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A SURVEY OF RECENT FIXED POINT THEORY IN FRÉCHET SPACES

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Abstract. This paper presents some generalizations of two modern approaches to fixed point theory of maps defined on Fréchet spaces.

1. Introduction.

This paper presents some generalizations of recent fixed point theorems for maps defined on Fréchet spaces (complete metrizable locally convex linear topological spaces). The literature on fixed point theory in Fréchet spaces usually begins with the Schauder–Tychonoff theorem (or its multivalued analogue).

Theorem 1.1. *Let C be a convex subset of a Fréchet space and $F : C \to C$ a compact, continuous map. Then F has a fixed point in C.*

In applications to construct a set C so that F takes C back into C is very difficult and sometimes impossible. As a result it makes sense to discuss maps $F : C \to E$. In the literature to discuss maps $F : C \to E$ many authors present variations of the Leray–Schauder alternative. A typical theorem is the following (see for example [17]).

Theorem 1.2. *Let E be a Fréchet space, C a convex subset of E, U an open subset of C and $0 \in U$. Suppose $F : \overline{U} \to C$ (here $\overline{U}$ denotes the closure of U in C) is a continuous, compact map. Then either*

(A1). F has a fixed point in $\overline{U}$; or

(A2). there exists $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

PROOF: Suppose $(A2)$ does not occur and F has no fixed points in ∂U. Let

$$A = \left\{ x \in \overline{U} : \; x = t\,F(x) \quad \text{for some} \quad t \in [0,1] \right\}.$$

Now $A \neq \emptyset$ since $0 \in A$ and A is closed since F is continuous. Also notice $A \cap \partial U = \emptyset$. Thus there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(A) = 1$ and $\mu(\partial U) = 0$. Let

$$N(x) = \begin{cases} \mu(x)\, F(x), & x \in \overline{U} \\ 0, & x \in C \setminus \overline{U}. \end{cases}$$

Clearly $N : C \to C$ is a continuous, compact map. Theorem 1.1 guarantees the existence of an $x \in C$ with $x = N(x)$. Notice $x \in U$ since $0 \in U$. As a result $x = \mu(x)\, F(x)$, so $x \in A$. Thus $\mu(x) = 1$ and so $x = F(x)$. □

In the Banach space setting Theorem 1.2 is applicable to wide classes of problems. However in the non–normable situation Theorem 1.2 is rarely of interest from an application viewpoint (this point seems to be overlooked by many authors) since in applications usually $C = E$ and the set U constructed is usually bounded, and so has empty interior. As a result from an application viewpoint, Theorem 1.2 needs to be adjusted.

There are only a handful of "applicable" results in the literature. The first applicable result was due to Furi and Pera [11] in 1987 and we present for completeness this result at the end of this section. This paper presents two recent approaches which are based on the fact that a Fréchet space can be viewed as a projective limit of a sequence of Banach spaces $\{E_n\}_{n \in N}$ (here $N = \{1, 2, ...\}$). Both approaches are based on constructing maps F_n defined on subsets of E_n whose fixed points converge to a fixed point of F. In the first approach [6–10] for $n \in N$ a specific map F_n is discussed. This differs from the second approach [1, 4, 15] where the maps $\{F_n\}_{n \in N}$ only need to satisfy a closure type property. Both approaches have advantages and disadvantages over the other. For example the results in approach one are easier and nicer to state than the results in approach two (compare, for example, Theorem 2.3 with Theorem 2.5). On the other hand even though the conditions in the second approach seem quite technical they are in fact easier to check in practice than the conditions in the first approach and moreover the second approach seems to apply to a wider class on problems.

For the remainder of this section we gather together some definitions and known results. Let (X, d) be a metric space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty]$ defined by

$$\alpha(B) = \inf\{r > 0 : \ B \subseteq \cup_{i=1}^{n} B_i \ \text{ and } \ diam\,(B_i) \leq r\}.$$

Let S be a nonempty subset of X. For each $x \in X$, define $d(x, S) = \inf_{y \in S}\ d(x, y)$. We say a set is countably bounded if it is countable and bounded. Now suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then $G : S \to 2^X$ is

(i). countably k–set contractive (here $k \geq 0$) if $G(S)$ is bounded and $\alpha(G(W)) \leq k\, \alpha(W)$ for all countably bounded sets W of S,

(ii). countably condensing if $G(S)$ is bounded, G is countably 1–set contractive and $\alpha(G(W)) < \alpha(W)$ for all countably bounded sets W of S with $\alpha(W) \neq 0$.

(iii). hemicompact if each sequence $\{x_n\}_{n\in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now present a result from the literature [1, 18] which will be needed in Section 2 (this result was first established in [12]).

Theorem 1.3. *Let (X, d) be a metric space, D a nonempty, complete subset of X, and $G : D \to 2^X$ a countably condensing map. Then G is hemicompact.*

PROOF: Let $\{x_n\}_{n\in N}$ be a sequence in D such that $d(x_n, G(x_n)) \to 0$ as $n \to \infty$. Note since $G(D)$ is bounded and $d(x_n, G(x_n)) \to 0$ as $n \to \infty$ then $\{x_n\}_{n\in N}$ is bounded. Let

$$M = \{x_n : \; n \in N\} \quad \text{so} \quad G(M) = \cup_{n=1}^{\infty} G(x_n).$$

Next let $\epsilon > 0$ be given. Then there exists $n_0 \in N$ such that for each $n > n_0$ there is a $y_n \in G(x_n)$ with $d(x_n, y_n) < \epsilon$. Now let

$$M^\star = \cup_{n=1}^{\infty} \{y_n \in G(x_n) : \; d(x_n, y_n) < \epsilon\}.$$

Then $B(M^\star, \epsilon)$ contains all but a finite number of elements of M. Also since $B(M^\star, \epsilon) \subseteq M^\star + \epsilon B(0,1)$ and $M^\star \subseteq G(M)$ we have

$$\alpha(M) \le \alpha(B(M^\star, \epsilon)) \le \alpha(M^\star) + 2\epsilon \le \alpha(G(M)) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary we have

$$\alpha(M) \le \alpha(G(M)).$$

Now G is countably condensing and M is countable, so $\alpha(M) = 0$ i.e. $\overline{M}$ is compact. Since D is complete we deduce that $\{x_n\}_{n\in N}$ has a convergent subsequence. □

Finally in this section, as promised, we present the Furi–Pera fixed point theorem [11, 14].

Theorem 1.4. *Let E be a Fréchet space, Q a closed, convex subset of E and $0 \in Q$. Suppose $F : Q \to E$ is a continuous, compact map with the following condition holding:*

$$(1.1) \qquad \begin{cases} \text{if } \{(x_j, \lambda_j)\}_{n\in N} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{converging to } (x, \lambda) \text{ with } x = \lambda F(x) \text{ and } 0 \le \lambda < 1, \\ \text{then } \lambda_j F(x_j) \in Q \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then F has a fixed point in Q.

PROOF: Let $r : E \to Q$ be a continuous retraction (the existence of r is immediate from Dugundji's extension theorem). Note if $0 \in int\, Q$ we may take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } x \in E,$$

where μ is the Minkowski functional on Q. On the other hand if $int\, Q = \emptyset$ then $\partial Q = Q$. So we choose the retraction r so that $r(z) \in \partial Q$ for

$z \in E \setminus Q$. Consider

$$B = \{x \in E : \ x = F \circ r\,(x)\,\}.$$

Notice $F \circ r : E \to E$ is a continuous, compact map, so Theorem 1.1 guarantees that $F \circ r$ has a fixed point. Thus $B \neq \emptyset$. In addition B is closed and in fact compact since $B \subseteq F \circ r\,(B) \subseteq F\,(Q)$. It remains to show $B \cap Q \neq \emptyset$. Suppose this is not true i.e. suppose $B \cap Q = \emptyset$. Then since B is compact and Q is closed there exists $\delta > 0$ with $d(B, Q) > \delta$, here d is the metric associated with E. Choose $m \in N$ such that $1 < \delta\, m$. Let

$$U_i = \left\{x \in E : \ d(x, Q) \ < \ \frac{1}{i}\right\} \quad \text{for} \quad i \in \{m, m+1, ...\}.$$

Fix $i \in \{m, m+1, ...\}$. Since $d\,(B, Q) > \delta$ then $B \cap \overline{U_i} = \emptyset$. In addition U_i is open, $0 \in U_i$ and $F \circ r : \overline{U_i} \to E$ is a continuous, compact map. Theorem 1.2 guarantees, since $B \cap \overline{U_i} = \emptyset$, that there exists

$$(y_i, \lambda_i) \in \partial U_i \times (0,1) \quad \text{with} \quad y_i = \lambda_i\, F \circ r\,(y_i).$$

Consequently

$$\lambda_j\, F \circ r\,(y_j) \notin Q \quad \text{for each} \quad j \in \{m, m+1, ...\}. \tag{1.2}$$

We now look at

$$D = \{x \in E : \ x = \lambda\, F \circ r\,(x) \ \text{ for some } \ \lambda \in [0,1]\,\}.$$

Notice $D \neq \emptyset$ is closed and compact. This together with

$$d(y_j, Q) = \frac{1}{j} \quad \text{and} \quad |\lambda_j| \leq 1 \ \text{ for } \ j \in \{m, m+1, ...\}$$

implies that we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \in [0,1] \quad \text{and} \quad y_j \to y^\star \in \partial Q.$$

In addition we have

$$y_j = \lambda_j\, F \circ r\,(y_j) \to \lambda^\star\, F \circ r\,(y^\star),$$

so $y^\star = \lambda^\star\, F \circ r\,(y^\star)$. Note $\lambda^\star \neq 1$ since $B \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. However (1.1) with

$$x_j = r(y_j) \in \partial Q \quad \text{and} \quad x = y^\star = r(y^\star)$$

implies that $\lambda_j\, F \circ r\,(y_j) \in Q$ for j sufficiently large. This contradicts (1.2). Thus $B \cap Q \neq \emptyset$ i.e. there exists $x \in Q$ with $x = F \circ r\,(x) = F(x)$. □

2. Fixed point theory in Fréchet spaces.

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \ldots\ldots \quad \text{for every} \quad x \in E. \tag{2.1}$$

To E we associate a sequence of Banach spaces $\{(\mathbf{E}^n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x-y|_n = 0. \tag{2.2}$$

We denote by $E^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}^n, |\cdot|_n)$ the completion of E^n with respect to $|\cdot|_n$ (the norm on E^n induced by $|\cdot|_n$ and its extension to $\mathbf{E}^n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}^n$. For each $X \subseteq E$ and each $n \in N$ we set $X^n = \mu_n(X)$, and we let $\overline{X^n}$ and ∂X^n denote respectively the closure and the boundary of X^n with respect to $|\cdot|_n$ in $\mathbf{E}^n$. Also for $X \subseteq E$ and $n \in N$ we let

$$diam_n(X) = \sup\{|x-y|_n : \; x, y \in X\}$$

and we define a multimap $S_n : X \to X$ by

$$S_n(x) = \{y \in X : \; |x-y|_n = 0\}.$$

For $r > 0$ and $x \in \mathbf{E}^n$ we let

$$B_n(x,r) = \{y \in \mathbf{E}^n : \; |x-y|_n \leq r\}.$$

Also the pseudo-interior of X is defined by [7]

$$pseudo-int\,(X) = \{x \in X : \; \mu_n(x) \in \overline{X^n} \setminus \partial X^n \;\; \text{for every} \;\; n \in N\}.$$

Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}^m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}^m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}^m \to \mathbf{E}^n$ since $E_m/\sim_n$ can be regarded as a subset of $\mathbf{E}^n$. Observe that E is the projective limit of $\{\mathbf{E}^n\}_{n \in N}$ i.e. $E = \cap_{n=1}^{\infty} \mathbf{E}^n$ where $\cap_1^{\infty}$ is the generalized intersection as described in [13 pp. 439].

We will need the following lemma [7].

Lemma 2.1. *Let X be a closed subset of E. Then for every sequence $\{z_n\}_{n \in N}$ with $z_n \in \overline{X^n}$ such that for every $n \in N$, $\{\mu_{n,m}(z_m)\}_{m \geq n}$ is a Cauchy sequence in $\overline{X^n}$, there exists $x \in X$ such that $\{\mu_{n,m}(z_m)\}_{m \geq n}$ converges to $\mu_n(x) \in X^n$ for every $n \in N$.*

Next we give a slight generalization of the notion of an admissible map introduced in [7].

Definition 2.1. Let X be a subset of E. A map $f : X \to E$ is called *admissible* if for every $n \in N$,

(i). the multimap $F^n : X^n \to \mathbf{E}^n$ defined by

$$F^n(\mu_n(x)) = \overline{co}\,(\mu_n \circ f \circ S_n(x))$$

admits an upper semicontinuous countably condensing extension $\mathbf{F}^n : \overline{X^n} \to \mathbf{E}^n$ with convex, compact values,

(ii). for every $\epsilon > 0$, $\exists m \geq n$ such that for every $x \in X$ we have $diam_n(f(S_m(x))) < \epsilon$.

Following the ideas in [7], we now present a Schauder–Tychonoff type theorem and a nonlinear alternative of Leray–Schauder type for admissible maps.

Theorem 2.2. *Let X be a closed subset of a Fréchet space $E = (E, \{|\cdot|_n\}_{n\in N})$ and $f : X \to X$ an admissible map. Then f has a fixed point.*

PROOF: From [2, Theorem 2.2] we know for each $n \in N$ that $\mathbf{F}^n$ has a fixed point $z_n \in \overline{X^n}$.

Fix $n \in N$. It is easy to check that for $m \geq n$ we have $\mu_{n,m}(z_m) \in \mathbf{F}^n(\mu_{n,m}(z_m))$. Now $\mathbf{F}^1$ is upper semicontinuous and countably condensing (so in particular hemicompact) so we can deduce that the sequence $\{\mu_{1,m}(z_m)\}_{m\in N}$ has a subsequence $\{\mu_{1,m}(z_m)\}_{m\in N_1}$ which converges to $x_1 \in \overline{X^1}$ with $x_1 \in \mathbf{F}^1(x_1)$. Take $N_1^\star = \{m \in N_1 : m \geq 2\}$. The same argument applied to $\{\mu_{2,m}(z_m)\}_{m\in N_1^\star}$ guarantees the existence of a subsequence $\{\mu_{2,m}(z_m)\}_{m\in N_2}$ which converges to $x_2 \in \overline{X^2}$ with $x_2 \in \mathbf{F}^2(x_2)$. Moreover (by uniqueness of limits) $\mu_{1,2}(x_2) = x_1$. Repeating this argument we obtain

$$\ldots\ldots\ldots\ldots\ \subseteq N_3 \subseteq N_2^\star \subseteq N_2 \subseteq N_1^\star \subseteq N_1 \subseteq N$$

and for every $n \in N$, $x_n \in \overline{X^n}$ with

$$x_n \in \mathbf{F}^n(x_n) \ \text{ and } \ \{\mu_{n,m}(z_m)\}_{m\in N_n} \ \text{ converges to } x_n.$$

By a diagonalization process we deduce the existence of a sequence $\{z_m\}_{m\in N_0}$ such that $\{\mu_{n,m}(z_m)\}_{m\in N_0}$ converges to x_n for every $n \in N$. Lemma 2.1 guarantees the existence of a $x \in X$ with $\mu_n(x) \in \mathbf{F}^n(\mu_n(x))$ for every $n \in N$.

To finish the proof it remains to show $x = f(x)$. If it was false then there exists a $n \in N$ with $|x - f(x)|_n = d > 0$. Now since f is admissible there exists $m \geq n$ with

$$\frac{d}{2} > diam_n(f(S_m(x)) = diam_n(co(f(S_m(x))).$$

Also $\mu_m(x) \in \mathbf{F}^m(\mu_m(x))$ guarantees that there exists $y \in co(f(S_m(x))$ with $|x - y|_m < \frac{d}{2}$. Thus

$$d = |x - f(x)|_n \leq |x - y|_n + |y - f(x)|_n < |x - y|_m + \frac{d}{2} < d,$$

a contradiction. □

Theorem 2.3. *Let X be a closed subset of a Fréchet space $E = (E, \{|\cdot|_n\}_{n\in N})$, $f : X \to E$ an admissible map and $0 \in pseudo-int\,(X)$. In addition assume for each $n \in N$ that*

$$z \notin \lambda\mathbf{F}^n(z) \quad \textit{for} \ \lambda \in [0,1] \ \textit{and} \ z \in \partial X^n.$$

Then f has a fixed point.

PROOF: From [2, Theorem 2.5] we know for each $n \in N$ that $\mathbf{F}^n$ has a fixed point $z_n \in \overline{X^n}$. Essentially the same argument as in Theorem 2.2 guarantees the existence of an $x \in X$ with $x = F(x)$. $\square$

Next we present a Krasnoselskii type theorem for k-admissible maps (described below). We use the notation introduced in [10]. For every $n \in N$, let $D(n) \subseteq \mathbf{E}^n$ and we define

$$D(\infty) = \{x \in E : \exists N_0 \subseteq N \text{ infinite and } z_n \in D(n) \text{ for } n \in N_0 \text{ such that } \forall n \in N,\ \mu_{n,m}(z_m) \to \mu_n(x) \text{ as } m \to \infty \text{ with } m \in N_0 \text{ and } m \geq n\}.$$

Definition 2.2. Let X be a subset of E. A map $f : X \to E$ is called *k-admissible* if $0 \leq k < 1$ and if for every $n \in N$,

(i). the multimap $F^n : X^n \to \mathbf{E}^n$ defined by

$$F^n(\mu_n(x)) = \overline{co}\,(\mu_n \circ f \circ S_n(x))$$

admits an upper semicontinuous countably k-set contractive extension $\mathbf{F}^n : \overline{X^n} \to \mathbf{E}^n$ with convex, compact values,

(ii). for every $\epsilon > 0$, $\exists m \geq n$ such that for every $x \in X$ we have $diam_n(f(S_m(x))) < \epsilon$.

Remark 2.1. Notice that if Y is a closed convex subset of E and $f : X \to E$ an admissible (or k-admissible) map such that $f(X) \subseteq Y$, then for every $n \in N$, the extension $\mathbf{F}^n$ can be chosen such that $\mathbf{F}^n(\overline{X^n}) \subseteq \overline{Y^n}$ since $F^n(X^n) \subseteq \overline{Y^n}$. Indeed, otherwise, its intersection with $\overline{Y^n}$ is also an extension of F^n.

We now present a slight generalization of a result in [10] (the argument is essentially the same).

Theorem 2.4. *Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space, C a closed cone in E, $f : C \to C$ a k-admissible map, and assume for every $n \in N$ that $|\cdot|_n$ is increasing with respect to $\overline{C^n}$. Suppose there exists $R > r > 0$ such that for every $n \in N$,*

(1). $|y|_n > r \ \ \forall\, y \in \mathbf{F}^n(x), \ \ \forall\, x \in \partial B_n(0, r) \cap \overline{C^n}$;

(2). $|y|_n \leq R \ \ \forall\, y \in \mathbf{F}^n(x), \ \ \forall\, x \in \partial B_n(0, R) \cap \overline{C^n}$;

or

(1'). $|y|_n \leq r \ \ \forall\, y \in \mathbf{F}^n(x), \ \ \forall\, x \in \partial B_n(0, r) \cap \overline{C^n}$;

(2'). $|y|_n > R \ \ \forall\, y \in \mathbf{F}^n(x), \ \ \forall\, x \in \partial B_n(0, R) \cap \overline{C^n}$.

Then there exists $x \in C \cap B(0, R) \cap D(\infty)$ with $x = f(x)$, here $D(n) = \overline{C^n} \cap \overline{B_n(0, R) \setminus B_n(0, r)}$ and $B(0, R) = \{x : |x|_n \leq r \ \forall n \in N\}$.

PROOF: Now since $f : C \to C$ is k-admissible, for every $n \in N$ we have $\mathbf{F}^n$ an upper semicontinuous countably k-set contractive extension of F^n defined by

$$F^n(\mu_n(x)) = \overline{co}\,(\mu_n \circ f \circ S_n(x)) \subseteq \overline{C^n}.$$

Observe for every $n \in N$ that $\overline{C^n}$ is a cone and from Remark 2.1 we can consider $\mathbf{F}^n : B_n(0,R) \cap \overline{C^n} \to \overline{C^n}$. From [3, Theorem 3.3 or Theorem 3.6] we have for each $n \in N$ that $\mathbf{F}^n$ has a fixed point $z_n \in D(n)$. Essentially the same argument as in Theorem 2.2 guarantees the existence of an $x \in D(\infty) \cap C \cap B(0,R)$ with $x = f(x)$. □

Remark 2.2. The last theorem is true for f defined on $C \cap B(0,R)$ if

(a). $\overline{C^n} \cap B_n(0,R) \subseteq \overline{\mu_n(C \cap B(0,R))}$ for every $n \in N$;

or

(b). the condition (i) in Definition 2.2 is replaced by:

(i'). $C^n \cap B_n(0,R) \subseteq \mu_n(C \cap B(0,R))$ and the multimap $F^n : C^n \cap B_n(0,R) \to \mathbf{E}^n$ defined by

$$F^n(\mu_n(x)) = \overline{co}\,(\mu_n \circ f \circ S_n(x))$$

admits an upper semicontinuous countably k-set contractive extension $\mathbf{F}^n : \overline{C^n} \cap B_n(0,R) \to \mathbf{E}^n$ with convex, compact values.

For the remainder of this section we present the second approach from the literature which is based also on assuming that there exists a sequence of maps, $\{F_n\}$, whose fixed points converge to a fixed point of the map, F, we are examining; to fulfill this a particular closure type property (see, for example, (2.7)) must be satisfied. In this approach this closure type property specifies the relationship between the maps F_n and F. This differs from the first approach where a specific map F_n was chosen for each $n \in N$.

Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with (2.1) holding. Assume for each $n \in N$ that $(E_n, |\cdot|_n)$ is a Banach space and suppose

$$E_1 \supseteq E_2 \supseteq \ldots\ldots\ldots\ldots$$

with $|x|_n \le |x|_{n+1}$ for all $x \in E_{n+1}$. Also assume $E = \cap_{n=1}^{\infty} E_n$ where $\cap_1^{\infty}$ is the generalized intersection as described in [13 pp 439] (i.e. E is the projective limit of $\{E_n\}_{n\in N}$) with the embedding $\mu_n : E \to E_n$. We are interested in showing that the inclusion

$$y \in F(y) \tag{2.3}$$

has a solution in E.

We begin by presenting a Leray–Schauder nonlinear alternative [1].

Theorem 2.5. *For each $n \in N$, let U_n be an open subset of E_n with $0 \in U_n$ and*

$$\overline{U_1} \supseteq \overline{U_2} \supseteq \ldots\ldots\ldots\,;$$

here $\overline{U_n}$ denotes the closure of U_n in E_n. Let $F : Y \to E$ for some $Y \subseteq E$ and suppose the following conditions are satisfied:

$$\left\{\begin{array}{l} \textit{for each } n \in N,\ F_n : \overline{U_n} \to AC(E_n) \textit{ is upper semicontinuous} \\ \textit{(here } AC(E_n) \textit{ denotes the family of nonempty, compact,} \\ \textit{acyclic subsets of } E_n\textit{);} \end{array}\right. \tag{2.4}$$

$$(2.5)\quad \begin{cases} \textit{for each } n \in N,\ y \notin \lambda F_n(y) \ \textit{ in } E_n \textit{ for all } \lambda \in (0,1) \textit{ and} \\ y \in \partial U_n \ (\textit{ here } \partial U_n \textit{ denotes the boundary of } U_n \textit{ in } E_n); \end{cases}$$

$$(2.6)\quad \begin{cases} \textit{for each } n \in N, \textit{ the map } \mathcal{K}_n : \overline{U_n} \to 2^{E_n}, \textit{ given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \ \textit{(see Remark 2.3), is} \\ \textit{countably condensing;} \end{cases}$$

and

$$(2.7)\quad \begin{cases} \textit{if there exists a } w \in E \textit{ and a sequence } \{y_n\}_{n\in N} \\ \textit{with } y_n \in \overline{U_n} \textit{ and } y_n \in F_n(y_n) \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \textit{ of } N \textit{ with } y_n \to w \textit{ in } E_k \\ \textit{as } n \to \infty \textit{ in } S, \textit{ then } w \in F(w) \textit{ in } E. \end{cases}$$

Then (2.3) *has a solution* y *in* E *(in fact* $\mu_n(y) \in \overline{U_n}$ *for every* $n \in N$*).*

Remark 2.3. The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{U_n}$ and $y \notin \overline{U_{n+1}}$ then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{U_{n+1}}$ and $y \notin \overline{U_{n+2}}$ then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

PROOF: From [5] we know for each $n \in N$ that F_n has a fixed point $y_n \in \overline{U_n}$ i.e $y_n \in F_n(y_n)$ in E_n. Lets look at $\{y_n\}_{n\in N}$. Now Theorem 1.3 (with $X = E_1$, $G = \mathcal{K}_1$, $D = \overline{U_1}$ and note $d_1(y_n, \mathcal{K}_1(y_n)) = 0$ for each $n \in N$ since $|x|_1 \leq |x|_n$ for all $x \in E_n$ and $y_n \in F_n(y_n)$ in E_n; here $d_1(x, Z) = \inf_{y\in Z} |x-y|_1$ for $Z \subseteq X$) guarantees that there exists a subsequence $N_1^\star$ of N and $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n\in N_1}$. Now Theorem 1.3 (with $X = E_2$, $G = \mathcal{K}_2$ and $D = \overline{U_2}$) guarantees that there exists a subsequence $N_2^\star$ of N_1 and $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $|z_2 - z_1|_1 = 0$ since $N_2^\star \subseteq N_1^\star$ and $E_1 \supseteq E_2$. Thus $z_2 = z_1$ in E_1. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ......, \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k = 1, 2, ...$ Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k (i.e. $\mu_k(y) = z_k$). Notice y is well defined and $y \in E$. Now $y_n \in F_n(y_n)$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (2.7) implies $y \in F(y)$ in E. □

Let F be defined on E_1. If $F_n = F|_{E_n}$ for each $n \in N$ then a slight modification of the argument in Theorem 2.5 yields the following result.

Theorem 2.6. *For each* $n \in N$, *let* U_n *be an open subset of* E_n *with* $0 \in U_n$. *Let* $F : E_1 \to E_1$ *and suppose the following conditions are satisfied:*

$$(2.8)\quad \begin{cases} \textit{for each } n \in N,\ F : \overline{U_n} \to AC(E_n) \textit{ is upper semicontinuous} \\ \textit{and countably condensing;} \end{cases}$$

$$(2.9) \qquad \begin{cases} \text{for each } n \in N, \ y \notin \lambda F(y) \ \text{ in } E_n \text{ for all} \\ \lambda \in (0,1) \ \text{ and } \ y \in \partial U_n; \end{cases}$$

and

$$(2.10) \qquad \begin{cases} \text{for each } \ n \in \{2, 3, \ldots.\} \ \text{ if } \ y \in \overline{U_n} \ \text{ solves } \ y \in F(y) \ \text{ in } \ E_n \\ \text{then } \ y \in \overline{U_k} \ \text{ for } \ k \in \{1, \ldots, n-1\}. \end{cases}$$

Then (2.3) *has a solution* y *in* E *(in fact* $\mu_n(y) \in \overline{U_n}$ *for every* $n \in N$*).*

PROOF: As in Theorem 2.5 we know for each $n \in N$ that F has a fixed point $y_n \in \overline{U_n}$. Lets look at $\{y_n\}_{n \in N}$. Now $y_1 \in \overline{U_1}$ and $y_k \in \overline{U_1}$ for $k \in N \setminus \{1\}$ from (2.10). Now Theorem 1.3 (with $X = E_1$, $D = \overline{U_1}$, $G = F$ and note $d_1(y_n, F(y_n)) = 0$ for each $n \in N$ since $|x|_1 \leq |x|_n$ for all $x \in E_n$ and $y_n \in F(y_n)$ in E_n; here $d_1(x, Z) = \inf_{y \in Z} |x - y|_1$ for $Z \subseteq X$) guarantees that there exists a subsequence $N_1^\star$ of N and $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Essentially the same reasoning as in Theorem 2.5 now establishes the result. □

Remark 2.4. The map in (2.4) (or (2.8)) can be replaced by any other map where there is a corresponding Leray–Schauder alternative in the Banach space setting (for example we could consider approximable maps); see [5] for other classes of maps.

Next we present a Schauder–Tychonoff type theorem in this setting. The proof is the same as that in Theorem 2.5 except in this case we quote the corresponding result for self maps in the Banach space setting [16].

Theorem 2.7. *For each* $n \in N$, *let* C_n *be a closed convex subset of* E_n *with*

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots\ldots\ldots .$$

Let $F : Y \to E$ *for some* $Y \subseteq E$ *and suppose the following conditions are satisfied:*

$$(2.11) \qquad \text{for each } \ n \in N, \ F_n : C_n \to AC(C_n) \ \text{ is upper semicontinuous};$$

$$(2.12) \qquad \begin{cases} \text{for each } \ n \in N, \ \text{ the map } \ \mathcal{K}_n : C_n \to 2^{E_n}, \ \text{ given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \ \text{ is countably condensing}; \end{cases}$$

and

$$(2.13) \qquad \begin{cases} \text{if there exists a } \ w \in E \ \text{ and a sequence } \ \{y_n\}_{n \in N} \\ \text{with } \ y_n \in C_n \ \text{ and } \ y_n \in F_n(y_n) \ \text{ in } \ E_n \ \text{ such that} \\ \text{for every } \ k \in N \ \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \ldots..\} \ \text{ of } \ N \ \text{ with } \ y_n \to w \ \text{ in } \ E_k \\ \text{as } \ n \to \infty \ \text{ in } \ S, \ \text{ then } \ w \in F(w) \ \text{ in } \ E. \end{cases}$$

Then (2.3) *has a solution* y *in* E *(in fact* $\mu_n(y) \in C_n$ *for every* $n \in N$*).*

Theorem 2.8. *For each $n \in N$, let C_n be a closed convex subset of E_n, $F : E_1 \to E_1$ and suppose the following conditions are satisfied:*

$$(2.14) \quad \begin{cases} \text{for each } n \in N,\ F : C_n \to AC(C_n) \text{ is upper semicontinuous} \\ \text{and countably condensing;} \end{cases}$$

and

$$(2.15) \quad \begin{cases} \text{for each } n \in \{2,3,....\} \text{ if } y \in C_n \text{ solves } y \in F(y) \text{ in } E_n \\ \text{then } y \in C_k \text{ for } k \in \{1,...,n-1\}. \end{cases}$$

Then (2.3) has a solution y in E (in fact $\mu_n(y) \in C_n$ for every $n \in N$).

Finally we present a Krasnoselskii–Petryshyn type result [15] in this setting. For this result for $n \in N$, C_n will be a cone in E_n and for $\rho > 0$ we will let

$$U_{n,\rho} = \{x \in E_n : \ |x|_n < \rho\} \quad \text{and} \quad \Omega_{n,\rho} = U_{n,\rho} \cap C_n.$$

Notice

$$\partial_{C_n} \Omega_{n,\rho} = \partial_{E_n} U_{n,\rho} \cap C_n \quad \text{and} \quad \overline{\Omega_{n,\rho}} = \overline{U_{n,\rho}} \cap C_n$$

(the first closure is with respect to C_n whereas the second is with respect to E_n).

Theorem 2.9. *For each $n \in N$, let C_n be a cone in E_n and assume $|\cdot|_n$ is increasing with respect to C_n and also that*

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq ....$$

Also $F : Y \to E$ for some $Y \subseteq E$. Let γ, r, R be constants with $0 < \gamma < r < R$ and we suppose the following conditions are satisfied:

$$(2.16) \quad \begin{cases} \text{for each } n \in N,\ F_n : \overline{U_{n,R}} \cap C_n \to CK(C_n) \\ \text{is a upper semicontinuous map (here } CK(C_n) \text{ denotes the} \\ \text{family of nonempty, convex, compact subsets of } C_n\text{);} \end{cases}$$

$$(2.17) \quad \text{for each } n \in N,\ |y|_n \le |x|_n \ \forall\, y \in F_n(x),\ \forall x \in \partial_{E_n} U_{n,r} \cap C_n;$$

$$(2.18) \quad \text{for each } n \in N,\ |y|_n > |x|_n \ \forall\, y \in F_n(x),\ \forall x \in \partial_{E_n} U_{n,R} \cap C_n;$$

$$(2.19) \quad \begin{cases} \text{for each } n \in N, \text{ the map } K_n : \overline{U_{n,R}} \cap C_n \to 2^{C_n} \\ \text{given by } K_n(y) = \cup_{m=n}^{\infty} F_m(y) \text{ is countably } k\text{-set} \\ \text{contractive (here } 0 \le k < 1\text{);} \end{cases}$$

$$(2.20) \quad \begin{cases} \text{for every } k \in N \text{ and any subsequence } A \subseteq \{k, k+1, ...\} \\ \text{if } x \in C_n,\ n \in A, \text{ is such that } R \ge |x|_n \ge r \\ \text{then } |x|_k \ge \gamma; \end{cases}$$

and

$$
(2.21)\qquad \begin{cases} \textit{if there exists a } w \in E \textit{ and a sequence } \{y_n\}_{n\in N} \\ \textit{with } y_n \in \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \cap C_n \textit{ and } y_n \in F_n(y_n) \textit{ in } E_n \\ \textit{such that for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \textit{ of } N \textit{ with } y_n \to w \textit{ in } E_k \\ \textit{as } n \to \infty \textit{ in } S, \textit{ then } w \in F(w) \textit{ in } E. \end{cases}
$$

Then (2.3) *has a solution* y *in* E *(in fact* $\mu_n(y) \in \left(\overline{U_{n,R}} \setminus U_{n,\gamma}\right) \cap C_n$ *for every* $n \in N$*).*

PROOF: From [3] we have for each $n \in N$ that F_n has a fixed point $y_n \in \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \cap C_n$. Lets look at $\{y_n\}_{n\in N}$. Note $y_n \in \overline{U_{1,R}} \setminus U_{1,\gamma}$ for each $n \in N$. To see this notice $|y_n|_n \leq R$ and $|x|_1 \leq |x|_n$ for all $x \in E_n$ implies $|y_n|_1 \leq R$, and so $y_n \in \overline{U_{1,R}}$ for each $n \in N$. On the other hand $|y_n|_n \geq r$, $y_n \in C_n$ together with (2.20) implies $|y_n|_1 \geq \gamma$. Now Theorem 1.3 (with $X = E_1$, $G = \mathcal{K}_1$, $D = \left(\overline{U_{1,R}} \setminus U_{1,\gamma}\right) \cap C_1$ and note $d_1(y_n, \mathcal{K}_1(y_n)) = 0$ for each $n \in N$) guarantees that there exists a subsequence $N_1^\star$ of N and $z_1 \in \left(\overline{U_{1,R}} \setminus U_{1,\gamma}\right) \cap C_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Notice in particular that $\gamma \leq |z_1|_1 \leq R$.

Proceed inductively to obtain subsequences of integers

$$
N_1^\star \supseteq N_2^\star \supseteq ......, \quad N_k^\star \subseteq \{k, k+1, ...\}
$$

and $z_k \in \left(\overline{U_{k,R}} \setminus U_{k,\gamma}\right) \cap C_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k = 1, 2, ...$ Also let $N_k = N_k^\star \setminus \{k\}$.

Essentially the same reasoning as in Theorem 2.5 now establishes the result. □

Remark 2.5. Of course there is an analogue of Theorem 2.9 when $U_{n,r}$ is replaced by $U_{n,R}$ in (2.17) and $U_{n,R}$ is replaced by $U_{n,r}$ in (2.18).

Remark 2.6. It is also possible to replace (2.18) in Theorem 2.9 by a Leggett Williams condition; see [4].

Remark 2.7. Applications of special cases of the theorems in this paper may be found in [1, 4, 7, 10, 15].

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