ON SOME GOODNESS-OF-FIT TESTS FOR THE POISSON DISTRIBUTION

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ABSTRACT

We develop new goodness-of-fit tests for the hypothesis of the Poisson model to a set of data and derive the asymptotic distribution of the test statistics. The null hypothesis may be unrestricted, giving an omnibus test; the alternative hypothesis may be that the data come from the negative binomial distribution, yielding an overdispersion test. We also present other tests for the Poisson distribution which have appeared in the statistical literature. Using simulations, we calculate the confidence level of these tests and their empirical power against some alternatives.

1 INTRODUCTION

Many authors have studied the subject of the fit of the Poisson distribution to a set of data, from Pearson (1900) and Fisher and al. (1922) to Pothoff and Whittinghill (1966). We will pursue this study by presenting a new goodness-of-fit test for that model. The Poisson distribution will be included in a two-parameter family of discrete distributions which also contains the negative binomial. By testing for a specific value for one of the parameters, we derive a test for the Poisson distribution. With a certain choice for the alternative hypothesis, we can obtain a test for overdispersion (variance larger than the mean) within that family.

The paper is organized as follows. In section 2, we present a family of discrete distributions defined by a recursive relationship for the probability function, family which contains the Poisson, binomial and negative binomial as members. We show how its parameters can be estimated by an iteratively reweighted least-squares method, and give the asymptotic distribution of the estimators. We also define a distance between the theoretical model and the data and show that the test statistic has an asymptotic chi-square

Table 1: Members of the (a, b) family

Distributions	a	b
Poisson (λ)	0	λ
Binomial (m, q)	$-q/(1-q)$	$-(m+1)a$, $m = 1, 2, \ldots$
Negative binomial $(r, 1/(1+p))$	$p/(1+p)$, $p > 0$	$(r-1)a$, $r > 0$

distribution (section 3). We review briefly other tests which have been used as goodness-of-fit or overdispersion tests for the Poisson distribution. Finally, with simulations, we compute the empirical confidence level of these various tests and their power when the alternative is a negative binomial distribution.

2 A GENERALIZED FAMILY

Panjer (1981) studied the two-parameter family of discrete distributions with probability function defined by the recursive relationship

$$p_{x+1} = (a + \frac{b}{x+1})p_x, \qquad x = 0, 1, 2, \ldots, \tag{1}$$

where $p_x = \Pr\{X = x\}$. He showed that the binomial, negative binomial and Poisson distributions are members of this family. Sundt and Jewell (1981) showed that there were no other members. Panjer and Willmot (1992) called this family the (a, b) family. Table 1 contains the admissible values for the parameters a and b and the corresponding distribution.

Let $\hat{p}_i$ be the observed percentage of the observations taking value i in the sample, for $i = 0, 1, \ldots, k$. From (1), Luong and Garrido (1993) obtain the linear model

$$\hat{p}_{i+1} = \left(a + \frac{b}{i+1}\right)\hat{p}_i + \epsilon_{i+1}, \qquad i = 0, 1, \ldots, k-1, \tag{2}$$

where $\hat{p}_i = f_i/n$ is the maximum likelihood estimator of p_i, and f_i is the observed frequency of $\{x = i\}$. We set $k = M > 0$. If we observe values larger than M, we treat them as outliers and reject them.

In matrix notation, the model can be rewritten

$$Y = \hat{X}\theta + \epsilon,$$

where $Y = \frac{1}{n}(f_1, \cdots, f_k)'$, $\theta = (a, b)'$, $\epsilon = (\epsilon_1, \ldots, \epsilon_k)'$, and

$\hat{X}' = \frac{1}{n}\begin{pmatrix} f_0 & f_1 & \cdots & f_{k-1} \\ f_0 & f_1/2 & \ldots & f_{k-1}/k \end{pmatrix}.$

Using the fact that $(f_0, \ldots, f_{k-1})$ follows a multinomial distribution, it can be shown that $E(\epsilon) = 0$ and

$$Var(\epsilon) = \Sigma_\theta = \frac{1}{n}\begin{pmatrix} \frac{p_0p_1+p_1^2}{p_0} & -p_2 & 0 & \cdots & 0 \\ -p_2 & \frac{p_1p_2+p_2^2}{p_1} & -p_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -p_{k-1} & \frac{p_{k-2}p_{k-1}+p_{k-1}^2}{p_{k-2}} & -p_k \\ 0 & \cdots & 0 & -p_k & \frac{p_{k-1}p_k+p_k^2}{p_{k-1}} \end{pmatrix}.$$

Let $\Sigma_\theta^* = n\Sigma_\theta$, $\theta_0 = (a,b)'$, the vector of the true parameter values a and b, and $Z_{(a,b)} = [\hat{p}_1 - (a+b)\hat{p}_0, \ldots, \hat{p}_k - (a+b/k)\hat{p}_{k-1}]'$, a vector of dimension k. Luong and Garrido (1993) proposed a quadratic distance estimator (QDE), easier to compute than the maximum likelihood estimator (MLE). They showed that the estimator $\hat{\theta}$ of θ which minimizes the distance $Z'_{(a,b)}\Sigma_{\theta_0}^{-1}Z_{(a,b)}$ can be written as $\hat{\theta} = (\hat{X}'\Sigma_{\theta_0}^{-1}\hat{X})^{-1}(\hat{X}'\Sigma_{\theta_0}^{-1}Y)$; it is consistent, efficient when $k \to \infty$ and has an asymptotic binormal distribution, since $\sqrt{n}(\hat{\theta} - \theta_0)$ converges in law to a $N(0, [X'\Sigma_{\theta_0}^{-1}X]^{-1})$ distribution, where X, the expected value of $\hat{X}$ equals $\begin{pmatrix} p_0 & p_1 & \cdots & p_{k-1} \\ p_0 & p_1/2 & \ldots & p_{k-1}/k \end{pmatrix}'$.

To estimate a and b, Luong and Garrido propose an iteratively reweighted least-squares method. The matrix Σ is replaced by the identity matrix in the above expression for $\hat{\theta}$, giving a consistent estimator $\hat{\theta}_0 = (\hat{X}'\hat{X})^{-1}(\hat{X}'Y)$. The value of $\hat{\theta}_0$ is used to obtain a first estimate for the variance-covariance matrix of ϵ, $\Sigma_{\hat{\theta}_0}$; from it, a new estimator $\hat{\theta}_2$ can then be calculated. This procedure is repeated until convergence of $\hat{a}$ and $\hat{b}$ to the required accuracy.

3 HYPOTHESIS TESTING

The family of discrete distributions defined recursively by (1) suggests a test for overdispersion within the (a, b) family. Testing the hypothesis $a = 0$ versus $a > 0$ will permit to distinguish between the Poisson and the negative binomial distributions. For members of the (a, b) family, $Var(X) = (a + b)/(1-a)^2$ and $E(X) = (a+b)/(1-a)$, so that $Var(X)/E(X) = (1-a)^{-1} > 1$ if $a > 0$. The quadratic distance $Z'_{(a,b)}\Sigma_{\theta_0}^{-1}Z_{(a,b)}$ can also provide a goodness-of-fit test for the Poisson distribution. We also present other tests which have appeared in the literature to test the hypothesis.

3.1 Quadratic distance test

Let $\tilde{\theta}$ be the vector $(0, \tilde{b})'$, where $\tilde{b}$ minimizes the distance $Z'_{(0,b)}\Sigma^{-1}_{(0,b)}Z_{(0,b)}$. Using the following theorem proved in Moore (1978), we will show that the quadratic distance between the sample and the Poisson distribution

$$d(F_n, F_{\tilde{\theta}}) = (Y - X\tilde{\theta})'\Sigma^{-1}_{\tilde{\theta}}(Y - X\tilde{\theta}) = n(Y - X\tilde{\theta})'\Sigma^{-1*}_{\tilde{\theta}}(Y - X\tilde{\theta}).$$

has an asymptotic χ^2 distribution with $(k-1)$ degrees of freedom.
Theorem: Let the vector Y, of dimension p, have a $N_p(0, \Sigma)$ distribution and C be a $p \times p$ positive definite symmetric matrix. If ΣC is idempotent with trace $(\Sigma C) = k$, then the quadratic form $Y'CY$ follows a χ^2 distribution with k degrees of freedom. This result is valid asymptotically if C is replaced by a consistent estimator $\hat{C}$.

The asymptotic distribution of $\sqrt{n}(Y - X\tilde{\theta})$ is $N(0, \Sigma_2^*)$, where

$$\Sigma_2^* = nVar\{[I - X_2(X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2)^{-1}(X_2'\Sigma^{-1*}_{\tilde{\theta}})]Y\}$$

and X_2 is the second column of X. We obtain

$$\Sigma_2^* = [I - X_2(X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2)^{-1}(X_2'\Sigma^{-1*}_{\tilde{\theta}})]\Sigma^*_{\tilde{\theta}}[I - \Sigma^{-1*}_{\tilde{\theta}}X_2(X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2)^{-1}X_2']$$

and so,

$$\Sigma_2^*\Sigma^{-1*}_{\tilde{\theta}} = [I - X_2(X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2)^{-1}(X_2'\Sigma^{-1*}_{\tilde{\theta}})][I - X_2(X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2)^{-1}(X_2'\Sigma^{-1*}_{\tilde{\theta}})].$$

The matrix $\Sigma_2^*\Sigma^{-1*}_{\tilde{\theta}}$ is idempotent and has a trace equal to
$\text{tr}(\Sigma_2^*\Sigma^{-1*}_{\tilde{\theta}}) = \text{tr}([I - X_2(X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2)^{-1}(X_2'\Sigma^{-1*}_{\tilde{\theta}})])$
$= k - \text{tr}([X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2]^{-1}[X_2'\Sigma^{-1*}_{\tilde{\theta}}X_2]$
$= k - 1.$

Using the previous theorem, it follows that $d(F_n, F_{\tilde{\theta}})$ has an asymptotic χ^2_{k-1} distribution. We reject the null hypothesis $H_0 : X \sim$ Poisson (b) at the approximate $(1-\alpha)$ confidence level when $d(F_n, F_{\tilde{\theta}}) > \chi^2_{k-1,1-\alpha}$, the $1-\alpha$ quantile of a chi-square distribution with $k-1$ degrees of freedom.

3.2 Test based on the asymptotic normality of $\hat{a}$

In section 2, we have seen that the QDE $\hat{\theta}$ has an aymptotic binormal distribution and $\sqrt{n}(\hat{\theta} - \theta_0)$ converges in law to a $N(0, [X'\Sigma^{-1}_{\theta_0}X]^{-1})$. It results that $\sqrt{n}(\hat{a} - a)$ will also have an asymptotic normal distribution with mean 0 and variance σ^2 equal to element (1,1) of the variance-covariance matrix $[X'\Sigma^{-1}_{\theta_0}X]^{-1}$. Since the Poisson distribution is the only member of the (a, b) family for which $a = 0$, we can construct a test of the Poisson fit to a set of data based on the test statistic $t = \hat{a}/\hat{\sigma}$.

We will reject the hypothesis $H_0 : X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$ at the $(1-\alpha)$ confidence level if $|t| > z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ quantile of a $N(0,1)$ distribution.

A test for overdispersion $H_0 : a = 0$ vs $H_a : a > 0$ can be obtained with a one-sided test; the hypothesis of the Poisson fit will be rejected at the $(1-\alpha)$ confidence level in favour of the negative binomial distribution if the test statistic t exceeds $z_{1-\alpha}$.

3.3 Index of dispersion

This is the usual test to measure the distance between the Poisson distribution and a set of data. The test, developed by Fisher et al. (1922) is a Poisson homogeneity test. It is based on the test statistic

$$S = \sum_{i=1}^{n} (x_i - \bar{X})^2 / \bar{X}.$$

Kendall and Stuart (1961, Vol. 2, p. 599) have shown that S has an approximate χ^2 distribution with $n-1$ degrees of freedom; the null hypothesis $H_0 : X_i \sim$Poisson(λ) will be rejected at the approximative confidence level $(1-\alpha)$ when $S > \chi^2_{n-1,1-\alpha}$. Darwin (1957) finds the limiting distribution of the index of dispersion when the X_i's follow a negative binomial, a Neyman or a Thomas distribution, from which the power can be calculated.

3.4 Deviance test

To test the null hypothesis $H_0 : X_i \sim$Poisson(λ), $\quad i = 1, \ldots, n$ versus the alternative hypothesis $H_a : X_i \sim$Poisson(λ_i) for each i, we can use the deviance statistic as a measure of dispersion. The deviance statistic is defined as (see McCullagh et Nelder (1989))

$$D = 2[l_{H_a} - l_{H_0}],$$

where l_H is the maximum of the loglikelihood function under hypothesis H. Under the null hypothesis, $\hat{\lambda} = \bar{X}$ and under the alternative hypothesis $\hat{\lambda}_i = x_i$, so that D becomes

$$D = 2 \sum_{i=1}^{n} x_i \ln(x_i / \bar{X}).$$

McCullagh et Nelder have shown that D has an asymptotical χ^2 distribution with $n-1$ degrees of freedom. We will reject the null hypothesis at the approximate $(1-\alpha)$ confidence level when when $D > \chi^2_{n-1,1-\alpha}$.

3.5 Katz test

Katz (1963) has studied the family of discrete distributions with probability function satisfying the recursive relationship

$$\frac{p_{x+1}}{p_x} = \frac{\alpha + \beta x}{x+1}, \quad x = 0, 1, 2, \ldots.$$

The constraint $\sum_{x=0}^{\infty} p_x = 1$ will provide the initial value p_0 to start the recursive calculation of the probability mass function.

From the above equation, we can see that the (a, b) family is just a reparametrization of Katz family, with $a = \beta$ and $b = \alpha - \beta$. The Poisson distribution now corresponds to $\beta = 0$ and $\alpha > 0$ while the negative binomial distribution $(r, 1/(1+p))$ corresponds to $\beta = p/(1+p) \in (0, 1)$ and $\alpha = r\beta$, $r \geq 0$.

To distinguish between the Poisson and the negative binomial distribution, Katz (1963) considered the ratio $(Var(X) - E(X))/E(X)$. Let $g = Var(X)/E(X)$, so that $g - 1$ equals zero for the Poisson distribution, is positive for the negative binomial and is negative for the binomial. With a sample of size n, the mean and variance of X are estimated with the unbiased estimators $\bar{X}$ and $S^2 = \sum_{i=1}^{n}(x_i - \bar{X})^2/(n-1)$. Katz has shown that the statistic $(S^2 - \bar{X})/\bar{X}$ follows an asymptotic $N(0, 2/n)$ distribution when the null hypothesis is true ($g = 1$). With Katz test, we reject the null hypothesis $H_0 : X \sim$Poisson(λ) at the approximate $(1 - \alpha)$ confidence level when

$$\left| \frac{S^2 - \bar{X}}{\bar{X}} \right| > z_{1-\alpha/2}\sqrt{\frac{2}{n}}.$$

A test of the Poisson distribution versus the negative binomial distribution can be obtained with the one-sided test: Reject H_0 if

$$\frac{S^2 - \bar{X}}{\bar{X}} > z_{1-\alpha}\sqrt{\frac{2}{n}}.$$

4 SIMULATIONS

For the five tests for the Poisson distribution presented in section 2, we have seen how to calculate the critical value corresponding to a given confidence level. The critical value for a test was obtained by deriving the asymptotic distribution of the statistic associated with that test. The statistics for the quadratic distance, the index of dispersion and the deviance tests followed an asymptotic χ^2 distribution while the statistics based on the asymptotic distribution of the estimator $\hat{a}$ and the one for Katz test followed a normal distribution. Those asymptotic distributions for the test statistics were obtained by letting the sample size n tend to infinity. But how large should the sample size be before those distributions become approximately true and the critical value could be used in a finite sample problem? With the help of simulations, we will answer this question in subsection 4.1.

In subsection 4.2, we look at the empirical power of these tests. We generate samples from the negative binomial distribution with the same mean as the Poisson distribution (1, 3, 5 and 10) and we calculate the probability that each test rejects the hypothesis of the Poisson distribution for the data.

Empirical confidence levels and powers were calculated by simulating 1000 samples of size 20, 50 and 100. All calculations were performed with the language SPLUS. We denote the five tests as:

• QD: indicates the quadratic distance test with the null hypothesis $H_0 : \theta_0 = (0, b)$.
• Normality: test of subsection 2.2, with the null hypothesis $H_0 : a = 0$.
• Dispersion : test of subsection 2.3, with the null hypothesis $H_0 : X \sim$Poisson(λ).
• Deviance: indicates the test of subsection 2.4 with the null hypothesis $H_0 : X \sim$Poisson(λ).
• Katz: indicates the test of subsection 2.5 with the null hypothesis $H_0 : X \sim$Poisson(λ).

4.1 Confidence level

With simulations, we compare the empirical confidence level of each test with the theoretical confidence level. Samples of size 20, 50 and 100 Poisson distributions were generated, with parameter 1, 3, 5, and 10. For the quadratic distance test, we have fixed k to 9, 20 and 30, corresponding to sample sizes 20 50 and 100 respectively. We obtained the results in table 2 for $\alpha = 0.05$ and in table 3 for $\alpha = 0.10$. The empirical confidence levels were obtained by dividing by 1000 the number of times the sample gave a test statistic smaller than the critical value obtained form the asymptotic distribution.

The variance of the empirical confidence level is $\sigma^2_\alpha = \alpha(1-\alpha)/N$, where N is the number of samples generated. With N=1000, we obtain $\sigma_{0.05} = 0.0069$ and $\sigma_{0.10} = 0.0095$. A 95% confidence interval for the confidence level corresponding to α=0.05 is [0.936, 0.964]; for α=0.10, it is [0.881, 0.919].

The quadratic distance test produces an empirical confidence level very close to the theoretical one even for sample sizes as small as 20. However, the test based on the asymptotic normality of the estimator $\hat{a}$ requires much larger samples (at least 100) to attain such results and it works best when the Poisson parameter is at least equal to 5. The dispersion and Katz test produce similar results, which are the best for all sample sizes and parameter values investigated. For samples of size 100, the deviance test was the worst:

Table 2: Empirical confidence levels ($\alpha = 0.05$)

Distribution	Poisson (1)			Poisson (3)		
Test / Sample size	20	50	100	20	50	100
QD	0.949	0.948	0.939	0.929	0.948	0.928
Normality	0.632	0.793	0.874	0.769	0.848	0.920
Dispersion	0.956	0.947	0.952	0.951	0.945	0.946
Deviance	0.940	0.885	0.792	0.914	0.865	0.840
Katz	0.955	0.948	0.956	0.952	0.944	0.937
	Poisson (5)			Poisson (10)		
QD	0.933	0.926	0.932	0.944	0.942	0.945
Normality	0.840	0.907	0.931	0.771	0.920	0.937
Dispersion	0.960	0.938	0.940	0.949	0.949	0.945
Deviance	0.928	0.893	0.906	0.936	0.934	0.929
Katz	0.961	0.946	0.943	0.952	0.946	0.946

Table 3: Empirical confidence levels ($\alpha = 0.10$)

Distribution	Poisson (1)			Poisson (3)		
Test / Sample size	20	50	100	20	50	100
QD	0.940	0.934	0.923	0.914	0.927	0.900
Normality	0.600	0.761	0.814	0.735	0.792	0.871
Dispersion	0.909	0.908	0.905	0.914	0.896	0.892
Deviance	0.850	0.766	0.619	0.841	0.797	0.721
Katz	0.913	0.908	0.910	0.918	0.880	0.896
	Poisson (5)			Poisson (10)		
QD	0.926	0.903	0.918	0.922	0.926	0.929
EMCa	0.800	0.866	0.882	0.724	0.873	0.883
Dispersion	0.905	0.871	0.898	0.900	0.906	0.886
Deviance	0.861	0.803	0.840	0.891	0.868	0.861
Katz	0.918	0.881	0.888	0.898	0.900	0.893

for all parameter values, the confidence level was lower with a sample of size 100 than with a sample of size 20.

4.2 Empirical power

The empirical power of each of the 5 tests was calculated by generating 1000 samples from the negative binomial distribution with the same mean as the Poisson distribution (1, 3, 5 and 10). The probability that each test rejects the hypothesis of the Poisson distribution for the data was obtained by dividing by 1000, the number of samples yielding a test statistic greater than than the critical value at the appropriate level. The results can be found in table 4 for $\alpha = 0.05$, and in table 5 for $\alpha = 0.10$. The notation used in those tables is the following; NB$x(r)$ will represent a negative binomial distribution with mean x and parameter r, so that $p = x/r$.

With small sample size (20), the test based on the normality of $\hat{a}$ produced better results than the quadratic distance test. The power was comparable to that attained by the other three tests. With samples of size 100, all 5 tests had very high power (over 95%).

From other simulations, we also noticed that if the parameter a is close to 0, the tests have a small empirical power. This is normal since the negative binomial distribution is then very similar to a Poisson distribution. As the parameter a increases, the empirical powers of the tests increase.

5 CONCLUSION

We have analyzed two new tests for the hypothesis of the fit of the Poisson distribution to a set of data. The quadratic distance test was based on a measure of distance between the data and the distribution, which followed an asymptotic χ^2 distribution, while the other test was based on the asymptotic normal distribution of the parameter estimator. We compared the empirical confidence level and power of the two tests with those of other well-known tests, the dispersion, deviance and Katz tests. The quadratic distance test produced excellent results for the confidence level, even for sample sizes as small as 20. For samples of size 100, the power of all 5 tests was very high.

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Table 4: Empirical powers (α =0.05)

Test	n	NB1(1)	NB3(1)	NB5(1)	NB 10(1)
QD	20	0.472	0.572	0.626	0.602
	50	0.784	0.852	0.862	0.855
	100	0.956	0.0.972	0.977	0.967
Normality	20	0.572	0.680	0.686	0.695
	50	0.848	0.915	0.941	0.936
	100	0.976	0.994	0.997	0.997
Dispersion	20	0.572	0.688	0.700	0.659
	50	0.899	0.942	0.960	0.954
	100	0.990	0.997	0.999	1.000
Deviance	20	0.573	0.785	0.756	0.674
	50	0.923	0.980	0.974	0.966
	100	0.998	0.999	0.999	1.000
Katz	20	0.562	0.667	0.683	0.637
	50	0.872	0.928	0.950	0.944
	100	0.986	0.995	0.996	0.998

Table 5: Empirical powers (α =0.10)

Test	n	NB1(1)	NB3(1)	NB5(1)	NB 10(1)
QD	20	0.505	0.609	0.655	0.626
	50	0.813	0.877	0.888	0.873
	100	0.962	0.979	0.988	0.976
Normality	20	0.640	0.736	0.733	0.738
	50	0.894	0.942	0.0.960	0.955
	100	0.990	0.997	0.998	0.998
Dispersion	20	0.0.659	0.792	0.799	0.769
	50	0.938	0.966	0.974	0.981
	100	0.999	0.999	0.999	1.000
Deviance	20	0.706	0.872	0.843	0.773
	50	0.964	0.989	0.986	0.984
	100	1.000	1.000	0.999	1.000
Katz	20	0.606	0.722	0.741	0.694
	50	0.911	0.949	0.964	0.965
	100	0.993	0.997	0.998	0.999

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