EFFICIENT ESTIMATORS FOR THE GOOD FAMILY

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ABSTRACT

We consider the problem of estimating the two parameters of the discrete Good distribution. We first show that the sufficient statistics for the parameters are the arithmetic and the geometric means. The maximum likelihood estimators (MLE's) of the parameters are obtained by solving numerically a system of equations involving the Lerch zeta function and the sufficient statistics. We find an expression for the asymptotic variance-covariance matrix of the MLE's, which can be evaluated numerically. We show that the probability mass function satisfies a simple recurrence equation linear in the two parameters, and propose the quadratic distance estimator (QDE) which can be computed with an iteratively reweighted least-squares algorithm. The QDE is easy to calculate and admits a simple expression for its asymptotic variance-covariance matrix. We compute this matrix for the MLE's and the QDE for various values of the parameters and see that the QDE has very high asymptotic efficiency. Finally, we present a numerical example.

1 INTRODUCTION

There has been increased attention in the last few years toward the Good distribution with probability mass function (pmf)

$$p_i = \frac{q^i i^a}{\sum_{i=1}^{\infty} q^i i^a}, \quad i = 1, 2, \ldots, \quad 0 < q < 1, \quad a \epsilon \Re,$$

(see Kulasekera and Tonkyn (1992)). If q equals 1, we obtain the zeta distribution; in this case, the parameter $a < -1$. The discrete Good distribution, introduced by Good (1953), is a special case of the Lerch distribution (see Zörnig and Altman (1995)) which has the pmf

$$p_i = \frac{q^i}{\Phi(q, b, c) \times (b + i)^c}, \quad i = 1, 2, \ldots, \quad b \neq -1, -2, \ldots,$$

with the parameter b set equal to 0 and $c = -a$ and where $\Phi(q, b, c)$ is the Lerch zeta function (see Gradshteyn and Ryzhik (1980), p. 1075) defined by

$$\Phi(q, b, c) = \sum_{i=1}^{\infty} q^i/(b + i)^c, \quad b \neq -1, -2, \ldots.$$

In this paper, we will use the following parametrization

$$p_i = \frac{e^{\alpha i} i^{\beta}}{\sum_{i=1}^{\infty} e^{\alpha i} i^{\beta}}, \quad i = 1, 2, \ldots, \quad \alpha < 0, \quad \beta \epsilon \Re, \qquad (1)$$

obtained by letting $q = e^{\alpha}$ and $a = \beta$. We should note that this pmf is similar to the probability density function of the gamma distribution $\Gamma(\alpha, \beta)$, $f(x) \propto x^{\alpha-1} e^{-\beta x}$, evaluated at positive integers; however, for a $\Gamma(\alpha, \beta)$ distribution, both parameters must be positive.

The Good distribution has been used in ecology to model the distribution of certain species of animals, birds or trees in territories as well as the occurences of words in linguistics.

Kulasekera and Tonkyn (1992) found that it is a very flexible family since it can be used to represent processes which have a monotone increasing, constant or decreasing hazard rate function, depending on the sign of β. It can also describe counts that are overdispersed or underdispersed (relation between the variance and the mean), with potential applications in actuarial science, depending on the relative value of the parameters.

Despite this flexibility, use of the Good distribution has remained rather limited in practical applications, probably because of the difficulty in estimating the parameters of the model by traditional maximum likelihood. Some ad hoc estimation methods have been proposed by Kulasekera and Tonkyn (1992). However, the asymptotic properties (bias, variance) of those estimators have not been investigated. In this paper, we develop a quadratic distance estimator for the two parameters of the Good distribution, easy to compute, asymptotically unbiased and with high asymptotic efficiency.

The paper is organized as follows. In section 2, we look at some properties of the Good distribution; in particular the sufficient statistics for the parameters are equal to the arithmetic and the geometric means. We then show that the MLE's are the solution of a system of equations involving the Lerch zeta function and the sufficient statistics (section 3); we also find the variance-covariance matrix of the MLE's. In section 4, we use the fact that the probability mass function satisfies a simple recurrence equation linear in the two parameters and propose an iteratively reweighted least-squares algorithm to obtain the quadratic distance estimator (QDE). The QDE is easy to calculate and admits a simple expression for its asymptotic variance-covariance matrix. We compute it for the MLE's and the QDE for various values of the parameters (section 5); the QDE has very high asymptotic efficiency for the usual range of parameter values of interest. Finally, we present a real data set, which can be modelled with the Good distribution and estimate the parameters of the model and their variance-covariance matrix with both methods.

2 PROPERTIES

For an exhaustive review of the properties of the Good distribution, the reader is referred to Kulasekera and Tonkyn (1992). In addition, it is easily shown that it is a member of the exponential family.

Let $X_1, \ldots, X_n$ be i.i.d. random variables from the distribution with pmf (1), and $x_1, \ldots, x_n$ be the observed values. The likelihood function equals

$$L(\alpha, \beta) = [\Phi(e^{\alpha}, 0, -\beta)]^{-n} \exp[\alpha \sum_{j=1}^{n} x_j + \beta \ln \prod_{j=1}^{n} x_j],$$

so that $(\bar{X}, \tilde{X})$ is a joint sufficient statistic for the parameters (α, β), where $\bar{X}$ is the sample arithmetic mean and $\tilde{X}$, the sample geometric mean. Because the Good distribution is a member of the exponential family, $(\bar{X}, \tilde{X})$ is also a complete statistic. Kulasekera and Tonkyn (1992) remarked that, for fixed β, the distribution belongs to the family of power series distributions.

Siromoney (1964) considered the general Dirichlet's series distribution with pmf

$$p_i = \frac{a_i \exp(-\lambda_i \theta)}{\sum_{i=1}^{\infty} a_i \exp(-\lambda_i \theta)}, \quad i = 1, 2, \ldots$$

and derived its entropy and moments. The Good distribution is a member of this family with $a_i = i^{\beta}$ and $\lambda_i = -i$.

3 MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimators $(\hat{\alpha}, \hat{\beta})$ of the parameters (α, β) maximize the log-likelihood function

$$l(\alpha, \beta) = -n \ln \Phi(e^{\alpha}, 0, -\beta) + \alpha n \bar{X} + \beta n \ln \tilde{X},$$

so that $(\hat{\alpha}, \hat{\beta})$ are the solution of the system of equations

$$\bar{X} = \frac{\partial}{\partial \alpha} \ln \Phi(e^{\alpha}, 0, -\beta) = \frac{\Phi(e^{\alpha}, 0, -(\beta + 1))}{\Phi(e^{\alpha}, 0, -\beta)}$$

$$\ln \tilde{X} = \frac{\partial}{\partial \beta} \ln \Phi(e^\alpha, 0, -\beta) = \frac{\sum_{i=1}^{\infty} e^{\alpha i} i^\beta \ln i}{\Phi(e^\alpha, 0, -\beta)}$$

Those equations must be solved iteratively for α and β. It is sometimes difficult to solve this system, even with a powerful symbolic programming language like MATHEMATICA. Good starting values are required. The choice of initial values for the iterative algorithm is discussed further in section 6, where we fit the Good distribution to one data set. With another data set however, we had difficulty obtaining the MLE's.

The asymptotic variance-covariance matrix of $(\hat{\alpha}, \hat{\beta})$ is equal to $I_{(\alpha,\beta)}^{-1}/n$ where $I_{(\alpha,\beta)}$, the Fisher information matrix, is easily found to be

$$I_{(\alpha,\beta)} = -E(\frac{\partial^2 \ln p_i}{\partial\alpha\partial\beta}) = \frac{\partial^2}{\partial\alpha\partial\beta} \ln \Phi(e^\alpha, 0, -\beta)$$

$$= [\Phi(e^\alpha, 0, -\beta)]^{-2} \begin{pmatrix} i_{11} & i_{12} \\ i_{12} & i_{22} \end{pmatrix},$$

with $i_{11} = \Phi(e^\alpha, 0, -\beta - 2)\Phi(e^\alpha, 0, -\beta) - (\Phi(e^\alpha, 0, -\beta - 1))^2$,
$i_{12} = \Phi(e^\alpha, 0, -\beta) \sum_{i=1}^{\infty} e^{\alpha i} i^{\beta+1} \ln i - \Phi(e^\alpha, 0, -\beta - 1) \sum_{i=1}^{\infty} e^{\alpha i} i^\beta \ln i$ and
$i_{22} = \Phi(e^\alpha, 0, -\beta) \sum_{i=1}^{\infty} e^{\alpha i} i^\beta (\ln i)^2 - (\sum_{i=1}^{\infty} e^{\alpha i} i^\beta \ln i)^2$.
Note that with the Good distribution, the observed and expected information matrix are equal, since the second derivative matrix of $\ln p_i$ with respect to the parameters, depending only on α and β, is a matrix of constants.

4 QUADRATIC DISTANCE ESTIMATION

Kulasekera and Tonkyn (1992) remarked that

$$\ln \frac{p_{i+1}}{p_i} = \ln q + \beta \ln \frac{i+1}{i}$$

and proposed to estimate β and $\ln q$ by a simple ordinary least-squares method after estimating p_k by the observed relative frequency of k's in the sample, f_k/n. They have suggested to exclude classes with a few observations, because

$\ln(f_{k+1}/f_k)$ is sensitive to small changes in the observed frequencies, when k is large. Here, we propose the QDE based on an iteratively reweighted least-squares method. The least-squares estimator of Kulasekera and Tonkyn can be viewed as a special case of the QDE. We give the properties of the estimator so obtained, and, in section 5, compare the asymptotic variance-covariance matrix of this estimator of (α, β) with that of the MLE's.

The following model, linear in α and β, will be used

$$\ln \frac{f_{i+1}}{f_i} = \alpha + \beta \ln \frac{i+1}{i} + \epsilon_i, \quad i = 1, \ldots, k,$$

where ϵ_i is a random error, $f_1, \ldots, f_{k+1}$ are assumed different from zero and f_{k+2} equals 0.

The model can be rewritten in matrix form as

$$Y = X\theta + \epsilon,$$

where

$$Y_{k\times 1} = \left(\ln \frac{f_2}{f_1}, \ldots, \ln \frac{f_{k+1}}{f_k} \right)',$$

$$X_{k\times 2} = \begin{pmatrix} 1 & \ldots & 1 \\ \ln 2 & \ldots & \ln(k+1)/k \end{pmatrix}',$$

$$\theta = (\alpha, \beta)'$$

$$\text{and } \epsilon_{k\times 1} = (\epsilon_1, \ldots, \epsilon_k)'.$$

Doray and Luong (1995) used a similar model for estimating the parameter of the zeta distribution, which is a special case of the Good distribution with α set equal to 0. We will show that asymptotically, we have the following two results:

1. $E(\epsilon_i) = 0, \quad i = 1, \ldots, k.$

2. $E(\epsilon\epsilon') = \Sigma = \frac{1}{n}\begin{pmatrix} \frac{p_1+p_2}{p_1p_2} & -\frac{1}{p_2} & 0 & 0 & \dots & 0 \\ -\frac{1}{p_2} & \frac{p_2+p_3}{p_2p_3} & -\frac{1}{p_3} & 0 & \dots & 0 \\ 0 & -\frac{1}{p_3} & \frac{p_3+p_4}{p_3p_4} & -\frac{1}{p_4} & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & -\frac{1}{p_{k-1}} & \frac{p_{k-1}+p_k}{p_{k-1}p_k} & -\frac{1}{p_k} \\ 0 & 0 & \dots & 0 & -\frac{1}{p_k} & \frac{p_k+p_{k+1}}{p_kp_{k+1}} \end{pmatrix} = \frac{1}{n}\Sigma^*.$

Proof: As $n \to \infty$, $\ln(f_{i+1}/f_i)$ converges in probability to $\ln(p_{i+1}/p_i)$ which equals $\alpha + \beta\ln(i+1)/i$, yielding the first result.

Let Σ be the variance-covariance matrix of the vector ϵ. Then,
$\text{Var}(\epsilon_i)$= $\text{Var}(\ln\frac{f_{i+1}/n}{f_i/n}) = \text{Var}(\ln f_{i+1}/n) + \text{Var}(\ln f_i/n) - 2\text{Cov}(\ln f_{i+1}/n, \ln f_i/n)$.
Since $f_i \sim \text{Bin}(n, p_i)$ and $(f_i, f_j) \sim$ Trinomial (n, p_i, p_j), $i \neq j$, the approximate variance

$$\text{Var}(\ln f_i/n) = [\frac{d\ln x}{dx}\ |_{x=E(f_i/n)}]^2 \times \text{Var}(f_i/n)$$
$$= (1/p_i)^2 \times (1/n^2)np_i(1-p_i) = \frac{1-p_i}{np_i},$$

while the approximate covariance

$$\text{Cov}(\ln f_i/n, \ln f_j/n) = [\frac{d\ln x}{dx}\ |_{x=E(f_i/n)}] \times [\frac{d\ln y}{dy}\ |_{y=E(f_j/n)}] \times \text{Cov}(f_i/n, f_j/n)$$
$$= \frac{1}{p_i}\frac{1}{p_j}\frac{-np_ip_j}{n^2} = -1/n.$$

Simplifying, we obtain $\text{Var}(\epsilon_i) = \frac{p_i+p_{i+1}}{np_ip_{i+1}}$. Similarly, it can be shown that asymptotically, $\text{Cov}(\epsilon_i, \epsilon_{i+1}) = \frac{-1}{np_{i+1}}$ and $\text{Cov}(\epsilon_i, \epsilon_j)$=0 if $|i-j| > 1$, so that $\Sigma = E(\epsilon\epsilon')$ is the above tridiagonal matrix.

The quadratic distance estimator (QDE) of the vector parameter θ, denoted by $\tilde{\theta}$, is obtained by minimizing the quadratic form

$$[Y - X\theta]'\Sigma^{-1}[Y - X\theta].$$

Explicitly, $\tilde{\theta}$ can be expressed as

$$\tilde{\theta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y. \qquad (2)$$

It is a consistent estimator of vector θ and asymptotically has a bivariate normal distribution with variance-covariance matrix

$$\mathrm{Var}(\tilde{\theta}) = (X'\Sigma^{-1}X)^{-1} = \frac{1}{n}(X'\Sigma^{*-1}X)^{-1}.$$

Note that no matter what value we choose for k, the QDE remains consistent.

There is some arbitrariness in choosing a value for k. For efficiency sake, we should let k be large to make use of all the data frequency classes; however, for robustness sake, as mentioned by Kulasekera and Tonkyn (1992), we should discard small observed frequencies at the tail in order to avoid extreme sensitiveness. We observe that the MLE is not robust since its score functions are based on $\bar{X}$ and $\tilde{X}$ which are not robust statistics. Consequently, this flexibility of trading off efficiency versus robustness is a special characteristic of the QDE, which is not shared by the MLE.

In the next section, we calculate the efficiency of the QDE $\tilde{\theta}$ compared to the MLE $\hat{\theta}$ for various values of the parameters and see that $\tilde{\theta}$ has very high asymptotic efficiency for parameter values of interest.

Since Σ depends on the unknown parameters α and β through the p_j's, an iterative algorithm will be necessary to estimate them. An initial consistent estimator of vector θ, such as

$$\tilde{\theta}_0 = (X'X)^{-1}X'Y$$

obtained by replacing Σ by the identity matrix I, needs to be computed first. With this $\tilde{\theta}_0$, a first estimate $\Sigma_{(\tilde{\theta}_0)}$ of the variance-covariance matrix can be calculated, from which a new estimate of $\tilde{\theta}$ can be obtained from (2). This iterative scheme is repeated until $\tilde{\theta}$ is obtained with the desired accuracy.

In section 6, we present an example of application of the Good distribution; in practice, only a few iterations are needed to get the QDE $\tilde{\theta}$ with good accuracy.

5 ASYMPTOTIC EFFICIENCY OF QDE

In this section, we calculate the aymptotic efficiency of the quadratic distance estimator $\tilde{\theta}$, compared to the maximum likelihood estimator $\hat{\theta}$ for various values of the parameters α and β.

The asymptotic efficiency of estimator $\tilde{\theta}$ is defined as

$$\text{eff}(\tilde{\theta}) = \frac{|I_{(\theta)}^{-1}/n|}{|(X'\Sigma^{-1}X)^{-1}|} = \frac{|(X'\Sigma^{*-1}X)|}{|I_{(\theta)}|}, \qquad (3)$$

which is a notion of efficiency based on Bhapkar (1972), where $|A|$ denotes the determinant of matrix A.

Calculations were performed with the symbolic programming language MATHEMATICA. It should be noted that as n tends to infinity, k also tends to infinity with probability 1, and we need to take the inverse of a matrix of dimension $k \times k$. Even though Σ is a tridiagonal matrix and has a lot of 0's, its inverse Σ^{-1} can only be calculated numerically, with given values of k, α and β. In practice, we were able to perform those matrix inversions with a maximum value of $k = 200$. Very little difference is observed in $(X'\Sigma^{*-1}X)^{-1}$ calculated with $k = 100$ or $k = 200$. For example, with $\alpha = -0.105, \beta = -2.0$ and $k = 100$, we obtained $\text{Var}(\hat{\alpha}) = 1.17164/n$ and $\text{Var}(\hat{\beta}) = 13.334/n$, while the corresponding values with $k = 200$ gave $\text{Var}(\hat{\alpha}) = 1.17147/n$ and $\text{Var}(\hat{\beta}) = 13.3326/n$.

Table 1 contains the asymptotic variance-covariance matrix of the QDE $(\tilde{\alpha}, \tilde{\beta})$. Note that the values of α in the table correspond to values of $a = 0.50, 0.55, \ldots, 0.95$. Table 2 contains the asymptotic efficiency of $\tilde{\theta}$ defined by (3). The efficiency of the QDE is very high for all the values of (α, β) in Table 1. For $\alpha = -0.051$ and $\beta \geq -1.0$, we were unable to calculate the elements of the information matrix because the series for i_{11}, i_{12} and i_{22} converge too slowly.

In all data sets we looked at, the value of β was negative, which is why

Table 1: Asymptotic variance of QDE: $n\times$ (Var($\tilde{\alpha}$), Cov($\tilde{\alpha}, \tilde{\beta}$), Var($\tilde{\beta}$))$'$

α	β -3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0
-.693	221.425	113.206	57.839	29.7507	15.5771	8.42974	4.80616
	-364.975	-192.949	-102.73	-55.5661	-30.9139	-17.9877	-11.1759
	626.046	345.355	193.611	111.397	66.6745	42.2632	29.0051
-.598	157.754	79.025	39.5473	19.9475	10.2754	5.50415	3.13444
	-265.223	-138.161	-72.5588	-38.8147	-21.4609	-12.5064	-7.86599
	467.246	255.832	142.706	82.0316	49.3605	31.7423	22.3674
-.511	112.642	55.1214	26.9348	13.2828	6.7156	3.55607	2.02285
	-193.578	-99.1499	-51.2568	-27.0719	-14.8621	-8.67669	-5.53424
	351.39	190.765	105.8	60.7436	36.7664	24.0287	17.44
-.431	80.1259	38.1532	18.1298	8.70741	4.30748	2.25101	1.27926
	-141.126	-70.8744	-35.9702	-18.7164	-10.19	-5.96215	-3.86528
	265.046	142.48	78.49	44.9935	27.4119	18.2477	13.6968
-.357	56.3921	25.9922	11.9448	5.55807	2.67907	1.37865	0.782943
	-102.143	-50.1087	-24.877	-12.7147	-6.85403	-4.02126	-2.65789
	199.542	106.032	57.943	33.1456	20.343	13.8341	10.7957
-.288	38.9242	17.2393	7.60251	3.40196	1.58845	0.802489	0.455605
	-72.8323	-34.7197	-16.7763	-8.38722	-4.46639	-2.62961	-1.77952
	149.088	78.1228	42.272	24.111	14.9231	10.4091	8.50644
-.223	26.0152	10.9506	4.5803	1.94912	0.873858	0.431513	0.245091
	-50.6027	-23.2588	-10.8556	-5.27539	-2.76565	-1.63593	-1.14061
	109.693	56.4867	30.1828	17.1429	10.7148	7.71081	6.66863
0.85	16.4798	6.4755	2.5197	1.00094	0.424691	0.203553	0.115821
	-33.6375	-14.7189	-6.55359	-3.06372	-1.57223	-0.936245	-0.679324
	78.5126	39.5174	20.7609	11.7137	7.40743	5.55135	5.16577
-.163	9.47594	3.35915	1.17147	0.419069	0.16365	0.0751654	0.0429734
	-20.6202	-8.3852	-3.4781	-1.53315	-0.761684	-0.458451	-0.352819
	53.4039	26.0207	13.3326	7.43456	4.76916	3.78662	3.90686
-.051	4.38192	1.28345	0.363682	0.106513	0.0356088	0.0152215	0.00886028
	-10.5103	-3.73361	-1.35683	-0.53527	-0.25015	-0.154135	-0.132319
	32.455	14.975	7.3385	3.98288	2.60096	2.28537	2.81516

Table 2: Asymptotic efficiency of QDE

β	α $\leq$-0.163	-0.105	-0.051
$\leq$-1.5	1.00	1.00	1.00
-1.0	1.00	0.99	–
-0.5	1.00	0.99	–
0.0	1.00	0.92	–

we reported the calculations only for $\beta < 0$ in Table 1. From various computations we made, we observed the following:

1- For a given value of β, the asymptotic variances of $\hat{\alpha}$ and $\tilde{\alpha}$ decrease as α increases.

2- For a given value of α, the asymptotic variances of $\hat{\beta}$ and $\tilde{\beta}$ also decrease as β increases.

3- The covariance between $\hat{\alpha}$ and $\hat{\beta}$ is always negative, as is the covariance between $\tilde{\alpha}$ and $\tilde{\beta}$.

For the zeta distribution ($\alpha = 0$), Doray and Luong (1995) computed the efficiency of the QDE.

6 EXAMPLE

In this section, we present a data set which can be modelled with the Good distribution. We calculate the estimates of the parameters obtained by maximum likelihood and by the quadratic distance method. The goodness-of-fit of the model is tested with the Pearson's χ^2 test.

Table 3 contains the number of boards which contained at least one sowbug (see Kulasekera and Tonkyn (1992)). The complete data (except the observed frequency at 0), which can be found in Janardan et al. (1979), was used to estimate the parameters. The second column represents the observed number of boards having j sowbugs ($j \geq 1$) and the last two columns, the expected

Table 3: Fit of the Good distribution

j	observed #	expected # (MLE)	expected # (QDE)
1	28	26.01	26.87
2	14	16.67	14.63
3	11	11.83	9.87
4	8	8.76	7.27
5	11	6.63	5.61
6	2	5.10	4.47
7	3	3.95	3.63
8	3	3.09	3.00
9	3	2.43	2.51
10	3	1.92	2.11
11	2	1.52	1.79
12	0	1.21	1.53
13	1	0.96	1.32
14	2	0.77	1.14
15	1	0.62	0.98
16	0	0.49	0.86
17	2	0.40	0.75
≥ 18	0	1.64	5.65
total	94	94	94
	χ^2_{obs}	6.04	12.84
	$\chi^2_{df,0.95}$	14.07	15.51

number, calculated with the MLE's and the QDE of the parameters of the Good distribution.

The MLE's are the solution of the following system of equations

$$\frac{\Phi(e^{\alpha}, 0, -(\beta + 1))}{\Phi(e^{\alpha}, 0, -\beta)} = 4.2765957$$

$$\frac{\sum_{i=1}^{\infty} e^{\alpha i} i^{\beta} \ln i}{\Phi(e^{\alpha}, 0, -\beta)} = 1.0718644.$$

With the FindRoot procedure in MATHEMATICA, we obtained the MLE's appearing in the first half of Table 4. Starting values must be specified for this procedure. For β, we suggest using the value obtained by fitting first the zeta distribution to the data and for α, 0. The approximate variance-covariance

Table 4: Estimated values

	α	β	Var.-cov. matrix
MLE	-0.1987095	-0.355221	0.0029009 -0.0125346 -0.0125346 0.0684158
QDE	-0.0923463	-0.743481	0.0088311 -0.03252 -0.03252 0.1357904

matrix of the MLE's also appears in Table 4.

To calculate the QDE, we used the values $f_1, \ldots, f_{11}$ and $f_{13}, f_{14}.f_{15}$, since f_{12} and $f_{16} = 0$. The second half of Table 4 contains the values of the QDE and its approximate variance-covariance matrix. Convergence to the indicated values was reached in only 5 iterations.

Note that in the example, we did not use the observed frequency for $j = 0$ as Kulasekera and Tonkyn (1992) did, since the Good distribution is defined for $j \epsilon \{1, 2, 3, \ldots\}$; we only modelled boards containing sowbugs.

The Good ditribution gives a good fit. The observed χ^2 is smaller than the critical value at a significance level of 0.05 for both the MLE's and the QDE, with 7 and 8 degrees of freedom respectively. However, the fit in the tail is better when using the MLE's.

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