat{\lambda}^*, \hat{\theta}^*)$ may also be defined implicitly as a root of the two-dimensional system of estimating equations

$$\frac{\partial}{\partial(\lambda, \theta)}\{[\mathbf{Z}_n - \mathbf{Z}(\lambda, \theta)]' \mathbf{\Sigma}^{-1}(\lambda, \theta)[\mathbf{Z}_n - \mathbf{Z}(\lambda, \theta)]\} = 0.$$

Using Lemmas (2.4.2) and (3.4.1) proposed in Luong and Thompson [24], we can conclude that

(i) $(\hat{\lambda}^*, \hat{\theta}^*) \xrightarrow{\mathcal{P}} (\lambda_0, \theta_0)$, that is, $(\hat{\lambda}^*, \hat{\theta}^*)$ is a consistent estimator of (λ_0, θ_0),
(ii) $(\hat{\lambda}^*, \hat{\theta}^*)$ satisfies $(\partial \mathbf{Z}'(\hat{\lambda}^*, \hat{\theta}^*)/\partial(\lambda, \theta))\{\mathbf{\Sigma}^{-1}(\hat{\lambda}^*, \hat{\theta}^*)\mathbf{Y}_n(\hat{\lambda}^*, \hat{\theta}^*)\} + o_p(1) = 0$,
(iii) $\sqrt{n}[(\hat{\lambda}^*, \hat{\theta}^*) - (\lambda_0, \theta_0)] = (\mathbf{S}'\mathbf{\Sigma}^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{\Sigma}^{-1}\mathbf{Y}_n + o_p(1)$,
(iv) $\mathbf{Y}_n(\hat{\lambda}^*, \hat{\theta}^*) = \mathbf{Y}_n - \{\mathbf{S} + o_p(1)\}\sqrt{n}[(\hat{\lambda}^*, \hat{\theta}^*) - (\lambda_0, \theta_0)]$,
(v) $\sqrt{n}[(\hat{\lambda}^*, \hat{\theta}^*) - (\lambda_0, \theta_0)] \xrightarrow{\mathcal{D}} N(0, (\mathbf{S}'\mathbf{\Sigma}^{-1}\mathbf{S})^{-1})$.

The symbol $o_p(1)$ denotes an expression converging to 0 in probability (i.e. $o_p(1) \xrightarrow{\mathcal{P}} 0$), and $\mathbf{S}$ is a matrix of dimension $k \times 2$ defined as

$$\mathbf{S} = \begin{pmatrix} \dfrac{\partial Z_1(\lambda, \theta)}{\partial \lambda} & \dfrac{\partial Z_1(\lambda, \theta)}{\partial \theta} \\ \vdots & \vdots \\ \dfrac{\partial Z_k(\lambda, \theta)}{\partial \lambda} & \dfrac{\partial Z_k \lambda, \theta)}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \phi(t_1)}{\partial \lambda} & \dfrac{\partial \phi(t_1)}{\partial \theta} \\ \vdots & \vdots \\ \dfrac{\partial \phi(t_k)}{\partial \lambda} & \dfrac{\partial \phi(t_k)}{\partial \theta} \end{pmatrix},$$

where

$$\frac{\partial \phi(t)}{\partial \lambda} = \frac{(1 + \lambda\theta t^2)\ln(1 + \lambda\theta t^2) - \lambda\theta t^2}{\lambda^2(1 + \lambda\theta t^2)^{1+1/\lambda}} \quad \text{and} \quad \frac{\partial \phi(t)}{\partial \theta} = -\frac{t^2}{(1 + \lambda\theta t^2)^{1+1/\lambda}},$$

with all quantities being evaluated at (λ_0, θ_0). Thus, the estimator $(\hat{\lambda}^*, \hat{\theta}^*)$ is consistent and asymptotically normally distributed with variance–covariance matrix $(\mathbf{S}'\mathbf{\Sigma}^{-1}\mathbf{S})^{-1}$. The same results hold for the estimator obtained by minimizing Equation (5).

The choice of points $t_1, \ldots, t_k$ affects $(\mathbf{S}'\mathbf{\Sigma}^{-1}\mathbf{S})^{-1}$, and thus, we must choose them with care. Feuerverger and McDunnough [12] showed that by using a sufficiently extensive grid $\{t_i\}_{i=1}^k$, $(\mathbf{S}'\mathbf{\Sigma}^{-1}\mathbf{S})^{-1}$ can be made arbitrarily close to the Cramér–Rao bound. However, by choosing more points, the $k \times k$ matrix $\mathbf{\Sigma}$ can become near singular and computational problems may arise. For our simulation study, we will consider sets of points having the general form

$$\{t_i\}_{i=1}^k = \left\{\frac{Mi}{k}\right\}_{i=1}^k = \left\{\frac{M}{k}, \frac{2M}{k}, \ldots, M\right\}, \tag{7}$$

where M is an arbitrary number. More precisely, we will use values of $M = 0.01, 0.1, 1, 2$ and 3 and examine the effect on our estimation when $k = 5, 10, 20$ or 30. We will determine the choices of points for which the variances of the estimated parameters are a minimum.

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3. Hypothesis testing

3.1. *Goodness of fit*

3.1.1. *Introduction*

Since we built statistics based on a minimum distance between empirical and theoretical parts, it is natural to use them for testing goodness of fit. Luong and Thompson [24] developed a unified theory for estimation and goodness of fit when quadratic distances are employed. They showed that test statistics for goodness of fit follow a chi-square distribution asymptotically. Their results generalize the tests based on the characteristic function proposed by Koutrouvelis [27] and Koutrouvelis and Kellermeir [28]. We now present the test statistics for the simple and composite hypotheses, respectively. The following theorem appearing in Luong and Doray [25] is needed; its proof can be found in Rao [29].

THEOREM 1 *Suppose that the random vector $\mathbf{Y}_n$ of dimension k is $N(0, \mathbf{\Sigma})$ and $\mathbf{Q}$ is any $k \times k$ symmetric positive semi-definite matrix, then the quadratic form $\mathbf{Y}_n'\mathbf{Q}\mathbf{Y}_n$ is chi-square distributed with ν degrees of freedom if $\mathbf{\Sigma Q}$ is idempotent and* $\mathrm{trace}(\mathbf{\Sigma Q}) = \nu$. (*The same result holds asymptotically if $\mathbf{Q}$ is replaced by a consistent estimate $\hat{\mathbf{Q}}$ and $\mathbf{Y}_n \xrightarrow{\mathcal{D}} N(0, \mathbf{\Sigma})$*).

3.1.2. *Simple hypothesis*

To test the simple hypothesis $H_0 : Z_1, \ldots, Z_n$ come from a specified DGD distribution with parameters (λ_0, θ_0), the following test statistic can be used:

$$\begin{aligned} nd(\lambda_0, \theta_0) &= n[\mathbf{Z}_n - \mathbf{Z}(\lambda_0, \theta_0)]'\mathbf{\Sigma}_0^{-1}[\mathbf{Z}_n - \mathbf{Z}(\lambda_0, \theta_0)] \\ &= \mathbf{Y}_n'\mathbf{\Sigma}_0^{-1}\mathbf{Y}_n, \end{aligned}$$

where $\mathbf{\Sigma}_0$ equals $\mathbf{\Sigma}$ evaluated at (λ_0, θ_0). It follows from Equation (6) and Theorem 1 that $nd(\lambda_0, \theta_0) \xrightarrow{\mathcal{D}} \chi^2_\nu$, where

$$\nu = \mathrm{trace}(\mathbf{\Sigma}\mathbf{\Sigma}^{-1}) = \mathrm{trace}(\mathbf{I}_k) = k,$$

and $\mathbf{I}_k$ is the $k \times k$ identity matrix. Thus, the test statistic follows a limiting chi-square distribution with $\nu = k$ degrees of freedom under H_0. To test the hypothesis H_0 at significance level α, compute the value of the test statistic $nd(\lambda_0, \theta_0)$ from the sample. The null hypothesis H_0 should be rejected if $nd(\lambda_0, \theta_0) > \chi^2_{k,1-\alpha}$, where $\chi^2_{k,1-\alpha}$ is the $100(1-\alpha)$th quantile of a chi-square distribution with k degrees of freedom.

3.1.3. *Composite hypothesis*

To test the composite hypothesis $H_0 : Z_1, \ldots, Z_n$ come from a DGD distribution where the values of the parameters are not specified, we first calculate the QDE $(\hat{\lambda}^*, \hat{\theta}^*)$ by minimizing Equation (4) with $\mathbf{Q}(\lambda, \theta) = \mathbf{\Sigma}^{-1}(\lambda, \theta)$. Luong and Thompson [24] showed that the test statistic

$$\begin{aligned} nd(\hat{\lambda}^*, \hat{\theta}^*) &= n[\mathbf{Z}_n - \mathbf{Z}(\hat{\lambda}^*, \hat{\theta}^*)]'\mathbf{\Sigma}^{-1}(\hat{\lambda}^*, \hat{\theta}^*)[\mathbf{Z}_n - \mathbf{Z}(\hat{\lambda}^*, \hat{\theta}^*)] \\ &= \mathbf{Y}_n'(\hat{\lambda}^*, \hat{\theta}^*)\mathbf{\Sigma}^{-1}(\hat{\lambda}^*, \hat{\theta}^*)\mathbf{Y}_n(\hat{\lambda}^*, \hat{\theta}^*) \end{aligned}$$

follows an asymptotic chi-square distribution with $\nu = k - 2$ degrees of freedom under H_0. Again, $\mathbf{\Sigma}(\hat{\lambda}^*, \hat{\theta}^*)$ can be replaced by a consistent estimate $\hat{\mathbf{\Sigma}}$. Analogous to the case for the simple null hypothesis, a significance level α test can be performed to test H_0.

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3.2. *Test of normality*

In Section 2.3.4, we showed that the estimator $(\hat{\lambda}^*, \hat{\theta}^*)$ is asymptotically normally distributed with variance–covariance matrix $(\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S})^{-1}$. Thus, we can easily construct individual and joint $(1-\alpha)\%$ confidence intervals for the parameters λ and θ.

Of more practical interest is testing for the parameter λ. In Section 1, we saw that particular values of λ define specific distributions within the DGD distribution family. For example, when $\lambda = 0$ or $\lambda = 1$, we obtain the normal and the Laplace distributions, respectively. This suggests using the parameter λ to test distributional assumptions. A test of normality of the data can be constructed by testing

$$H_0 : \lambda = 0 \quad \text{versus} \quad H_a : \lambda > 0. \tag{8}$$

In Section 1, we noted that the kurtosis of a DGD(λ, θ) random variable Z is $(3 + 3\lambda)$. Thus, if we have a sample from Z, $\hat{\beta_2} = (3 + 3\hat{\lambda}^*)$ is a consistent estimate of the population kurtosis β_2. Moreover, since β_2 is a linear function of λ, the hypotheses identified in Equation (8) are equivalent to

$$H_0 : \beta_2 = 3 \quad \text{versus} \quad H_a : \beta_2 > 3.$$

This implies that in Equation (8) we are testing the normal distribution against symmetric distributions with heavier tails. Thus, it would be interesting to compare the power of this test to that of a normality test based on the sample kurtosis. D'Agostino and Pearson [30] described such a test. Moreover, when the alternative is the Laplace distribution, the power of the test can be compared to that of the likelihood ratio test. Kotz *et al.* [2] asserted that the likelihood ratio test is the most powerful scale-invariant test for testing the normal against the Laplace when the centre of symmetry is known. In Section 4, we provide a simulation study for estimating parameters and testing hypotheses with the methods presented previously.

4. Simulation study

4.1. *Parameter estimation*

While the expressions for QDEs may seem complex, they are relatively simple to implement using a computer software with built-in statistical functions. The QDE can be computed numerically using a non-linear least-squares method. All our simulations were completed using Maple 11.0.

We first generated 100 random samples from a DGD$(\lambda = 1, \theta = 1)$ random variable of sizes 100, 500 and 1000. For each sample, we estimated the parameters using the method of moments (MOM), ordinary least squares (OLS) (i.e. using Equation (4) with $\mathbf{Q}(\lambda, \theta) = \mathbf{I}$, the identity matrix) and WLS (i.e. using Equation (4) with an appropriate choice of $\mathbf{Q}(\lambda, \theta)$). The OLS and WLS methods were implemented using 20 different sets of points, $\{t_i\}_{i=1}^k$, in order to determine which are the best choices. All the sets have the general form defined by Equation (7). Values of $M = 0.01, 0.1, 1, 2$ and 3 and values of $k = 5, 10, 20$ and 30 were used to define $\{t_i\}_{i=1}^k$.

Tables 1–3 summarize the pertinent results for samples sizes of 100, 500 and 1000, respectively. Each table provides the mean and the standard error based on 100 random samples of the estimated values of λ and θ using the MOM, OLS and WLS. The WLS estimates were obtained using the iterative procedure to estimate $\boldsymbol{\Sigma}$ presented in Section 2.3.3. The results for the values of $M = 0.01$ and 0.1 are not presented as the WLS method rarely found an improved estimate over the OLS method. Consequently, we do not recommend using those choices of M. All the other values of M yielded good estimates, but we suggest using $M = 3$, as the standard errors of the estimates were generally the lowest for this choice. Moreover, increasing the value of k (i.e. increasing the number of points in the sets) generally improved the estimates. However, when using $k = 30$, the

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Table 1. Estimates based on 100 random samples of size 100.

		λ		θ	
True values		1		1	
MOM (s.e.)		0.6853 (0.4099)		1.0287 (0.2184)	
k	M	OLS (s.e.)	WLS (s.e.)	OLS (s.e.)	WLS (s.e.)
5	1	1.1111 (0.5926)	1.0877 (0.5026)	1.0714 (0.2388)	1.0465 (0.2221)
5	2	1.0642 (0.4332)	1.0434 (0.3926)	1.0622 (0.2592)	1.0477 (0.2232)
5	3	1.0371 (0.4332)	1.0684 (0.3477)	1.0563 (0.2928)	1.0697 (0.2358)
10	1	1.1038 (0.6062)	1.0642 (0.4280)	1.0698 (0.2388)	1.0446 (0.2204)
10	2	1.0688 (0.4333)	1.0314 (0.3707)	1.0647 (0.2610)	1.0415 (0.2157)
10	3	1.0342 (0.4362)	1.0270 (0.3414)	1.0556 (0.2913)	1.0419 (0.2185)
20	1	1.0990 (0.6142)	1.0298 (0.3974)	1.0688 (0.2389)	1.0418 (0.2203)
20	2	1.0705 (0.4323)	1.0264 (0.3602)	1.0653 (0.2606)	1.0410 (0.2159)
20	3	1.0340 (0.4363)	1.0297 (0.3387)	1.0550 (0.2882)	1.0421 (0.2180)

Table 2. Estimates based on 100 random samples of size 500.

		λ		θ	
True values		1		1	
MOM (s.e.)		0.8841 (0.2745)		1.0112 (0.0991)	
k	M	OLS (s.e.)	WLS (s.e.)	OLS (s.e.)	WLS (s.e.)
5	1	1.0441 (0.2449)	1.0161 (0.2194)	1.0232 (0.1064)	1.0151 (0.1007)
5	2	1.0083 (0.2329)	1.0221 (0.2024)	1.0153 (0.1185)	1.0184 (0.1003)
5	3	1.0294 (0.2095)	1.0314 (0.1981)	1.0263 (0.1260)	1.0244 (0.1108)
10	1	1.0439 (0.2470)	1.0030 (0.2051)	1.0231 (0.1058)	1.0140 (0.1003)
10	2	1.0072 (0.2337)	1.0124 (0.1923)	1.0151 (0.1188)	1.0150 (0.0988)
10	3	1.0255 (0.2132)	1.0198 (0.1841)	1.0244 (0.1268)	1.0163 (0.1006)
20	1	1.0437 (0.2485)	1.0025 (0.1997)	1.0230 (0.1054)	1.0139 (0.0994)
20	2	1.0065 (0.2337)	1.0117 (0.1877)	1.0148 (0.1187)	1.0149 (0.0990)
20	3	1.0240 (0.2145)	1.0188 (0.1780)	1.0236 (0.1259)	1.0160 (0.1004)

Table 3. Estimates based on 100 random samples of size 1000.

		λ		θ	
True values		1		1	
MOM (s.e.)		0.9428 (0.2606)		1.0106 (0.0771)	
k	M	OLS (s.e.)	WLS (s.e.)	OLS (s.e.)	WLS (s.e.)
5	1	1.0197 (0.1816)	1.0058 (0.1662)	1.0143 (0.0791)	1.0116 (0.0769)
5	2	1.0180 (0.1664)	1.0173 (0.1395)	1.0162 (0.0979)	1.0144 (0.0779)
5	3	1.0017 (0.1635)	1.0053 (0.1379)	1.0088 (0.0974)	1.0121 (0.0801)
10	1	1.0205 (0.1835)	1.0094 (0.1544)	1.0144 (0.0788)	1.0119 (0.0773)
10	2	1.0190 (0.1655)	1.0087 (0.1367)	1.0166 (0.0966)	1.0118 (0.0771)
10	3	1.0037 (0.1680)	1.0032 (0.1297)	1.0099 (0.1016)	1.0111 (0.0763)
20	1	1.0209 (0.1848)	1.0176 (0.1458)	1.0145 (0.0786)	1.0128 (0.0773)
20	2	1.0192 (0.1652)	1.0106 (0.1335)	1.0166 (0.0958)	1.0120 (0.0768)
20	3	1.0051 (0.1686)	1.0021 (0.1284)	1.0107 (0.1019)	1.0108 (0.0762)

estimation process was slow and the improvement over $k = 10$ or 20 was not substantial and not worthy of the additional computation time. Thus, we suggest using values of $k = 10$ or 20 for a fast and efficient estimation. Moreover, the asymptotic standard deviations of the estimators that

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can be calculated from the results given in Section 2.3.4 are very close to the standard errors that were observed in our samples.

We also performed WLS estimation with choices of $\mathbf{Q}(\lambda, \theta) = (\boldsymbol{\Sigma}_n^{\mathrm{Re}})^{-1}$ and $\boldsymbol{\Sigma}_0^{-1}$. With $(\boldsymbol{\Sigma}_n^{\mathrm{Re}})^{-1}$, we obtained poor estimates, and they often did not converge to a solution. The choice of $\boldsymbol{\Sigma}_0^{-1}$ produced estimates that were comparable to the ones obtained in the tables under the WLS columns. However, the choice of $\boldsymbol{\Sigma}_0^{-1}$ is not a viable selection in practice as the true parameters (λ_0, θ_0) are unknown.

4.2. *Goodness of fit testing*

We performed goodness of fit testing for the simple hypothesis as presented in Section 3.1.2. First, we wish to determine if the test has a correct size when using critical values (CVs) from the chi-square distribution for sample sizes of $n = 100, 500$ and 1000. For each sample size n, we generated 5000 samples from a DGD$(\lambda = 1, \theta = 1)$ random variable and calculated the test statistics $nd(\lambda_0 = 1, \theta_0 = 1)$. We repeated the procedure for samples from a DGD$(\lambda = 2, \theta = 1)$. We were thus able to obtain simulated CVs for a level α test by taking the $100(1-\alpha)$th quantiles from the empirical distributions of the test statistics. All the test statistics were obtained using the set of points defined by Equation (7) with values of $M = 3$ and $k = 10$. We present our results in Tables 4 and 5.

The results given in Table 4 indicate that the goodness of fit test has an incorrect size that is severe enough to warrant a recommendation that the test should not be used without appropriately sized CVs. Based on Table 5, we remark that the test statistic $nd(\lambda_0, \theta_0)$ converges very slowly to a chi-square random variable. Even for sample sizes of 1000, the approximation is not satisfactory. The real distribution of the test statistic will generally have a heavier right tail than the chi-square

Table 4. Actual sizes of the test using $\chi^2_{10,1-\alpha}$ with 5000 simulation runs.

		Actual sizes of the test		
α	(λ_0, θ_0)	$n = 100$	$n = 500$	$n = 1000$
0.100	(1,1)	0.1250	0.1242	0.1168
	(2,1)	0.1420	0.1350	0.1194
0.050	(1,1)	0.0966	0.0834	0.0764
	(2,1)	0.1138	0.0898	0.0738
0.025	(1,1)	0.0798	0.0594	0.0544
	(2,1)	0.0930	0.0628	0.0448
0.010	(1,1)	0.0658	0.0406	0.0310
	(2,1)	0.0756	0.0412	0.0264

Table 5. CVs obtained for various sample sizes.

		CVs			
α	(λ_0, θ_0)	$n = 100$	$n = 500$	$n = 1000$	$\chi^2_{10,1-\alpha}$
0.100	(1,1)	17.9453	17.1660	16.8809	15.9872
	(2,1)	19.6814	17.5147	16.9050	15.9872
0.050	(1,1)	27.2592	21.5835	20.9238	18.3070
	(2,1)	27.6104	21.9962	20.0182	18.3070
0.025	(1,1)	37.0803	27.4926	24.4911	20.4832
	(2,1)	37.6163	25.9949	23.5219	20.4832
0.010	(1,1)	68.5516	39.5960	29.3234	23.2093
	(2,1)	58.1972	32.5478	28.3944	23.2093

Table 6. Power of the test ($\alpha = 0.05$) with 5000 simulation runs.

		Alternatives (λ_a, θ_a)				
n	CV	$(0, 1)$	$(0.5, 1)$	$(1, 1)$	$(1.5, 1)$	$(2, 1)$
100	27.4348	0.0626	0.0164	0.0518	0.1070	0.1852
500	21.7899	0.9986	0.2692	0.0516	0.3656	0.9394
1000	20.4710	1.0000	0.8000	0.0572	0.7524	1.0000

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distribution, even for large sample sizes, and thus the test will always be oversized when using the CV $\chi^2_{k,1-\alpha}$.

Next, we assessed the power of the goodness of fit test for the simple hypothesis $H_0 : (\lambda_0 = 1, \theta_0 = 1)$ against alternatives $H_a : (\lambda_a, \theta_a = 1)$, where $\lambda_a = 0, 0.5, 1, 1.5$ and 2. A level $\alpha = 0.05$ and sample sizes of $100, 500$ and 1000 were employed. We determined the power of the test by generating 5000 samples for each of the alternatives considered. Appropriately sized CVs were calculated by taking the average of the two CVs obtained in Table 5 for each sample size. The results are given in Table 6.

The goodness of fit test performed poorly in rejecting the selected alternatives for a sample size of 100. When $n = 500$, the test did very well for alternatives of λ_a one unit away of $\lambda_0 = 1$ but not so well when λ_a was half a unit away. For a large sample size of 1000, the test was powerful for all alternatives considered. For sample sizes of 500 and 1000, the recorded powers for alternatives $(0, 1)$ and $(2, 1)$ were close to or equal to 100%. This suggests that the test is well suited for discriminating between the fits of normal, Laplace and heavier tailed symmetric distributions for a large enough sample size. Moreover, by using adjusted CVs instead of $\chi^2_{k,1-\alpha}$, the tests had an adequate size. The discrepancies between the actual sizes and $\alpha = 0.05$ are due to the precision of the simulated CVs and to the large variability of the test statistic.

5. Conclusion

We have introduced the double gamma difference family, which is a family of leptokurtic symmetric distributions. The Laplace, the sums of Laplace and the normal distributions all arise as special cases of this family. While there is no general closed-form expression for the density function, the characteristic function is simple to work with. Parameters can be estimated through a minimum quadratic distance method based on the characteristic function. The estimators obtained were shown to be consistent, robust and asymptotically normally distributed. Goodness of fit tests for the simple and composite hypotheses were presented and the test statistics were shown to follow a chi-square distribution asymptotically. Moreover, we suggested employing the parameter λ to test for distributional assumptions. Simulations revealed that large sample sizes are required to get a reasonable amount of precision for estimating the parameters. Also, the goodness of fit tests must be carried out with appropriate simulated CVs for the tests to have a correct size because the convergence to the chi-square distribution is slow.

The family can be extended by adding a third parameter for location μ. The characteristic function $\phi^*(t)$ would then both have real and imaginary parts, where

$$\phi^*(t) = \begin{cases} \mathrm{e}^{\mathrm{i}t\mu} \cdot \left(\dfrac{1}{1 + t^2\lambda\theta} \right)^{1/\lambda} & \text{for } \lambda > 0, \theta \geq 0, \\ \mathrm{e}^{\mathrm{i}t\mu - t^2\theta} & \text{for } \lambda = 0, \theta \geq 0. \end{cases}$$

Parameter estimation could still be achieved through a minimum-distance method based on the characteristic function. However, both real and imaginary parts would have to be taken into account. For more details on the minimum-distance method when the real and imaginary parts are involved, see [21]. Before fitting data to this family, it is still necessary to verify symmetry. For testing symmetry around the unknown median μ, we suggest using the triples test introduced by Randles *et al*. [31].

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Appendix

Proof of Proposition 1 From the generalized binomial theorem, we obtain the binomial series for $\phi(t)$, where

$$\phi(t) = (1 + t^2\lambda\theta)^{-1/\lambda} = \sum_{k=0}^{\infty} \binom{-1/\lambda}{k} (t^2\lambda\theta)^k.$$

From the relationship between moments of a random variable and the derivatives of its characteristic function, we have

$$\mathrm{E}[Z^{2k}] = i^{2k} \left. \frac{\mathrm{d}^{2k}}{\mathrm{d}t^{2k}} \phi(t) \right|_{t=0} = (-1)^k \phi^{(2k)}(0).$$

$\phi^{(2k)}(0)$ corresponds to the $(k+1)$th term from the binomial series for $\phi(t)$ differentiated $2k$ times. Thus,

$$\mathrm{E}[Z^{2k}] = (-1)^k \binom{-1/\lambda}{k} (2k)!\lambda^k\theta^k.$$

Since the binomial coefficients admit the representation

$$\binom{-1/\lambda}{k} = \frac{1}{k!}\prod_{j=0}^{k-1}\left(-\frac{1}{\lambda} - j\right) = \frac{(-1)^k}{k!\lambda^k}\prod_{j=0}^{k-1}(1 + j\lambda),$$

we get the following expression:

$$\mathrm{E}[Z^{2k}] = (-1)^k (2k)!\lambda^k\theta^k \left[\frac{(-1)^k}{k!\lambda^k} \prod_{j=0}^{k-1}(1 + j\lambda) \right] = \frac{\theta^k (2k)!}{k!} \prod_{j=0}^{k-1}(1 + j\lambda).$$

We note here that the proof only applies for values of $\lambda > 0$. However, the same result holds for $\lambda = 0$. The expression for $\lambda = 0$ can be derived similarly by using the series expansion for $\mathrm{e}^{-t^2\theta}$. ■