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A Note On Upper And Lower Solutions For First Order Inclusions Of Upper Semicontinuous Or Lower Semicontinuous Type

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Abstract. Existence results based on fixed point theorems for self maps are used to establish an upper and lower solutions theory for first order inclusions.

1 Introduction

This paper presents an upper and lower solutions theory for the first order inclusions

$$\begin{cases} x'(t) \in F(t, x(t)) \quad \text{a.e. } t \in [0,T] \\ x(0) = r \end{cases} \tag{1.1}$$

where $F : [0,T] \times \mathbf{R} \to K(\mathbf{R})$; here $K(\mathbf{R})$ denotes the family of nonempty, compact subsets of $\mathbf{R}$. Throughout the paper our map $x \mapsto F(t,x)$ is upper semicontinuous or lower semicontinuous for a.e. $t \in [0,T]$. Recall a map $G : \mathbf{R} \to K(\mathbf{R})$ is lower semicontinuous (respectively upper semicontinuous) if the set $\{x \in \mathbf{R} : G(x) \cap A \neq \emptyset\}$ is open (respectively closed) for any open (respectively

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closed) set A in $\mathbf{R}$. Also a map $H : [0,T] \to K(\mathbf{R})$ is measurable if the set $\{t \in [0,T] : H(t) \cap A \neq \emptyset\}$ is measurable for any closed set A in $[0,T]$. The reader is referred to [1–4] for results on multivalued amppings.

We note that part of the theory presented in this paper was inspired by results in [5] for second order problems. Our results rely on the following existence results established in the literature [5, 6]. The first result follows immediately from Ky Fan's fixed point theorem and the second from the Bressan Colombo selection theorem and Schauder's fixed point theorem.

Theorem 1.1. *Suppose $F : [0,T] \times \mathbf{R} \to CK(\mathbf{R})$ (here $CK(\mathbf{R})$ denotes the family of nonempty, convex, compact subsets of $\mathbf{R}$) satisfies the following conditions:*

$$t \mapsto F(t,x) \text{ is measurable for every } x \in \mathbf{R} \tag{1.2}$$

$$x \mapsto F(t,x) \text{ is upper semicontinuous for a.e. } t \in [0,T] \tag{1.3}$$

and

$$\begin{cases} \exists\, h \in L^1[0,T] \text{ with } |F(t,x)| \leq h(t) \\ \text{for a.e. } t \in [0,T] \text{ and } x \in \mathbf{R}. \end{cases} \tag{1.4}$$

Then (1.1) *has a solution $y \in W^{1,1}[0,T]$.*

Theorem 1.2. *Suppose $F : [0,T] \times \mathbf{R} \to K(\mathbf{R})$ satisfies* (1.4) *and the following two conditions:*

$$(t,x) \mapsto F(t,x) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable} \tag{1.5}$$

and

$$x \mapsto F(t,x) \text{ is lower semicontinuous for a.e. } t \in [0,T]. \tag{1.6}$$

Then (1.1) *has a solution $y \in W^{1,1}[0,T]$.*

Remark 1.1. Recall $A \subseteq I \times \mathbf{R}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ algebra generated by all sets of the form $N \times D$ where N is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbf{R}$.

2 Upper and lower solutions.

In this section we present an upper and lower solutions result for the first order inclusion

$$\begin{cases} x'(t) \in F(t,x(t)) \quad \text{a.e. } t \in [0,T] \\ x(0) = r \end{cases} \tag{2.1}$$

where $F : [0,T] \times \mathbf{R} \to K(\mathbf{R})$.

Definition 2.1. A function $\beta \in W^{1,1}[0,T]$ is said to be an *upper solution* for (2.1) if for almost every $t \in [0,T]$ there exists $v \in F(t,\beta(t))$ with $v \leq \beta'(t)$ (i.e. $F(t,\beta(t)) \cap (-\infty, \beta'(t)] \neq \emptyset$) and $\beta(0) \geq r$. Similarly a function $\alpha \in W^{1,1}[0,T]$ is said to be a *lower solution* for (2.1) if for almost every $t \in [0,T]$ there exists $v \in F(t,\alpha(t))$ with $v \geq \alpha'(t)$ (i.e. $F(t,\alpha(t)) \cap [\alpha'(t), \infty) \neq \emptyset$) and $\alpha(0) \leq r$.

We begin with the upper semicontinuous situation.

Theorem 2.1. *Suppose $F : [0,T] \times \mathbf{R} \to CK(\mathbf{R})$ satisfies the following conditions:*

$$t \mapsto F(t,x) \text{ is measurable for every } x \in \mathbf{R} \tag{2.2}$$

$$x \mapsto F(t,x) \text{ is upper semicontinuous for a.e. } t \in [0,T] \tag{2.3}$$

and

$$\begin{cases} \text{for each } r > 0,\ \exists h_r \in L^1[0,T] \text{ with } |F(t,x)| \le h_r(t) \\ \text{for a.e. } t \in [0,T] \text{ and } x \in \mathbf{R} \text{ with } |x| \le r. \end{cases} \tag{2.4}$$

Also assume there exists α, $\beta \in W^{1,1}[0,T]$ respectively lower and upper solutions of (2.1) with $\alpha(t) \le \beta(t)$ for $t \in [0,T]$. Then (2.1) has a solution $y \in W^{1,1}[0,T]$ with $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0,T]$.

PROOF: Let

$$h(t,x) = \begin{cases} \alpha(t), & x < \alpha(t), \\ x, & \alpha(t) \le x \le \beta(t), \\ \beta(t), & x > \beta(t); \end{cases}$$

$$\Gamma_+(t,x) = \begin{cases} [\alpha'(t), \infty), & x < \alpha(t), \\ \mathbf{R}, & \alpha(t) \le x \le \beta(t), \\ (-\infty, \beta'(t)], & x > \beta(t); \end{cases}$$

and let

$$F_+(t,x) = F(t, h(t,x)) \cap \Gamma_+(t,x).$$

Notice $F_+ : [0,T] \times \mathbf{R} \to CK(\mathbf{R})$ (notice the values are nonempty from the definition of upper and lower solutions). Also it is easy to see that $x \mapsto \Gamma_+(t,x)$ is upper semicontinuous for a.e. $t \in [0,T]$ so the map $x \mapsto F_+(t,x)$ is upper semicontinuous for a.e. $t \in [0,T]$. In addition the map $t \mapsto F_+(t,x)$ is measurable for each $x \in \mathbf{R}$. Consider the modified problem

$$\begin{cases} x'(t) \in F_+(t, x(t)) \quad \text{a.e. } t \in [0,T] \\ x(0) = r. \end{cases} \tag{2.5}$$

Now Theorem 1.1 guarantees that (2.5) has a solution $y \in W^{1,1}[0,T]$. To finish the proof it suffices to show $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0,T]$. Suppose $y(t) \not\le \beta(t)$ for some $t \in [0,T]$. Since $y(0) = r \le \beta(0)$ there exists $t_1 < t_2 \in [0,T]$ with

$$y(t_1) = \beta(t_1) \quad \text{and} \quad y(t) > \beta(t) \quad \text{for} \quad t \in (t_1, t_2].$$

For almost every $t \in (t_1, t_2)$, since $y(t) > \beta(t)$ we have

$$y'(t) = w(t) \quad \text{where} \quad w(t) \in F_+(t, y(t)). \tag{2.6}$$

In particular $w(t) \in \Gamma_+(t, y(t))$ so $w(t) \in (-\infty, \beta'(t)]$. This together with (2.6) implies $y'(t) \le \beta'(t)$. Integration from t_1 to t_2 yields $y(t_2) \le \beta(t_2)$, a contradiction. Thus $y(t) \le \beta(t)$ for $t \in [0,T]$. A similar argument shows $y(t) \ge \alpha(t)$ for $t \in [0,T]$. □

Our next result concerns the lower semicontinuous situation.

Theorem 2.2. *Suppose $F : [0,T] \times \mathbf{R} \to K(\mathbf{R})$ satisfies (2.4) and the following two conditions:*

$$(t,x) \mapsto F(t,x) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable} \tag{2.7}$$

and

$$x \mapsto F(t,x) \text{ is lower semicontinuous for a.e. } t \in [0,T]. \tag{2.8}$$

Also assume there exists $\alpha, \beta \in W^{1,1}[0,T]$ respectively lower and upper solutions of (2.1) with $\alpha(t) \le \beta(t)$ for $t \in [0,T]$. Then (2.1) has a solution $y \in W^{1,1}[0,T]$ with $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0,T]$.

PROOF: Let

$$\Gamma_-(t,x) = \begin{cases} [\alpha'(t),\infty), & x \le \alpha(t) \\ \mathbf{R}, & \alpha(t) < x < \beta(t) \\ (-\infty, \beta'(t)], & x \ge \beta(t); \end{cases}$$

and let

$$F_-(t,x) = F(t,h(t,x)) \cap \Gamma_-(t,x).$$

Notice $F_- : [0,T] \times \mathbf{R} \to K(\mathbf{R})$. We claim $x \mapsto F_-(t,x)$ is lower semicontinuous for a.e. $t \in [0,T]$. From (2.8) there exists a null set N with $x \mapsto F(t,x)$ is lower semicontinuous for $t \in [0,T] \setminus N$. Fix $t \in [0,T] \setminus N$. To show $F(t,.)$ is lower semicontinuous for $x \in \mathbf{R}$ take a sequence $\{x_n\}_1^\infty$ in $\mathbf{R}$ with $x_n \to x$. Take any $y \in F_-(t,x)$. We must show that there exists a subsequence S of $N_0 = \{1,2,\ldots\}$ and elements $y_k \in F_-(t,x_k)$, $k \in S$, with $y_k \to y$ as $k \to \infty$ in S. The proof is broken into a number of cases. We discuss three such cases which illustrate the ideas involved. For the first case suppose $\alpha(t) < x < \beta(t)$ and $\alpha(t) < x_n < \beta(t)$ for $n \in N_0$. Then

$$y \in F(t,x) \cap \mathbf{R} = F(t,x).$$

Since $x \mapsto F(t,x)$ is lower semicontinuous then there exists a subsequence S of N_0 and elements $y_k \in F(t,x_k)$, $k \in S$, with $y_k \to y$ as $k \to \infty$ in S. Note $y_k \in F_-(t,x_k)$, $k \in S$, since $\Gamma_-(t,x_k) = \mathbf{R}$ and $F(t,h(t,x_k)) = F(t,x_k)$. For the second case suppose $x \ge \beta(t)$ and assume $x_n \ge \beta(t)$ for $n \in N_0$. Then

$$y \in F(t,\beta(t)) \cap (-\infty, \beta'(t)].$$

Choose $S = N_0$ and $y_k = y$, $k \in S$. Notice $y_k \in F_-(t,x_k)$, $k \in S$, since $x_n \ge \beta(t)$ for $n \in N_0$ implies $F(t,h(t,x_k)) = F(t,\beta(t))$ and $\Gamma_-(t,x_k) = (-\infty,\beta'(t)]$. For the third case suppose $x = \beta(t)$ and assume $\alpha(t) < x_n < \beta(t)$ for $n \in N_0$. Then

$$y \in F(t,\beta(t)) \cap (-\infty,\beta'(t)] = F(t,x) \cap (-\infty,\beta'(t)].$$

Since $x \mapsto F(t,x)$ is lower semicontinuous then there exists a subsequence S of N_0 and elements $y_k \in F(t,x_k)$, $k \in S$, with $y_k \to y$ as $k \to \infty$ in S. Note $y_k \in F_-(t,x_k)$, $k \in S$, since $\alpha(t) < x_n < \beta(t)$ for $n \in N_0$ implies $F_-(t,x_k) = F(t,x_k) \cap \mathbf{R} = F(t,x_k)$. The other cases are similar, so as a result $x \mapsto F_-(t,x)$ is lower semicontinuous for a.e. $t \in [0,T]$. Consider the modified problem

$$\begin{cases} x'(t) \in F_-(t,x(t)) & \text{a.e. } t \in [0,T] \\ x(0) = r. \end{cases} \tag{2.9}$$

Now Theorem 1.2 guarantees that (2.9) has a solution $y \in W^{1,1}[0,T]$, and essentially the same argument as in Theorem 2.1 yields $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0,T]$. □

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