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On a notion of category depending on a functional, Part I: Theory and application to critical point theory[☆]

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ABSTRACT

We introduce the notion of category depending on a fixed functional f defined on a topological space. This notion permits us to obtain a better lower bound to the number of critical points of f than the bound obtained with the Lusternik–Schnirelman category. We also introduce the notion of truncated category depending on the functional f which permits us to obtain a lower bound to the number of critical points of unbounded functionals. Finally, we extend our result after having introduced the notions of relative category and limit relative category depending on the functional f.

1. Introduction

Lusternik and Schnirelman [1] introduced in 1934 a topological invariant for compact manifolds X called *category* and noted $\text{cat}(X)$. They showed using minimax methods that $\text{cat}(X)$ gives a lower bound to the number of critical points of C^1 functionals $f : X \to \mathbb{R}$. This theory was extended to Riemannian manifolds by Schwartz [2] in 1964, and to Finsler manifolds by Palais [3] in 1966. The main ingredients to obtain critical point results to C^1 functionals bounded from below are a compactness condition (called Palais–Smale condition), and a deformation lemma ensuring the existence of a continuous deformation η such that $f(\eta(x,t)) \le f(x)$; see also [4]. The reader who wants to learn more on the Lusternik–Schnirelman category is referred to the monograph [5].

In order to consider functionals unbounded from below, Reeken [6] introduced a notion of relative category. This notion was forgotten and rediscovered by Fournier and Willem [7,8].

The relative category was hardly applicable to obtain a lower bound to the number of critical points of strongly indefinite functionals. This could be understood by the fact that, in an infinite dimensional Hilbert space, the relative category of a closed ball B relative to its boundary ∂B in B is trivial ($\text{cat}_{B,\partial B}(B) = 0$). To overcome this problem, Fournier, Lupo, Ramos and Willem [9] introduced the notion of limit relative category. Here the main ingredients were a Galerkin argument, a deformation lemma ensuring the existence of a suitable family of deformations, and the compactness condition (called Palais–Smale-star condition $(\text{PS})^*$) introduced by Bahri and Berestycki [10], and independently by Liu and Li [11].

The notion of relative category was also extended to metric spaces X and applied to the critical point theory of continuous functionals $f : X \to \mathbb{R}$ by Canino and Degiovanni [12]. In this context, x is a critical point of f if $|df|(x) = 0$, where $|df|$ is the weak slope introduced by Degiovanni and Marzocchi [13]. Deformation lemmas obtained by Corvellec, Degiovanni and Marzocchi [14] were used. In 1980, De Giorgi, Marino and Tosques [15] were the first to introduce a notion of slope for discontinuous functionals defined on metric space.

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All those notions of category depend only on the space. Hence the lower bound to the number of critical points of f is the same for every suitable functional $f : X \to \mathbb{R}$. So, in many cases, the number of critical points of f is much larger than the lower bound given by the category. In this paper, following [16], we introduce notions of category which take into account the functional. A first step was done in this direction by Szulkin [17]. Our category is always bigger or equal than Szulkin's one.

Our paper is organized as follows. In Section 3, we introduce the notion of category depending on a fixed functional f defined on a topological space X. Roughly speaking, we cover a set by contractible sets with suitable deformations η. In particular, it will be fine if $f(\eta(x,t)) \le f(x)$. We also introduce the notions of relative category and truncated category depending on the functional f. Their properties are studied. It is worthwhile to point out that with our notions, it cannot be shown that a compact subset has a closed neighborhood with the same category. This fact was fundamental with the Lusternik–Schnirelman category.

In Section 4, we discuss the relation between the category depending on f and the notion of linking introduced by Frigon [18]. This notion of linking contains and extends the classical notions of linking. A particular attention is given to the linking of $(B_1 \times M, S_1 \times M)$ with $(B_2 \times M, S_2 \times M)$, where M is a compact manifold, $X = E_1 \oplus E_2$ is a Banach space with $0 < \dim(E_1) < \infty$, B_i and S_i are respectively the closed unit ball and the unit sphere in E_i. In fact, the linking of (B_1, S_1) with (B_2, S_2) corresponds to the notion of splitting spheres introduced by Marino, Micheletti and Pistoia [19]. Let us mention that the notion of splitting spheres generalizes the notion of local linking; see [20,11].

In Section 5, we consider a metric space X and $f : X \to \mathbb{R}$ a continuous functional satisfying a suitable deformation property. We show that our notions of category depending on f permit to obtain a lower bound to the number of critical points of f. Let us mention that our notion of truncated category permits us to consider unbounded functionals (see Theorem 5.9); there are no analogous result in the literature.

Finally, we introduce the notion of limit relative category depending on the functional f and we study its properties. An application of our results to Hamiltonian systems is presented in [21] in which the existence and the multiplicity of solutions are obtained.

2. Preliminaries

Let X be a topological space. For $A, B \subset X$, $\eta_1 : A \times [0,1] \to X$ and $\eta_2 : B \times [0,1] \to X$ continuous such that $\eta_i(x,0) = x$ and $\eta_1(A,1) \subset B$, we define $\eta_2 \star \eta_1 : A \times [0,1] \to X$ by

$$\eta_2 \star \eta_1(x,t) = \begin{cases} \eta_1(x,2t), & \text{if } t \in [0,1/2], \\ \eta_2(\eta_1(x,1), 2t-1), & \text{if } t \in]1/2, 1]. \end{cases}$$

Let $f : X \to \mathbb{R}$ be a functional. For $a \in \mathbb{R}$, we denote

$$f^a = \{x \in X : f(x) \le a\},$$

and $f^\infty = X$.

We define

$$\mathcal{N}_f = \{\eta : X \times [0,1] \to X : \eta \text{ is continuous}, \eta(x,0) = x \text{ and}, f(\eta(x,t)) \le f(x)\ \forall x \in X,\ t \in [0,1]\}.$$

It is well known that the *Lusternik–Schnirelman category of* $B \subset X$, noted $\mathrm{cat}_X(B)$, is the smallest $n \in \mathbb{N}$ such that B can be covered by n closed contractible subsets of X. If such a n does not exist, set $\mathrm{cat}_X(B) = \infty$.

Let us recall the definition of relative category presented in [17] which is a variant of the relative category introduced by Reeken [6] and rediscovered by Fournier and Willem [7,8].

Definition 2.1. Let B, Y be closed in X. The *category of* B *relative to* Y *in* X, noted $\mathrm{cat}_{X,Y}(B)$, is the smallest $n \in \mathbb{N} \cup \{0\}$ such that there exist closed sets $A_0, \dots, A_n$ such that $B \subset \bigcup_{i=0}^n A_i$, $A_1, \dots, A_n$ are contractible in X, and there exists a continuous deformation $\eta_0 : A_0 \cup Y \times [0,1] \to X$ such that $\eta_0(x,0) = x$, $\eta_0(Y,t) \subset Y$, and $\eta_0(A_0, 1) \subset Y$.

The notion of relative cuplength was introduced by Fournier and Willem [7]; see also [9] or [17]. For $A \subset B$ two closed subsets of $\mathbb{R}^m$, we denote by $H^*(B,A)$ the Čech cohomology of (B,A) with coefficients in $\mathbb{Z}_2$. For arbitrary metrizable pair (X,Y), we use the Alexander cohomology as defined in [22, Chapter 6] with the fact that the Alexander and Čech cohomologies are isomorphic for paracompact spaces, and the tautness property for Alexander cohomology.

Definition 2.2. Let A, B be two closed subsets of $\mathbb{R}^m$. The *cuplength of* B *relative to* A, noted $\mathrm{cuplength}(B,A)$, is the largest $n \in \mathbb{N} \cup \{0\}$ for which there exist $\alpha_0, \dots, \alpha_n$ such that $\alpha_0 \in H^{q_0}(B,A)$, $\alpha_k \in H^{q_k}(B)$ and $q_k \ge 1$ for $k \ge 1$, and

$$\alpha_0 \bigcup \cdots \bigcup \alpha_n \ne 0 \quad \text{in } H^q(B,A) \text{ with } q = q_0 + \cdots + q_n,$$

where "$\bigcup$" denotes the cup product. If such a n does not exist, we define $\mathrm{cuplength}(B,A) = -1$.

Let $E = E_1 \oplus E_2$ be a Banach space with $0 < \dim(E_1) < \infty$, and M a compact manifold. Let $r_1, r_2 > 0$, we denote for $i = 1, 2$,

$$B_i = \{x \in E_i : \|x\| \le r_i\}$$
$$S_i = \{x \in E_i : \|x\| = r_i\}.$$

The next result follows from Marino [23].

Theorem 2.3. *Let Y be a closed set such that $S_1 \times M \subset Y \subset (E \setminus B_2) \times M$. Then*

$$\mathrm{cat}_{(E\setminus S_2)\times M, Y}(B_1 \times M) \geq \mathrm{cuplength}(B_1 \times M, S_1 \times M) + 1.$$

Proof. Assume that $\mathrm{cat}_{(E\setminus S_2)\times M,Y}(B_1 \times M) = m < \infty$. There exits $A_0, \ldots, A_m$ as in Definition 2.1. Without lost of generality, we can assume that $B_1 \times M = A_0 \cup \cdots \cup A_m$.

Let $\alpha_0, \ldots, \alpha_m$ be such that $\alpha_0 \in H^{q_0}(B_1 \times M, S_1 \times M)$, $\alpha_k \in H^{q_k}(B_1 \times M)$ with $q_k \geq 1$ for $k \geq 1$.

First of all, observe that the inclusion

$$l : (B_1 \times M, S_1 \times M) \to ((E \setminus S_2) \times M, (E \setminus B_2) \times M)$$

induces the isomorphism

$$l^{q_0} : H^{q_0}((E \setminus S_2) \times M, (E \setminus B_2) \times M) \to H^{q_0}(B_1 \times M, S_1 \times M).$$

Indeed, $B_1 \times M$ and $S_1 \times M$ are respectively deformation retracts of $(B_1 \oplus \mathrm{int}(B_2)) \times M$ and $((B_1 \oplus \mathrm{int}(B_2)) \setminus B_2) \times M$. Observe that

$$(B_1 \oplus \mathrm{int}(B_2), (B_1 \oplus \mathrm{int}(B_2)) \setminus B_2) = ((E \setminus S_2) \setminus Z, (E \setminus B_2) \setminus Z),$$

where

$$Z = ((E_1 \oplus (E_2 \setminus \mathrm{int}(B_2))) \setminus S_2) \cup ((E_1 \setminus B_1) \oplus E_2).$$

Those observations imply that l^{q_0} is an isomorphism since the closure of Z in $E \setminus S_2$ is contained in $E \setminus B_2$. The fact that l^{q_0} is an isomorphism, and the exactness of the sequence associated to the inclusions permit us to deduce that the following map κ^{q_0} is surjective

$$H^{q_0}((E \setminus S_2) \times M, (E \setminus B_2) \times M) \longrightarrow H^{q_0}((E \setminus S_2) \times M, Y) \xrightarrow{\kappa^{q_0}} H^{q_0}(B_1 \times M, S_1 \times M).$$

The properties of A_0 imply that the inclusion

$$H^{q_0}((E \setminus S_2) \times M, Y) \to H^{q_0}(A_0 \cup Y, Y)$$

is trivial. Thus, from the exact sequence

$$H^{q_0}((E \setminus S_2) \times M, A_0 \cup Y) \xrightarrow{h^{q_0}} H^{q_0}((E \setminus S_2) \times M, Y) \xrightarrow{0} H^{q_0}(A_0 \cup Y, Y),$$

we deduce that h^{q_0} is surjective.

Therefore from the inclusions, we obtain the commutative diagram

$$\begin{array}{ccccc} H^{q_0}\big((E\setminus S_2)^M, A_0 \cup Y\big) & \xrightarrow{h^{q_0}} & H^{q_0}\big((E\setminus S_2)^M, Y\big) & \xrightarrow{0} & H^{q_0}(A_0 \cup Y, Y) \\ \downarrow & & \downarrow \kappa^{q_0} & & \downarrow \\ H^{q_0}\big(B_1^M, A_0 \cup S_1^M\big) & \xrightarrow{j^{q_0}} & H^{q_0}(B_1^M, S_1^M) & \longrightarrow & H^{q_0}\big(A_0 \cup S_1^M, S_1^M\big) \end{array}$$

where for a set S, S^M means $S \times M$. So there exists $\widetilde{\alpha}_0 \in H^{q_0}(B_1^M, A_0 \cup S_1^M)$ such that $\alpha_0 = j^{q_0}(\widetilde{\alpha}_0)$.

Similarly, for $k \geq 1$, $H^{q_k}((E \setminus S_2) \times M) \to H^{q_k}(A_k)$ is trivial since A_k is contractible in $(E \setminus S_2) \times M$. Thus, from the exact sequence

$$H^{q_k}((E \setminus S_2) \times M, A_k) \xrightarrow{h^{q_k}} H^{q_k}((E \setminus S_2) \times M) \xrightarrow{0} H^{q_k}(A_k),$$

we deduce that h^{q_k} is surjective. Again, in the following commutative diagram

$$\begin{array}{ccccc} H^{q_k}\big((E\setminus S_2) \times M, A_k\big) & \xrightarrow{h^{q_k}} & H^{q_k}\big((E\setminus S_2) \times M\big) & \xrightarrow{0} & H^{q_k}(A_k) \\ \downarrow & & \downarrow \kappa^{q_k} & & \downarrow \\ H^{q_k}(B_1 \times M, A_k) & \xrightarrow{j^{q_k}} & H^{q_k}(B_1 \times M) & \longrightarrow & H^{q_k}(A_k) \end{array}$$

the contractibility of B_1 ensures that κ^{q_k} is surjective. Therefore, we deduce that there exists $\widetilde{\alpha}_k \in H^{q_k}(B_1 \times M, A_k)$ such that $\alpha_k = j^{q_k}(\widetilde{\alpha}_k)$.

Consequently, the compactness of $B_1, S_1, M, A_0, \ldots, A_m$, the theory of cohomology and the naturality of the cup product (see [24,22]) imply that

$$\alpha_0 \bigcup \cdots \bigcup \alpha_m = j^q \left(\widetilde{\alpha}_0 \bigcup \cdots \bigcup \widetilde{\alpha}_m\right) \in H^q(B_1 \times M, S_1 \times M), \quad q = \sum_{k=0}^{m} q_m,$$

where

$$j^q : H^q(B_1 \times M, A_0 \cup \cdots \cup A_m \cup (S_1 \times M)) \to H^q(B_1 \times M, S_1 \times M).$$

Since $A_0 \cup \cdots \cup A_m \cup (S_1 \times M) = B_1 \times M$, it follows that

$$\alpha_0 \bigcup \cdots \bigcup \alpha_m = 0. \quad \square$$

The following result is due to Szulkin [17].

Lemma 2.4. *Let B_1, S_1 and M be as above. Then*

$$\text{cuplength}(B_1 \times M, S_1 \times M) \geq \text{cuplength}(M).$$

3. Notions of category depending on a functional

We introduce notions of category depending on a given functional. Those notions will be useful to obtain a better lower bound to the number of critical points of the functional than the lower bound given by the classical Lusternik–Schnirelman category.

In what follows, X denotes a topological space, and $f : X \to \mathbb{R}$ is a functional fixed from the beginning.

Definition 3.1. Let A be a subset of X and $\varepsilon > 0$, we say that A is (f, ε)-*contractible in* X if there exist $\hat{x} \in X$ and a continuous deformation $\eta : A \times [0, 1] \to X$ such that for all $x \in A$,

(a) $\eta(x, 0) = x$;
(b) $\eta(x, 1) = \hat{x}$;
(c) $f(\eta(x, t)) \leq f(x) + \varepsilon$ for all $t \in [0, 1]$.

The previous notion of (f, ε)-contractibility permits us to introduce the category depending on the functional f.

Definition 3.2. Let $B \subset X$. We define the *f-category of B in X* by

$$f\text{-cat}_X(B) = \sup_{\varepsilon > 0} n_\varepsilon^f(B, X),$$

where $n_\varepsilon^f(B, X)$ is the smallest $n \in \mathbb{N}$ such that there exist closed subsets $A_1, \ldots, A_n$ satisfying:

(a) $B \subset \bigcup_{i=1}^n A_i$;
(b) A_i is (f, ε)-contractible in X, for all $i = 1, \ldots, n$.

If such a n does not exist, we set $n_\varepsilon^f(B, X) = \infty$, and if $B = \emptyset$, we set $n_\varepsilon^f(B, X) = 0$.

Remark 3.3. Observe that if $f\text{-cat}_X(B) < \infty$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in\ ]0, \varepsilon_0]$,

$$f\text{-cat}_X(B) = n_\varepsilon^f(B, X).$$

Notice also that $\varepsilon \mapsto n_\varepsilon^f(B, X)$ is nonincreasing and hence

$$f\text{-cat}_X(B) = \lim_{\varepsilon \to 0} n_\varepsilon^f(B, X).$$

Examples 3.4. Very often, $f\text{-cat}_X(B) > \text{cat}_X(B)$, we give two examples.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4 - 5x^2 + 4$. Then $f\text{-cat}_\mathbb{R}(\mathbb{R}) = 2$ and $\text{cat}_\mathbb{R}(\mathbb{R}) = 1$.
(2) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x \sin(1/x)$. Then $f\text{-cat}_\mathbb{R}([-r, r]) = \infty$ and $\text{cat}_\mathbb{R}([-r, r]) = 1$ for every $r > 0$.

Most of the usual properties of the category are satisfied by the f-category.

Theorem 3.5. *Let A, B be subsets of X. The following properties are satisfied:*

(i) $\text{cat}_X(B) \leq f\text{-cat}_X(B)$;
(ii) $f\text{-cat}_X(B) = 0$ *if and only if* $B = \emptyset$;
(iii) *if* $f\text{-cat}_X(B) = 1$, *then* $\overline{B}$ *is* (f, ε)-*contractible for every* $\varepsilon > 0$;
(iv) *if* $A \subset B$, *then* $f\text{-cat}_X(A) \leq f\text{-cat}_X(B)$;
(v) $f\text{-cat}_X(A \cup B) \leq f\text{-cat}_X(A) + f\text{-cat}_X(B)$;
(vi) *if* $\eta : \overline{B} \times [0, 1] \to X$ *is continuous and satisfies* $\eta(x, 0) = x$ *and* $f(\eta(x, t)) \leq f(x)$, *then* $f\text{-cat}_X(B) \leq f\text{-cat}_X(\eta(B, 1))$;
(vii) *if* $f\text{-cat}_X(B) < \infty$, *there exists* $\overline{\varepsilon} > 0$ *such that for every* $\varepsilon \in\ ]0, \overline{\varepsilon}[$, *and every* $\eta : \overline{B} \times [0, 1] \to X$ *continuous and satisfying* $\eta(x, 0) = x$ *and* $f(\eta(x, t)) \leq f(x) + \varepsilon$, *then* $f\text{-cat}_X(B) \leq f\text{-cat}_X(\eta(B, 1))$;
(viii) *if* $f\text{-cat}_X(B) = \infty$, *then for every* $k \in \mathbb{N}$, *there exists* $\overline{\varepsilon} > 0$ *such that for every* $\varepsilon \in\ ]0, \overline{\varepsilon}[$, *and every* $\eta : \overline{B} \times [0, 1] \to X$ *continuous and satisfying* $\eta(x, 0) = x$ *and* $f(\eta(x, t)) \leq f(x) + \varepsilon$, *one has* $f\text{-cat}_X(\eta(B, 1)) \geq k$.

Proof. (i) It is clear that for every $\varepsilon > 0$, $\text{cat}_X(B) \le n^f_\varepsilon(B, X) \le f\text{-cat}_X(B)$.

(ii) and (iii) are obvious.

(iv) and (v) follow directly from the fact that for every $\varepsilon > 0$, $n^f_\varepsilon(A, X) \le n^f_\varepsilon(B, X)$ when $A \subset B$, and $n^f_\varepsilon(A \cup B, X) \le n^f_\varepsilon(A, X) + n^f_\varepsilon(B, X)$.

(vi), (vii) and (viii): Observe that for every $\varepsilon \ge 0$ and every $\eta : \overline{B} \times [0, 1] \to X$ continuous such that $\eta(x, 0) = x$ and $f(\eta(x, t)) \le f(x) + \varepsilon$,

$$n^f_{\varepsilon+\delta}(B, X) \le n^f_\delta(\eta(B, 1), X) \quad \forall \delta > 0. \tag{3.1}$$

Indeed, if $n^f_\delta(\eta(B, 1), X) = k$, for $i = 1, \ldots, k$, there exists A_i closed and (f, δ)-contractible with a deformation η_i, and such that $\eta(B, 1) \subset \bigcup_{i=1}^k A_i$. Denote $C_i = \{x \in \overline{B} : \eta(x, 1) \in A_i\}$. It is easy to verify C_i is $(f, \varepsilon + \delta)$-contractible with the deformation $\eta_i \star \eta$ and $B \subset \bigcup_{i=1}^k C_i$. Hence $n^f_{\varepsilon+\delta}(B, X) \le k$.

Now, if for some $m \in \mathbb{N}$, $f\text{-cat}_X(B) \ge m$, there exists $\overline{\varepsilon} > 0$ such that for every $\varepsilon_1 \in\ ]0, \overline{\varepsilon}]$, $n^f_{\varepsilon_1}(B, X) \ge m$. For $\varepsilon \in [0, \overline{\varepsilon}[$ and $\eta : \overline{B} \times [0, 1] \to X$ continuous such that $\eta(x, 0) = x$ and $f(\eta(x, t)) \le f(x) + \varepsilon$, we deduce from (3.1) that for every $\delta \in\ ]0, \overline{\varepsilon} - \varepsilon]$,

$$n^f_\delta(\eta(B, 1)) \ge n^f_{\varepsilon+\delta}(B, X) \ge m.$$

Therefore $f\text{-cat}_X(\eta(B, 1)) \ge m$. □

Remark 3.6. Observe that $f\text{-cat}_X(B) = 1$ does not imply that there exists a continuous deformation $\eta : B \times [0, 1] \to X$ such that $\eta(x, 0) = x$, $\eta(x, 1) = \hat{x}$, and $f(\eta(x, t)) \le f(x)$. This can be easily seen with $X = B = \mathbb{R}$ and $f(x) = e^x$.

The Lusternik–Schnirelman category depends only on the space X. Here, on the contrary, our notion of category depends strongly on the functional. The next result establishes that if the $f\text{-cat}_X(B) < \infty$, then small perturbations of the functional increase the category of B associated to the perturbed functional.

Proposition 3.7. *Let $B \subset X$, and $f, g : X \to \mathbb{R}$.*

(i) *For every $k \in \mathbb{N}$ such that $f\text{-cat}_X(B) \ge k$, there exists $\delta > 0$ such that if $\sup\{|f(x) - g(x)| : x \in B\} \le \delta$, then $g\text{-cat}_X(B) \ge k$. In particular, if $f\text{-cat}_X(B) < \infty$, there exists $\delta > 0$ such that if $\sup\{|f(x) - g(x)| : x \in B\} \le \delta$, then*

$$f\text{-cat}_X(B) \le g\text{-cat}_X(B).$$

(ii) *If $\sup\{|f(x) - g(x)| : x \in B\} = \beta$, then for every $\varepsilon > 0$,*

$$n^g_\varepsilon(B, X) \ge n^f_{\varepsilon+2\beta}(B, X).$$

Proof. (i) If $f\text{-cat}_X(B) \ge k \in \mathbb{N}$, there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in\ ]0, \hat{\varepsilon}]$, $n^f_\varepsilon(B, X) \ge k$. Choose $\delta < \hat{\varepsilon}/2$. If

$$\sup\{|f(x) - g(x)| : x \in B\} \le \delta \quad \text{and} \quad g\text{-cat}_X(B) = m < k,$$

there exists $\bar{\varepsilon}$ such that for all $\varepsilon \in\ ]0, \bar{\varepsilon}]$, $n^g_\varepsilon(B, X) = m$. Let $\varepsilon < \min\{\bar{\varepsilon}, \hat{\varepsilon} - 2\delta\}$. For $i = 1, \ldots, m$, there exist a closed set A_i and an associated continuous deformation $\eta_i : A_i \times [0, 1] \to X$ given by Definition 3.2 such that $g(\eta_i(x, t)) \le g(x) + \varepsilon$, and $B \subset \bigcup_{i=1}^k A_i$. Denote $C_i = \overline{B} \cap A_i$. For every $x \in C_i$ and $t \in [0, 1]$,

$$f(\eta_i(x, t)) \le g(\eta_i(x, t)) + \delta \le g(x) + \varepsilon + \delta \le f(x) + \varepsilon + 2\delta.$$

Hence, $n^f_{\varepsilon+2\delta}(B, X) \le m$; contradiction.

(ii) The argument is analogous to the previous one. □

Examples 3.8. Here is a simple example to show that the previous proposition is optimal in the sense that equality cannot be obtained. Take $f(x) = x^2$ and $g(x) = x^2(1 + c\sin(1/x))$. Obviously $f\text{-cat}_{\mathbb{R}}([-1, 1]) = 1$ and $g\text{-cat}_{\mathbb{R}}([-1, 1]) = \infty$ for every $c \ne 0$.

Observe that if the functional f is such that $\inf f(B) = -\infty$, $f\text{-cat}_X(B) = \infty$. Indeed, for every $\varepsilon > 0$, $n^f_\varepsilon(B, X) = \infty$. Otherwise there would exist a finite subset of X, $S = \{x_1, \ldots, x_k\}$ such that $\min f(S) \le f(x) + \varepsilon$ for every $x \in B$. From this observation, we can expect that if the functional is not bounded from below, we will not be able to deduce a relation between the $f\text{-cat}_X(X)$ and the number of critical points of f; this is also the case for the classical category. This leads us to introduce the notion of truncated category depending on the functional f.

Definition 3.9. Let $B \subset X$. We define the *truncated f-category of B in X* by

$$f\text{-catt}_X(B) = \sup_{\varepsilon > 0} t^f_\varepsilon(B, X),$$

where $t^f_\varepsilon(B, X)$ is the smallest $n \in \mathbb{N} \cup \{0\}$ such that there exist closed subsets $A_0, \ldots, A_n$ satisfying:

(a) $B \subset \bigcup_{i=0}^n A_i$;

(b) A_i is (f, ε)-contractible in X, for all $i = 1, \dots, n$;
(c) there exists $\eta_0 \in \mathcal{N}_f$ such that $\eta_0(A_0, 1) \subset f^{-1/\varepsilon}$.

If such a n does not exist, we set $t^f_\varepsilon(B, X) = \infty$.

Examples 3.10. (1) Take $f(x) = x$, $f\text{-catt}_{\mathbb{R}}(\mathbb{R}) = 0$ while $f\text{-cat}_{\mathbb{R}}(\mathbb{R}) = \infty$.
(2) Take $f(x) = -x^2$, $f\text{-catt}_{\mathbb{R}}(\mathbb{R}) = 1$ while $f\text{-cat}_{\mathbb{R}}(\mathbb{R}) = \infty$.

The previous examples illustrate that the values of the two previous notions of f-category can be very different. However, they coincide when the functional is bounded from below.

Proposition 3.11. *Let $B \subset X$ be such that $\inf f(X) > -\infty$. Then*

$$f\text{-catt}_X(B) = f\text{-cat}_X(B).$$

Properties satisfied by the truncated f-category are analogous to those of the f-category. The proof of the following result is analogous to the proof of Theorem 3.5.

Theorem 3.12. *Let A, B be subsets of X, the following properties are satisfied:*

(i) *if $A \subset B$, then $f\text{-catt}_X(A) \le f\text{-catt}_X(B)$;*
(ii) *$f\text{-catt}_X(A \cup B) \le f\text{-catt}_X(A) + f\text{-cat}_X(B)$;*
(iii) *for every $\eta \in \mathcal{N}_f$, $f\text{-catt}_X(B) \le f\text{-catt}_X(\eta(B, 1))$.*

As before, small perturbations of the functional increase the value the truncated category.

Proposition 3.13. *Let $B \subset X$, and $f, g : X \to \mathbb{R}$. If $f\text{-catt}_X(B) < \infty$, there exists $\delta > 0$ such that if $\sup\{|f(x) - g(x)| : x \in B\} \le \delta$, then*

$$f\text{-catt}_X(B) \le g\text{-cat}_X(B).$$

Moreover, if $f\text{-catt}_X(B) = \infty$, for every $k \in \mathbb{N}$, there exists $\delta > 0$ such that if $\sup\{|f(x) - g(x)| : x \in B\} < \delta$, $g\text{-catt}_X(B) \ge k$.

Proof. If $f\text{-catt}_X(B) = k < \infty$, there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in \,]0, \hat{\varepsilon}]$, $t^f_\varepsilon(B, X) = k$. Choose $\delta < \hat{\varepsilon}/2$. If

$$\sup\{|f(x) - g(x)| : x \in B\} \le \delta \quad \text{and} \quad g\text{-catt}_X(B) = m < k,$$

there exists $\bar{\varepsilon}$ such that for all $\varepsilon \in \,]0, \bar{\varepsilon}]$, $t^g_\varepsilon(B, X) = m$.

Let $\varepsilon < \min\{\bar{\varepsilon}, \hat{\varepsilon} - 2\delta, 2\hat{\varepsilon}/(2 + \hat{\varepsilon}^2)\}$. For $i = 0, \dots, m$, there exist a closed set A_i and an associated continuous deformation η_i given by Definition 3.9. Denote $C_i = \overline{B} \cap A_i$. As in the proof of Proposition 3.7, $f(\eta_i(x, t)) \le f(x) + \varepsilon + 2\delta$ for every $x \in C_i$, $t \in [0, 1]$, and $i = 1, \dots, m$. Also,

$$f(\eta_0(x, 1)) \le g(\eta_0(x, 1)) + \delta \le \delta - \frac{1}{\varepsilon} \le -\frac{1}{\hat{\varepsilon}} \quad \forall x \in C_0.$$

Hence, $t^f_{\hat{\varepsilon}}(B, X) \le m$; contradiction. The case $f\text{-catt}_X(B) = \infty$ is analogous. □

A notion of relative category depending on the functional f can also be introduced.

Definition 3.14. Let Y be a closed subset of X, and $B \subset X$. We define the *f-category of B relative to Y in X* by

$$f\text{-cat}_{X,Y}(B) = \sup_{\varepsilon > 0} n^f_\varepsilon(B, X, Y),$$

where $n^f_\varepsilon(B, X, Y)$ is the smallest $n \in \mathbb{N} \cup \{0\}$ such that there exist closed subsets $A_0, \dots, A_n$ satisfying:

(a) $B \subset \bigcup_{i=0}^n A_i$;
(b) A_i is (f, ε)-contractible in X, for all $i = 1, \dots, n$;
(c) there exists $\eta_0 \in \mathcal{N}_f$ such that $\eta_0(Y, t) \subset Y$, and $\eta_0(A_0, 1) \subset Y$.

If such a n does not exist, we set $n^f_\varepsilon(B, X, Y) = \infty$.

Examples 3.15. (1) Take $X = \mathbb{R}^2$, $f(x, y) = (x^2 - 1)^2 - y^2$, $Y = f^a$ for some $a < 0$. Observe that $f\text{-cat}_{X,Y}(X) = 2$ while $\text{cat}_{X,Y}(X) = 1$.
(2) Take $X = \mathbb{R}$ and $f(x) = x^3 + 3x^2$.

$$f\text{-cat}_{X,f^a}(X) = \begin{cases} 1, & \text{if } a < 4, \\ 0, & \text{otherwise;} \end{cases} \quad \text{while} \quad \text{cat}_{X,f^a}(X) = \begin{cases} 1, & \text{if } a \in [0, 4[, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $f\text{-catt}_X(X) = 1$.

Again, most of the usual properties of the relative category are preserved in this new context.

Theorem 3.16. *Let Y be a closed subset of X, $A, B \subset X$. The following properties are satisfied:*

(i) $f\text{-cat}_{X,Y}(B) \geq \text{cat}_{X,Y}(\overline{B})$;
(ii) $f\text{-cat}_{X,Y}(B) \leq f\text{-cat}_X(B)$;
(iii) *if* $A \subset B$, *then* $f\text{-cat}_{X,Y}(A) \leq f\text{-cat}_{X,Y}(B)$;
(iv) $f\text{-cat}_{X,Y}(A \cup B) \leq f\text{-cat}_{X,Y}(A) + f\text{-cat}_X(B)$;
(v) *if* $\eta \in \mathcal{N}_f$ *satisfies* $\eta(Y, t) \subset Y$, *then* $f\text{-cat}_{X,Y}(B) \leq f\text{-cat}_{X,Y}(\eta(B, 1))$;
(vi) *if Y is such that for every $\eta \in \mathcal{N}_f$, one has $\eta(Y, t) \subset Y$, and if $Z \subset X$ is a closed set such that there exists $\hat{\eta} \in \mathcal{N}_f$ satisfying $\hat{\eta}(Z, 1) \subset Y$, then*

$$f\text{-cat}_{X,Y}(B) \leq f\text{-cat}_{X,Z}(B).$$

In particular, $f\text{-cat}_{X,Y}(B) \leq f\text{-cat}_{X,Z}(B)$ for every closed $Z \subset Y$.

Proof. (i) Observe that for every $\varepsilon > 0$, $\text{cat}_{X,Y}(\overline{B}) \leq n^f_\varepsilon(B, X, Y)$.

(ii) If $n^f_\varepsilon(B, X) = k < \infty$, there exist $A_1, \ldots, A_k$ given by Definition 3.2. Hence, $A_0, \ldots, A_k$ with $A_0 = Y$ satisfy conditions of Definition 3.14. So $n^f_\varepsilon(B, X, Y) \leq k$. Hence, $f\text{-cat}_{X,Y}(B) \leq f\text{-cat}_X(B)$.

(iii) is obvious.

(iv) If $f\text{-cat}_{X,Y}(A) = k < \infty$ and $f\text{-cat}_X(B) = m < \infty$, there exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in\ ]0, \hat{\varepsilon}]$, $n^f_\varepsilon(A, X, Y) = k$ and $n^f_\varepsilon(B, X) = m$. It follows that $n^f_\varepsilon(A \cup B, X, Y) \leq k + m$, and hence $f\text{-cat}_{X,Y}(A \cup B) \leq k + m$.

(v) The proof is analogous to the proof of (vi) of Theorem 3.5.

(vi) Assume that $f\text{-cat}_{X,Z}(B) = k < \infty$. There exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in\ ]0, \hat{\varepsilon}]$, there exist closed sets $A_0, \ldots, A_k$ with associated deformations $\eta_0, \ldots, \eta_k$ as in Definition 3.14. Observe that $\hat{\eta} \star \eta_0(A_0, 1) \subset \hat{\eta}(Z, 1) \subset Y$, and by assumption $\hat{\eta} \star \eta_0(Y, t) \subset Y$. So, $n^f_\varepsilon(B, X, Y) \leq k$, and hence $f\text{-cat}_{X,Y}(B) \leq k$. □

Remark 3.17. Assumptions of Theorem 3.16(vi) are satisfied for example with $Y = f^a$ and $Z \subset Y$.

Remark 3.18. A very important property of the Lusternik–Schnirelman category and the relative category is the following. We state it for the relative category.

If $Y \subset X$ is closed and both X, Y are absolute neighborhood extensor (ANE), then for every closed subset B of X, there exists a closed neighborhood V of B such that

$$\text{cat}_{X,Y}(B) = \text{cat}_{X,Y}(V);$$

see [9]. Recall that X is an ANE if for every metric space G, every F closed in G, and every continuous map $g : F \to X$ there exists a continuous extension of g on a neighborhood of F in G.

This property is no longer true for the f-category. For example, with $X = \mathbb{R}$, $Y = \emptyset$, $f(x) = x \sin(1/x)$,

$$f\text{-cat}_{\mathbb{R}}(\{0\}) = 0 \quad \text{and} \quad f\text{-cat}_{\mathbb{R}}(V) = \infty \quad \text{for every neighborhood } V \text{ of } 0.$$

Now, we would like to compare

$$f\text{-cat}_{X,Y}(B) \quad \text{with } f\text{-cat}_{\hat{X},\hat{Y}}(B \cap \hat{X})$$

for a closed pair $(\hat{X}, \hat{Y}) \subset (X, Y)$ satisfying some suitable properties.

Theorem 3.19. *Let Y and $\hat{Y} \subset \hat{X}$ be closed subsets of X, and let $B \subset X$.*

(i) *If $(\hat{X}, \hat{Y})$ is a retract of (X, Y) for some retraction $r : (X, Y) \to (\hat{X}, \hat{Y})$ such that $f(r(x)) \leq f(x)$ for every $x \in X$, then*

$$f\text{-cat}_{X,Y}(B) \geq f\text{-cat}_{\hat{X},\hat{Y}}(B \cap \hat{X}).$$

(ii) *If $(\hat{X}, \hat{Y})$ is a deformation retract of (X, Y) for some continuous deformation $\eta : (X, Y) \times [0, 1] \to (X, Y)$ such that $\eta \in \mathcal{N}_f$, $(\eta(X, 1), \eta(Y, 1)) = (\hat{X}, \hat{Y})$, and $\eta(x, t) = x$ for every $x \in \hat{X}$, $t \in [0, 1]$, then*

$$f\text{-cat}_{X,Y}(B) = f\text{-cat}_{\hat{X},\hat{Y}}(B \cap \hat{X}).$$

Proof. (i) If $f\text{-cat}_{X,Y}(B) = k < \infty$, there exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in\ ]0, \hat{\varepsilon}]$, there exist $A_0, \ldots, A_k$ with associated deformations $\eta_0, \ldots, \eta_k$ given by Definition 3.14. Define $\hat{A}_i = A_i \cap \hat{X}$ and $\hat{\eta}_i = r \circ \eta_i$ for $i = 0, \ldots, k$. Since r is a retraction of (X, Y) on $(\hat{X}, \hat{Y})$ such that $f(r(x)) \leq f(x)$, we verify that $\hat{A}_i$ is (f, ε)-contractible for $i = 1, \ldots, k$, and that $\hat{\eta}_0 \in \mathcal{N}_f$ satisfies $\hat{\eta}_0(\hat{Y}, t) \subset \hat{Y}$, and $\hat{\eta}_0(\hat{A}_0, 1) \subset \hat{Y}$. Therefore, $n^f_\varepsilon(B \cap \hat{X}, \hat{X}, \hat{Y}) \leq k$, and hence $f\text{-cat}_{\hat{X},\hat{Y}}(B \cap \hat{X}) \leq k$.

(ii) Assume that $f\text{-cat}_{\hat{X},\hat{Y}}(B \cap \hat{X}) = k < \infty$. As in (i), there exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in\ ]0, \hat{\varepsilon}]$, there exist $\hat{A}_0, \ldots, \hat{A}_k$ with associated deformations $\hat{\eta}_0, \ldots, \hat{\eta}_k$ given by Definition 3.14 such that $B \cap \hat{X} \subset \bigcup_{i=0}^k \hat{A}_i$. For $i = 0, \ldots, k$, define $A_i = \{x \in X : \eta(x, 1) \in \hat{A}_i\}$, and $\eta_i = \hat{\eta}_i \star \eta$. It is easy to verify that for $i = 1, \ldots, k$, A_i is (f, ε)-contractible, and that $\eta_0 \in \mathcal{N}_f$ satisfies $\eta_0(Y, t) \subset Y$, and $\eta_0(A_0, 1) \subset Y$. Therefore, $n^f_\varepsilon(B, X, Y) \leq k$, and hence $f\text{-cat}_{X,Y}(B) \leq k$. □

It is natural to compare the relative category of homotopy equivalent spaces in our context. Obviously, the functionals will be related by this equivalence.

Theorem 3.20. *Let Y be a closed subset of X, $B \subset X$. Assume that V is a topological space $A \subset V$ and W is a closed subset of V such that there exist continuous maps $\phi : (X, Y) \to (V, W)$, $\psi : (V, W) \to (X, Y)$ such that $\phi(B) \subset A$, and $\psi \circ \phi$ is homotopic to the identity map on (X, Y) with an homotopy $h : (X, Y) \times [0, 1] \to (X, Y)$ such that $h \in \mathcal{N}_f$. Then*

$$f\text{-cat}_{X,Y}(B) \le g\text{-cat}_{V,W}(A),$$

where $g = f \circ \psi$.

Moreover if $\psi(A) \subset B$ and (X, Y) and (V, W) are homotopy equivalent with ϕ and ψ as above and such that $\phi \circ \psi$ is homotopic to the identity map on (V, W) with an homotopy $k : (V, W) \times [0, 1] \to (V, W)$ such that $k \in \mathcal{N}_g$, then

$$f\text{-cat}_{X,Y}(B) = g\text{-cat}_{V,W}(A).$$

Proof. Assume that $g\text{-cat}_{V,W}(A) = m < \infty$. Let $\hat{\varepsilon} > 0$ be such that for every $\varepsilon \in \,]0, \hat{\varepsilon}]$, there exist closed sets $A_0, \ldots, A_m$ associated to deformation $\eta_0, \ldots, \eta_m$ given by Definition 3.14 such that $A \subset \bigcup_{i=0}^m A_i$. Denote $B_i = \phi^{-1}(A_i)$. Obviously $B \subset \bigcup_{i=0}^n B_i$. Define for $i = 0, \ldots, m$,

$$\tilde{\eta}_i(x, t) = \begin{cases} h(x, 2t), & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \psi(\eta_i(\phi(x), 2t - 1)), & \text{if } t \in \left]\frac{1}{2}, 1\right]. \end{cases}$$

Since $h(x, 1) = \psi \circ \phi$ and $f(h(x, t)) \le f(x)$, $\tilde{\eta}_i$ is continuous and $f(\tilde{\eta}_i(x, t)) \le f(x)$ for every x and $t \in [0, 1/2]$. For $t \in \,]1/2, 1]$,

$$\begin{aligned} f(\tilde{\eta}_i(x, t)) &= g(\eta_i(\phi(x), t)) \\ &\le \begin{cases} g(\phi(x)), & \text{if } i = 0, \\ g(\phi(x)) + \varepsilon, & \text{if } i = 1, \ldots, m \end{cases} \\ &\le f(x) + \begin{cases} 0, & \text{if } i = 0, \\ \varepsilon, & \text{if } i = 1, \ldots, m. \end{cases} \end{aligned}$$

On the other hand, for $t \in [0, 1/2]$,

$$\tilde{\eta}_0(Y, t) \subset h(Y, 2t) \subset Y,$$

and for $t \in \,]1/2, 1]$,

$$\tilde{\eta}_0(Y, t) \subset \psi(\eta_0(W, 2t - 1)) \subset \psi(W) \subset Y.$$

Also,

$$\tilde{\eta}_0(B_0, 1) \subset \psi(\eta_0(A_0, 1)) \subset \psi(W) \subset Y.$$

Therefore $n_\varepsilon^f(B, X, Y) \le m$ and hence $f\text{-cat}_{X,Y}(B) \le m$.

Now, if (X, Y) and (V, W) are homotopy equivalent and satisfy the assumptions, the first part of the proof ensures that $f\text{-cat}_{X,Y}(B) = g\text{-cat}_{V,W}(A)$. □

4. Linking and *f*-category

In this section, X is a normal topological space and $f : X \to \mathbb{R}$ is a continuous functional. We study the relations between the category depending on the functional f and the notion of linking. We use the general notion of linking introduced in [18]. By convention: $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$, and $\text{dist}(\emptyset, Q) = \infty$.

Definition 4.1. Let $A \subset B \subset X$, $P \subset Q \subset X$. Let us denote

$$\mathcal{N}(A) = \{\eta : X \times [0, 1] \to X \text{ continuous } : \eta(x, t) = x \,\forall (x, t) \in X \times \{0\} \cup A \times [0, 1]\}.$$

We say that (B, A) *links* (Q, P) if $B \cap Q \ne \emptyset$, $A \cap Q = \emptyset$, $B \cap P = \emptyset$ and for every $\eta \in \mathcal{N}(A)$, one of the following conditions is satisfied:

(a) $\eta(B, 1) \cap Q \ne \emptyset$;
(b) $\eta(B,]0, 1[) \cap P \ne \emptyset$.

This notion includes the classical notion of linking corresponding to the case where $(B, \partial B)$ links $(Q, \emptyset)$. It includes also the notion of *splitting spheres* introduced by Marino, Micheletti and Pistoia [19] corresponding to the case where (B_1, S_1) links (B_2, S_2) with $X = E_1 \oplus E_2$ a Banach space such that $0 < \dim(E_1) < \infty$, B_i and S_i are respectively the closed unit ball and the unit sphere in E_i.

Theorem 4.2. *Assume that* (B, A) *links* (Q, P), A *is closed, and*

$$f(x) < f(y) \quad \text{for every } x \in B, y \in P. \tag{4.1}$$

(i) *If there exists* $a \in \mathbb{R}$ *such that* $\sup f(A) \le a < \inf f(Q)$, *then*

$$f\text{-cat}_{X,f^a}(B) \ge 1 \quad \text{and} \quad f\text{-cat}_{X,A}(B) \ge 1.$$

(ii) *If* $\sup f(A) = \inf f(Q)$ *and* $A \cap \overline{Q} = \emptyset$, *then* $f\text{-cat}_{X,A}(B) \ge 1$.

Proof. (i) Let $\delta > 0$ be such that $a + \delta < \inf f(Q)$. Assume that $f\text{-cat}_{X,f^a}(B) = 0$. There exists a continuous deformation $\eta \in \mathcal{N}_f$ such that $\eta(f^a, t) \subset f^a$ and $\eta(B, 1) \subset f^a$.

Consider $\lambda : X \to [0, 1]$ an Urysohn function such that $\lambda(f^a) = \{0\}$ and $\lambda(\overline{X \setminus f^{a+\delta}}) = \{1\}$. Define a deformation $\hat{\eta}$ by $\hat{\eta}(x, t) = \eta(x, t\lambda(x))$. Obviously, $\hat{\eta} \in \mathcal{N}(A)$. Observe that $\hat{\eta}$ does not satisfy condition (b) of Definition 4.1. Indeed, $f(\hat{\eta}(x, t)) \le f(x) < f(y)$ for every $x \in B$ and $y \in P$. Therefore, since (B, A) links (Q, P), there exists $\hat{x} \in B$ such that $\hat{\eta}(\hat{x}, 1) \in Q$, and hence

$$a + \delta < \inf f(Q) \le f(\hat{\eta}(\hat{x}, 1)) \le f(\hat{x}).$$

So, $\lambda(\hat{x}) = 1$. Thus $\hat{\eta}(\hat{x}, 1) = \eta(\hat{x}, 1) \in f^a$. Contradiction.

It follows from Theorem 3.16(vi) that $f\text{-cat}_{X,A}(B) \ge f\text{-cat}_{X,f^a}(B) \ge 1$.

(ii) If $f\text{-cat}_{X,A}(B) = 0$, there exists $\eta \in \mathcal{N}_f$ such that $\eta(A, t) \subset A$ and $\eta(B, 1) \subset A$. Let us denote

$$Z = \overline{\{x \in X : \eta(x, t) \in Q \text{ for some } t \in [0, 1]\}}.$$

Observe that $A \cap Z = \emptyset$. Indeed, otherwise if $x_0 \in A \cap Z$, there exist sequences $x_n \to x_0$ and $t_n \to t_0$ such that $\eta(x_n, t_n) \in Q$, and by continuity, $\eta(x_0, t_0) \in \overline{Q} = \overline{Q} \setminus A$, which contradicts the fact that $\eta(A, t_0) \subset A$.

As in (i), we consider $\lambda : X \to [0, 1]$ an Urysohn function such that $\lambda(A) = \{0\}$, $\lambda(Z) = \{1\}$, and we define a deformation $\hat{\eta}$ by $\hat{\eta}(x, t) = \eta(x, t\lambda(x))$. Again, $\hat{\eta} \in \mathcal{N}(A)$ and there exists $\hat{x} \in B$ such that $\hat{\eta}(\hat{x}, 1) = \eta(\hat{x}, 1) \in Q$. It is a contradiction since $\eta(B, 1) \subset A$. $\square$

Examples 4.3. This result is obviously false for the classical relative category.

(1) Take $X = \mathbb{R}^2$, $f(x, y) = -x^2 + y^3 - y^2 - y + 1$, $B = S^1$, $A = \{(1, 0)\}$, $Q = \{0\} \times [0, \infty[$ and $P = (0, 0)$. Observe that (B, A) links (Q, P), $f\text{-cat}_{X,A}(B) = 1$ and $\text{cat}_{X,A}(B) = 0$.
(2) Take $X = \mathbb{R}^2$, $f(x, y) = 4x^2 - x^4 - y^2$, $B = \{0\} \times [-1, 1]$, $A = \{0\} \times \{-1, 1\}$, $Q = [-1, 1] \times \{0\}$, $P = \{-1, 1\} \times \{0\}$. Observe that (B, A) links (Q, P), $f\text{-cat}_{X,f^a}(B) = 1$, and $\text{cat}_{X,f^a}(B) = 0$ for every $a \in [-1, 0[\; = [\sup f(A), \inf f(Q)[$.

If we replace condition (4.1) by a stronger one, we can obtain results in the space $X \setminus P$.

Theorem 4.4. *Assume that* (B, A) *links* (Q, P), A *is closed, and assume that* $\sup f(B) < \inf f(P)$.

(i) *If there exists* $a \in \mathbb{R}$ *such that* $\sup f(A) \le a < \inf f(Q)$, *then*

$$f\text{-cat}_{X\setminus P,f^a}(B) \ge 1 \quad \text{and} \quad f\text{-cat}_{X\setminus P,A}(B) \ge 1.$$

(ii) *If* $\sup f(A) = \inf f(Q)$ *and* $A \cap \overline{Q} = \emptyset$, *then* $f\text{-cat}_{X\setminus P,A}(B) \ge 1$.

Proof. Let $b = \sup f(B)$ and $\gamma > 0$ such that $\inf f(P) > b + \gamma$. Consider $\sigma : X \to [0, 1]$ an Urysohn function such that $\sigma(f^b) = \{1\}$, $\sigma(\overline{X \setminus f^{b+\gamma}}) = \{0\}$. Observe that if $\eta : X \setminus P \times [0, 1] \to X \setminus P$ is a continuous deformation such that $\eta(x, 0) = x$ and $f(\eta(x, t)) \le f(x)$, then the deformation $\tilde{\eta} : X \times [0, 1] \to X$ defined by

$$\tilde{\eta}(x, t) = \begin{cases} x, & \text{if } x \in P, \\ \eta(x, t\sigma(x)), & \text{if } x \in X \setminus P; \end{cases}$$

is in $\mathcal{N}_f$. Moreover, $\tilde{\eta}$ satisfies $\tilde{\eta}(x, t) = \eta(x, t)$ for $x \in f^b$.

(i) (resp. (ii)) If $f\text{-cat}_{X\setminus P,f^a}(B) = 0$ (resp. $f\text{-cat}_{X\setminus P,A}(B) = 0$), there exists a continuous deformation $\eta : X \setminus P \times [0, 1] \to X \setminus P$ such that $\eta(x, 0) = x$, $f(\eta(x, t)) \le f(x)$, and $\eta(f^a, t) \subset f^a$, $\eta(B, 1) \subset f^a$ (resp. $\eta(A, t) \subset A$, $\eta(B, 1) \subset A$). Using $\tilde{\eta}$ defined above and arguing as in the proof of Theorem 4.2, we deduce the conclusion. $\square$

To conclude this section, we consider the particular case of linking of type splitting spheres [19]. For $E = E_1 \oplus E_2$ a Banach space, we denote

$$B_i = \{x \in E_i : \|x\| \le r_i\},$$
$$S_i = \{x \in E_i : \|x\| = r_i\},$$

for $i = 1, 2$.

Theorem 4.5. *Let $E = E_1 \oplus E_2$ be a Banach space with $0 < \dim(E_1) < \infty$. Let $f : E \to \mathbb{R}$ be continuous. Assume that there exist $r_1, r_2 > 0$ such that*

$$\sup f(S_1) < \inf f(B_2) \leq \sup f(B_1) < \inf f(S_2).$$

Then for all $a \in [\sup f(S_1), \inf f(B_2)[$,

$$f\text{-cat}_{E,S_1}(B_1) \geq f\text{-cat}_{E,f^a}(B_1) \geq \text{cat}_{B_1,S_1}(B_1).$$

Proof. Theorem 3.16(vi) implies that

$$f\text{-cat}_{E,S_1}(B_1) \geq f\text{-cat}_{E,f^a}(B_1).$$

Assume that $f\text{-cat}_{E,f^a}(B_1) = k < \infty$. There exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in\ ]0, \hat{\varepsilon}]$, there exist closed sets $A_0, \dots, A_k$ with associate deformations $\eta_0, \dots, \eta_k$ satisfying conditions of Definition 3.14, and in particular $B_1 \subset \bigcup_{i=0}^k A_i$.

Let $\rho : E \to B_1$ be a continuous retraction. Obviously, for $i = 1, \dots, k$, $A_i \cap B_1$ is contractible in B_1 with the homotopy $h_i : (A_i \cap B_1) \times [0, 1] \to B_1$ defined by $\rho \circ \eta_i$.

Let $\delta > 0$ be such that $a + \delta < \inf f(B_2)$. Consider $\lambda : E \to [0, 1]$ an Urysohn function such that $\lambda(f^a) = \{0\}$ and $\lambda(\overline{E \setminus f^{a+\delta}}) = \{1\}$. Define a deformation $\hat{\eta}$ by $\hat{\eta}(x, t) = \eta_0(x, t\lambda(x))$. Write $\hat{\eta} = \hat{\eta}_1 + \hat{\eta}_2$ with $\hat{\eta}_i(E \times [0, 1]) \subset E_i$. The fact that $\hat{\eta}(x, t) = x$ for every $x \in f^a$ implies that $\hat{\eta}_1(x, t) = x$ for every $x \in S_1$. The topological degree theory ensures the existence of a continuum $\mathcal{C} \subset (B_1 \setminus S_1) \times [0, 1]$ such that $\mathcal{C} \cap B_1 \times \{0\} \neq \emptyset$, $\mathcal{C} \cap B_1 \times \{1\} \neq \emptyset$, and $\hat{\eta}_1(x, t) = 0$ for every $(x, t) \in \mathcal{C}$; that is $\hat{\eta}(\mathcal{C}) \subset E_2$. We claim that

$$\hat{\eta}(\mathcal{C}) \subset B_2 \setminus S_2. \tag{4.2}$$

Otherwise, there would exists $(x_0, t_0) \in \mathcal{C}$ such that $\hat{\eta}(x_0, t_0) \in S_2$. Indeed, $\mathcal{C} \cap B_1 \times \{0\} = (0, 0)$ and $\hat{\eta}(\mathcal{C})$ is connected. This leads to a contradiction since

$$\inf f(S_2) \leq f(\hat{\eta}(x_0, t_0)) \leq f(x_0) \leq \sup f(B_1) < \inf f(S_2).$$

Let $(\bar{x}, 1) \in \mathcal{C}$. It follows from (4.2) that $a + \delta \leq f(\hat{\eta}(\bar{x}, 1)) \leq f(\bar{x})$. So $\lambda(\bar{x}) = 1$, and $\hat{\eta}(\bar{x}, 1) = \eta_0(\bar{x}, 1)$. Since $\eta_0(A_0, 1) \subset f^a$, $\bar{x} \notin A_0 \cap B_1$. To conclude, we notice that S_1 is a deformation retract of $B_1 \setminus \{\bar{x}\}$ and hence of $(A_0 \cap B_1) \cup S_1$ in B_1.

So $A_0 \cap B_1, \dots, A_k \cap B_1$ satisfy conditions of Definition 2.1, and hence

$$\text{cat}_{B_1,S_1}(B_1) \leq k. \quad \square$$

We conjecture that an analogous result holds in $E \times M$ for M a compact manifold.

Conjecture 4.6. *Let $E = E_1 \oplus E_2$ be a Banach space with $0 < \dim(E_1) < \infty$, and M a compact manifold. Let $f : E \times M \to \mathbb{R}$ be continuous. Assume that there exist $r_1, r_2 > 0$ such that*

$$\sup f(S_1 \times M) < \inf f(B_2 \times M) \leq \sup f(B_1 \times M) < \inf f(S_2 \times M). \tag{4.3}$$

Then for all $a \in [\sup f(S_1 \times M), \inf f(B_2 \times M)[$,

$$f\text{-cat}_{E \times M, f^a}(B_1 \times M) \geq \text{cat}_{B_1 \times M, S_1 \times M}(B_1 \times M).$$

Even though we cannot prove this conjecture, we can obtain a partial result.

Theorem 4.7. *Let $E = E_1 \oplus E_2$ be a Banach space with $0 < \dim(E_1) < \infty$, and M a compact manifold. Let $f : E \times M \to \mathbb{R}$ be continuous. Assume that (4.3) is satisfied for some $r_1, r_2 > 0$. Then for all $a \in [\sup f(S_1 \times M), \inf f(B_2 \times M)[$,*

$$\begin{aligned} f\text{-cat}_{E \times M, S_1 \times M}(B_1 \times M) &\geq f\text{-cat}_{E \times M, f^a}(B_1 \times M) \\ &\geq \text{cuplength}(B_1 \times M, S_1 \times M) + 1 \geq \text{cuplength}(M) + 1. \end{aligned}$$

In order to prove this theorem, the following result will be needed.

Proposition 4.8. *Let $E = E_1 \oplus E_2$ be a Banach space with $0 < \dim(E_1) < \infty$, and M a compact manifold. Let $f : E \times M \to \mathbb{R}$ be continuous. Assume that (4.3) is satisfied for some $r_1, r_2 > 0$. Then for all $a \in [\sup f(S_1 \times M), \inf f(B_2 \times M)[$,*

$$f\text{-cat}_{E \times M, f^a}(B_1 \times M) \geq \text{cat}_{E \setminus S_2 \times M, f^a}(B_1 \times M).$$

Proof. Assume that $f\text{-cat}_{E \times M, f^a}(B_1 \times M) = k < \infty$. There exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in\ ]0, \hat{\varepsilon}]$, there exist closed sets $A_0, \dots, A_k$ with associate deformations $\eta_0, \dots, \eta_k$ satisfying conditions of Definition 3.14, and in particular $B_1 \times M \subset \bigcup_{i=0}^k A_i$.

Let $\rho : E \times M \to E_1 \times M$ be a continuous retraction. Obviously, for $i = 1, \dots, k$, $\hat{A}_i := A_i \cap (B_1 \times M)$ is contractible in $E \setminus S^2$ with the homotopy $h_i : \hat{A}_i \times [0, 1] \to E \setminus S_2$ defined by $\rho \circ \eta_i$.

On the other hand, observe that since $f(\eta_0(x, t)) \leq f(x)$, $\eta_0 : (\hat{A}_0 \cup f^a) \times [0, 1] \to E \setminus S_2$ satisfies the desired properties. $\square$

Proof of Theorem 4.7. It follows from Theorem 3.16(vi) and the previous proposition that

$$f\text{-cat}_{E\times M,S_1\times M}(B_1\times M)\geq f\text{-cat}_{E\times M,f^a}(B_1\times M)\geq \text{cat}_{E\setminus S_2\times M,f^a}(B_1\times M).$$

Theorem 2.3 and Lemma 2.4 imply that

$$\text{cat}_{E\setminus S_2\times M,f^a}(B_1\times M)\geq \text{cuplength}(B_1\times M,S_1\times M)+1\geq \text{cuplength}(M)+1. \quad \square$$

Remark 4.9. From the previous results, it is easy to see that Conjecture 4.6 is true if we can show that for $S_1\times M\subset A\subset B_1\times M$ closed such that there exists a continuous deformation $\eta: E\times M\times[0,1]\to E\times M$ satisfying $\eta(x,t)=x\forall x\in S_1\times M$, $\eta(A,1)\subset(E\setminus B_2)\times M$ and $\eta(B_1\times M\times[0,1])\subset(E\setminus S_2)\times M$, there exists a continuous deformation $h: A\times[0,1]\to B_1\times M$ such that $h(x,0)=x$, $h(S_1\times M\times[0,1])\subset S_1\times M$, and $h(A,1)\subset S_1\times M$.

5. Critical point theory and *f*-category

In this section, we want to show that under suitable assumptions, the f-category gives a lower bound to the number of critical points of f. We present our results in an abstract context. In particular, they can be seen in the classical critical point theory of continuously differentiable maps, as well as in the generalized critical point theory of nondifferentiable functionals.

In this section X is a metric space, $f: X\to\mathbb{R}$ is continuous, and $K\subset X$ is given and called the set of *critical points of* f, and it satisfies

$$K\cap f^{-1}(C) \text{ is compact for every compact set } C\subset\mathbb{R}. \tag{5.1}$$

For $c\in\mathbb{R}$ and $I\subset\mathbb{R}$, we denote

$$K_c=K\cap f^{-1}(c)\quad\text{and}\quad K_I=K\cap f^{-1}(I).$$

We say c is a *critical value of* f if $K_c\neq\emptyset$.

5.1. Local contractibility and deformation properties

The metric space X will have to satisfy a condition of contractibility. This notion can be found for example in Borsuk [25].

Definition 5.1. We say that a metric space Y is *locally contractible* if for every $y\in Y$ and every $\mathcal{U}$ neighborhood of y, there exists a closed neighborhood $\mathcal{V}\subset\mathcal{U}$ of y contractible in $\mathcal{U}$; that is there exist $\hat{y}\in\mathcal{U}$ and a continuous deformation $h:\mathcal{V}\times[0,1]\to\mathcal{U}$ such that $h(v,0)=v$ and $h(v,1)=\hat{y}$ for every $v\in\mathcal{V}$.

Lemma 5.2. *Assume that X is locally contractible. Then for every $\varepsilon>0$ and every $x\in X$, there exists $\mathcal{V}$ a closed neighborhood of x which is (f,ε)-contractible.*

Proof. Let $\varepsilon>0$ and $x\in X$. The continuity of f guaranties that there exists $\mathcal{U}$, a neighborhood of x such that $f(\mathcal{U})\subset B(f(x),\varepsilon/2)$. Since X is locally contractible, there exist $\mathcal{V}$ a closed neighborhood of x contractible in $\mathcal{U}$ with a continuous deformation $h:\mathcal{V}\times[0,1]\to\mathcal{U}$. Observe that

$$f(h(y,t))\leq f(x)+\frac{\varepsilon}{2}\leq f(y)+\varepsilon\quad\forall y\in\mathcal{V},t\in[0,1]. \quad \square$$

It is well known that deformation lemmas play a key role in critical point theory. In this abstract context, we will assume that f satisfies a suitable deformation property.

Definition 5.3. Let $c\in\mathbb{R}$. We say that f satisfies $\mathcal{D}(X,K_c)$ if for every $\hat{\varepsilon},\rho>0$ and every neighborhood $\mathcal{U}$ of K_c, there exist $\varepsilon\in\ ]0,\hat{\varepsilon}[$ and a continuous deformation $\eta: X\times[0,1]\to X$ such that

(a) $\eta(x,t)=x$ for every $(x,t)\in X\times\{0\}\cup f^{c-\hat{\varepsilon}}\times[0,1]$;
(b) $f(\eta(x,t))\leq f(x)$ for every $(x,t)\in X\times[0,1]$;
(c) $\eta(f^{c+\varepsilon}\setminus\mathcal{U},1)\subset f^{c-\varepsilon}$;
(d) $\text{dist}(x,\eta(x,t))\leq\rho$ for every $(x,t)\in X\times[0,1]$.

This property permits to deduce a noncritical interval deformation lemma.

Lemma 5.4. *Let $\hat{\varepsilon}>0$, $a\in\mathbb{R}$, $b\in\ ]a,\infty]$. Assume that f satisfies $\mathcal{D}(X,K_c)$ and $K_c=\emptyset$ for every $c\in[a,b]\cap\mathbb{R}$. Then there exists a continuous deformation $\eta: X\times[0,1]\to X$ such that*

(i) *$\eta(x,t)=x$ for every $(x,t)\in X\times\{0\}\cup f^{a-\hat{\varepsilon}}\times[0,1]$;*
(ii) *$f(\eta(x,t))\leq f(x)$ for every $(x,t)\in X\times[0,1]$;*
(iii) *$\eta(f^b,1)\subset f^a$;*
(iv) *if $b<\infty$, there exists $\delta>0$ such that $\text{dist}(x,\eta(x,t))\leq\delta$ for every $(x,t)\in X\times[0,1]$.*

Proof. Assume that $b<\infty$. The property $\mathcal{D}(X,K_c)$ guaranties the existence of ε_c and η_c satisfying Definition 5.3 with $\rho=1$. The compactness of $[a,b]$ implies that we can choose $a\leq c_1<c_2\cdots<c_n\leq b$ such that $c_1-\varepsilon_{c_1}<a<b<c_n+\varepsilon_{c_n}$, and $c_{i+1}-\varepsilon_{c_{i+1}}<c_i+\varepsilon_{c_i}$ for $i=1,\ldots,n-1$. It is easy to verify that $\eta=\eta_{c_1}\star\cdots\star\eta_{c_n}$ satisfies (i)–(iv) with $\delta=n$.

Now consider the case $b = \infty$. From the previous case, for every $n \in \mathbb{N}$, there exists η_n satisfying (i)–(iv) on the interval $[a + (n-1)\hat{\varepsilon}, a + n\hat{\varepsilon}]$. Define

$$\eta(x,t) = \begin{cases} \eta_1(x,t), & \text{if } f(x) \leq a, \\ \eta_1 \star \eta_2(x,t), & \text{if } f(x) \in \ ]a, a+\hat{\varepsilon}], \\ \eta_1 \star \eta_2 \star \eta_3(x,t), & \text{if } f(x) \in \ ]a+\hat{\varepsilon}, a+2\hat{\varepsilon}], \\ \vdots \end{cases}$$

It is easy to verify that η is continuous and satisfies (i)–(iii). □

With an extra assumption, a can be a critical value.

Definition 5.5. Let $c \in \mathbb{R}$. We say that f *satisfies* $\mathcal{D}_2(X, K_c)$ if for every $\hat{\varepsilon} > 0$ such that $f(K) \cap [c, c+\hat{\varepsilon}] = \{c\}$, there exist $\varepsilon \in \ ]0, \hat{\varepsilon}[$ and a continuous deformation $\eta : X \times [0,1] \to X$ such that

(a) $\eta(x, 0) = x$ for every $(x,t) \in X \times \{0\} \cup f^{c-\hat{\varepsilon}} \times [0,1]$;
(b) $f(\eta(x,t)) \leq f(x)$ for every $(x,t) \in X \times [0,1]$;
(c) $\eta(f^{c+\varepsilon}, 1) \subset f^c$.

Lemma 5.6. *Let $a \in \mathbb{R}$, $b \in \ ]a, \infty]$. Assume that $K_c = \emptyset$ for every $c \in \ ]a,b] \cap \mathbb{R}$, and f satisfies $\mathcal{D}(X, K_c)$ and $\mathcal{D}_2(X, K_a)$. Then there exists a continuous deformation $\eta : X \times [0,1] \to X$ such that*

(i) *$\eta(x,t) = x$ for every $(x,t) \in X \times \{0\}$;*
(ii) *$f(\eta(x,t)) \leq f(x)$ for every $(x,t) \in X \times [0,1]$;*
(iii) *$\eta(f^b, 1) \subset f^a$.*

The next result follows directly from Theorem 3.16(vi), and Lemmas 5.4 and 5.6.

Proposition 5.7. *Under the assumptions of Lemma 5.4 or Lemma 5.6,*

$$f\text{-cat}_{X,f^a}(B) = f\text{-cat}_{X,f^b}(B) \quad \textit{for every } B \subset X.$$

5.2. Results with the f-category and the truncated f-category

The following result establishes that $f\text{-cat}_X(X)$ is a lower bound to the cardinality of the set of critical points of f when f is bounded from below. Let us recall that the property stated in Remark 3.18 is crucial in the proof of the analogous result for the Lusternik–Schnirelman category. This property is no longer true for the f-category.

Theorem 5.8. *Assume (5.1) and assume that X is locally contractible, and f satisfies $\mathcal{D}(X, K_c)$ for every $c \in \mathbb{R}$. If f is bounded from below, then f has at least $f\text{-cat}_X(X)$ critical points.*

An analogous result can be obtained in the case where f is not bounded from below considering the truncated f-category. It is worthwhile to point out that there is no analogous result with the classical Lusternik–Schnirelman category.

Theorem 5.9. *Assume (5.1) and assume that X is locally contractible, and f satisfies $\mathcal{D}(X, K_c)$ for every $c \in \mathbb{R}$. Then f has at least $f\text{-catt}_X(X)$ critical points.*

To prove those results, we need the following lemma with the following notation. For $n \in \mathbb{N}$, set

$$\Gamma_n = \{B \subset X : f\text{-cat}_X(B) \geq n\}, \quad (\text{resp. } \Gamma_n^t = \{B \subset X : f\text{-catt}_X(B) \geq n\}),$$

and

$$c_n = \inf_{B \in \Gamma_n} \sup f(B) \quad \left(\text{resp. } c_n^t = \inf_{B \in \Gamma_n^t} \sup f(B)\right).$$

Lemma 5.10. *Assume (5.1) and X is locally contractible. Assume also that $c = c_k = \cdots = c_{k+m}$ (resp. $c = c_k^t = \cdots = c_{k+m}^t$) is not an accumulation point of $f(K)$, and f satisfies $\mathcal{D}(X, K_c)$. Then $\operatorname{card} K_c \geq m+1$.*

Proof. By assumption, there exists $\delta > 0$ such that $[c-\delta, c+\delta] \cap f(K) = \{c\}$.

Suppose that $\operatorname{card} K_c \leq m$. Observe that $f^{c+\delta} \in \Gamma_{k+m}$ (resp. $f^{c+\delta} \in \Gamma_{k+m}^t$). Indeed, by definition, there exists $B \subset \Gamma_{k+m}$ (resp. $B \subset \Gamma_{k+m}^t$) such that $\sup f(B) < c + \delta$. Theorem 3.5(iv) (resp. Theorem 3.12(i)) implies that

$$f\text{-cat}_X(f^{c+\delta}) \geq f\text{-cat}_X(B) \quad (\text{resp. } f\text{-catt}_X(f^{c+\delta}) \geq f\text{-catt}_X(B)).$$

Therefore, there exists $\bar{\varepsilon} > 0$ such that

$$n_\varepsilon^f(f^{c+\delta}, X) \geq k+m (\text{resp. } t_\varepsilon^f(f^{c+\delta}, X) \geq k+m) \quad \text{for every } \varepsilon \in \ ]0, \bar{\varepsilon}]. \tag{5.2}$$

Lemma 5.2 implies that for every $y \in K_c$, there exists $\mathcal{V}_y$ a $(f, \bar{\varepsilon})$-contractible, closed neighborhood of y. Denote

$$\mathcal{U} = \bigcup_{y \in K_c} \operatorname{int}(\mathcal{V}_y).$$

Let $\varepsilon \in \;]0, \delta[$ and η be the continuous deformation given by Definition 5.3 such that

$$\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1) \subset f^{c-\varepsilon}. \tag{5.3}$$

We claim that

$$f\text{-cat}_X(\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1)) \geq k \quad (\text{resp. } f\text{-catt}_X(\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1)) \geq k). \tag{5.4}$$

Otherwise, if $f\text{-cat}_X(\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1)) = j < k$ (resp. $f\text{-catt}_X(\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1)) = j < k$), we can choose $\tilde{\varepsilon} \leq \bar{\varepsilon}$ such that

$$n^f_{\tilde{\varepsilon}}(\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1)) = j \quad (\text{resp. } t^f_{\tilde{\varepsilon}}(\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1)) = j).$$

According to Definition 3.2 (resp. Definition 3.9), there exist $B_1, \ldots, B_j$ $(f, \tilde{\varepsilon})$-contractible with deformations $\eta_1, \ldots, \eta_j$ (resp. and B_0 with a deformation η_0 such that $f(\eta_0(x, 1)) \leq -1/\tilde{\varepsilon}$ for all $x \in B_0$) such that $\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1) \subset B_1 \cup \cdots \cup B_j$ (resp. $\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1) \subset B_0 \cup \cdots \cup B_j$). On the other hand, let $\hat{\eta} : X \times [0, 1] \to X$ be a continuous deformation such that

$$\hat{\eta}(f^{c+\delta}, 1) \subset f^{c+\varepsilon}$$

given by Lemma 5.4 since $K_{[c+\varepsilon, c+\delta]} = \emptyset$. Set

$$A_i = \{x \in X : \eta \star \hat{\eta}(x, 1) \in B_i\} \quad \text{for } i = 1, \ldots, j \quad (\text{resp. } i = 0, \ldots, j),$$

and if card $K_c = l > 0$ and $K_c = \{y_1, \ldots, y_l\}$, set

$$A_i = \{x \in X : \hat{\eta}(x, 1) \in \mathcal{V}_{y_{i-j}}\} \quad \text{for } i = j+1, \ldots, j+l.$$

It is easy to verify that A_i is $(f, \bar{\varepsilon})$-contractible for $i = 1, \ldots, j + \text{card } K_c$ (resp. and $f(\eta_0 \star \eta \star \hat{\eta}(x, 1)) \leq -1/\bar{\varepsilon}$ for all $x \in A_0$). The covering $A_1, \ldots, A_{j+\text{card}K_c}$ (resp. $A_0, \ldots, A_{j+\text{card } K_c}$) permits us to deduce that

$$n^f_{\bar{\varepsilon}}(f^{c+\delta}, X) \leq j + \text{card } K_c < k + m, \quad (\text{resp. } t^f_{\bar{\varepsilon}}(f^{c+\delta}, X) < k + m),$$

which contradicts (5.2).

Finally, combining (5.3) and (5.4), we deduce that $c_k \leq c - \varepsilon$; contradiction. □

Proof of Theorem 5.8. We can assume that card $K < \infty$ since otherwise the conclusion is obviously true. Thus, $f(K)$ has no accumulation points and $f(K) \subset \;]-\infty, b]$ for some $b \in \mathbb{R}$.

It follows from Theorem 3.5(iv), (vi) and Lemma 5.4 that $f\text{-cat}_X(X) = f\text{-cat}_X(f^b)$. Thus, $c_i \leq b$ for every $i \in \mathbb{N}$ such that $i \leq f\text{-cat}_X(X)$. Observe that

$$c_1 \geq \inf f(X) > -\infty.$$

The conclusion follows from Lemma 5.10. □

Proof of Theorem 5.9. We can assume that $f\text{-catt}_X(X) \geq 1$ otherwise the result is trivial. Also, as in the proof of Theorem 5.8, we can assume that card $K < \infty$, and we deduce that $f(K)$ has no accumulation points, $f(K) \subset \;]-\infty, b]$ for some $b \in \mathbb{R}$, and $c_i \leq b$ for every $i \in \mathbb{N}$ such that $i \leq f\text{-catt}_X(X)$.

Let $a = \min f(K)$. We claim that $c_1 \geq a$. Otherwise, there exists $B \in \Gamma^t_1$ such that $\sup f(B) = m < a$. It follows from Lemma 5.4 that for $\varepsilon > 0$ sufficiently small, $t^f_\varepsilon(B, X) = 0$ since there are no critical values in $[-1/\varepsilon, m]$. Hence $f\text{-catt}_X(B) = 0$; contradiction.

The conclusion follows from Lemma 5.10. □

5.3. Results with the relative f-category

We already know that the cardinality of the set of critical points of f is bigger or equal to its truncated f-category. Using the relative f-category permits us to get more precision on the number of critical points in a given interval $[a, b]$.

Theorem 5.11. *Let $-\infty < a < b \leq \infty$. Assume (5.1), X is locally contractible, and f satisfies $\mathcal{D}(X, K_c)$ for every $c \in [a, b] \cap \mathbb{R}$. Then*

$$\text{card}(K_{[a,b]}) \geq f\text{-cat}_{X,f^a}(f^b).$$

Proof. We assume that $\text{card}(K_{[a,b]}) < \infty$ and $f\text{-cat}_{X,f^a}(f^b) \geq 1$, otherwise, the result is trivial. For $n \in \mathbb{N}$, set

$$\Gamma^a_n = \{B \subset f^b : f\text{-cat}_{X,f^a}(B) \geq n\},$$

and

$$c_n = \inf_{B \in \Gamma_n^a} \sup f(B).$$

Observe that $a \leq c_1 \leq c_n \leq b$ for every $n \leq f\text{-cat}_{X,f^a}(f^b)$ since $f\text{-cat}_{X,f^a}(B) = 0$ if $B \subset f^a$.

We claim that

$$\text{if } a = c_1 = \cdots = c_m, \quad \text{then card } K_a \geq m. \tag{5.5}$$

Indeed, otherwise, card $K_a < m$. There exists $\delta > 0$ such that $f(K) \cap [a, a+\delta] = \{a\}$. Let $B \in \Gamma_m^a$ be such that $\sup f(B) < a+\delta$. There exists $\bar{\varepsilon} > 0$ such that

$$n_\varepsilon^f(B, X, f^a) \geq m \quad \text{for every } \varepsilon \in \;]0, \bar{\varepsilon}].$$

Lemma 5.2 implies that for every $y \in K_a$, there exists $\mathcal{V}_y$ a $(f, \bar{\varepsilon})$-contractible closed neighborhood of y. Denote

$$\mathcal{U}_a = \bigcup_{y \in K_a} \text{int}(\mathcal{V}_y).$$

Let $\varepsilon \in \;]0, \delta[$ and η be the continuous deformation given by Definition 5.3, and $\hat{\eta}$ given by Lemma 5.4 such that

$$\eta(f^{a+\varepsilon} \setminus \mathcal{U}_a, 1) \subset f^{a-\varepsilon} \quad \text{and} \quad \hat{\eta}(f^{a+\delta}, 1) \subset f^{a+\varepsilon}.$$

Therefore, arguing as in the proof of Lemma 5.10 yields

$$n_{\bar{\varepsilon}}^f(B, X, f^a) \leq n_{\bar{\varepsilon}}^f(f^{a+\delta}, X, f^a) \leq \text{card } K_a < m;$$

contradiction.

Now, we claim that

$$\text{if } a < c = c_{k+1} = \cdots = c_{k+m} < b, \quad \text{then card } K_c \geq m. \tag{5.6}$$

Indeed, otherwise, card $K_c < m$. There exists $\delta > 0$ such that $f(K) \cap [c - \delta, c + \delta] = \{c\}$. Notice that $f^{c+\delta} \in \Gamma_{k+m}^a$. There exists $\bar{\varepsilon} > 0$ such that

$$n_\varepsilon^f(f^{c+\delta}, X, f^a) \geq k + m \quad \text{for every } \varepsilon \in \;]0, \bar{\varepsilon}].$$

Lemma 5.2 implies that for every $y \in K_c$, there exists $\mathcal{V}_y$ a $(f, \bar{\varepsilon})$-contractible, closed neighborhood of y. Denote

$$\mathcal{U}_c = \bigcup_{y \in K_c} \text{int}(\mathcal{V}_y).$$

Arguing as above, we deduce that there exists $\varepsilon \in \;]0, \delta[$ such that

$$n_{\bar{\varepsilon}}^f(f^{c+\delta}, X, f^{c-\varepsilon}) \leq \text{card } K_c < m. \tag{5.7}$$

Notice that $f\text{-cat}_{X,f^a}(f^{c-\varepsilon}) \leq k$ since $c - \varepsilon < c$. So,

$$n_{\bar{\varepsilon}}^f(f^{c-\varepsilon}, X, f^a) \leq k. \tag{5.8}$$

Combining (5.7) and (5.8) lead to

$$n_{\bar{\varepsilon}}^f(f^{c+\delta}, X, f^a) \leq k + \text{card } K_c < k + m;$$

contradiction.

Finally, we claim that

$$\text{if } b = c_{k+1} = \cdots < \infty, \quad \text{then card } K_b \geq f\text{-cat}_{X,f^a}(f^b) - k. \tag{5.9}$$

Indeed, otherwise, card $K_b = m < f\text{-cat}_{X,f^a}(f^b) - k$. Notice that $f^b \in \Gamma_{k+m+1}^a$. There exists $\bar{\varepsilon} > 0$ such that

$$n_\varepsilon^f(f^b, X, f^a) \geq k + m + 1 \quad \text{for every } \varepsilon \in \;]0, \bar{\varepsilon}].$$

Arguing as above, we can show that there exists $\varepsilon > 0$ such that $n_{\bar{\varepsilon}}^f(f^b, X, f^{b-\varepsilon}) \leq \text{card } K_b$ and $n_{\bar{\varepsilon}}^f(f^{b-\varepsilon}, X, f^a) \leq k$, hence

$$n_{\bar{\varepsilon}}^f(f^b, X, f^a) \leq k + \text{card } K_b = k + m;$$

contradiction.

To conclude, we combine (5.5), (5.6) and (5.9). □

Imposing a stronger deformation property permits to obtain a lower bound to the number of critical points in $]a, b]$.

Theorem 5.12. *Let $-\infty < a < b \leq \infty$. Assume (5.1) and assume that X is locally contractible, and f satisfies $\mathcal{D}_2(X, K_a)$ and $\mathcal{D}(X, K_c)$ for every $c \in \,]a, b] \cap \mathbb{R}$. Then*

$$\operatorname{card}(K_{]a,b]}) \geq f\text{-cat}_{X,f^a}(f^b).$$

Proof. If a is an accumulation point of $f(K) \cap [a, b]$, $\operatorname{card}(K_{]a,b]}) = \infty$. Otherwise, the proof is analogous to the previous one after noticing that $c_1 > a$. □

Now, we deduce the existence of critical points in presence of linking.

Theorem 5.13. *Assume (5.1), X is locally contractible, and assume that (B, A) links (Q, P) where B and Q are closed. If*

$$\sup f(A) \leq a \leq \inf f(Q) \leq \sup f(B) = b \leq \inf f(P),$$

$a, b \in \mathbb{R}$, $\operatorname{dist}(A, Q) > 0$ if $\sup f(A) = \inf f(Q)$, $\operatorname{dist}(B, P) > 0$ if $\sup f(B) = \inf f(P)$, and f satisfies $\mathcal{D}(X, K_c)$ for every $c \in [a, b]$, then

$$k := \operatorname{card}(K_{[a,b]}) \geq f\text{-cat}_{X,f^a}(B). \tag{5.10}$$

Moreover,

(i) *if $a < \inf f(Q) \leq b < \inf f(P)$, then $k \geq 1$;*
(ii) *if $a < \inf f(Q) < b = \inf f(P)$, and $B \cap K_b = \emptyset$, then $k \geq 1 + \operatorname{card} K_b$;*
(iii) *if $a = \inf f(Q) \leq b < \inf f(P)$, then $k \geq 1 + \operatorname{card}(K_a \cap (X \setminus Q))$;*
(iv) *if $a = \inf f(Q) < b = \inf f(P)$, and $B \cap K_b = \emptyset$, then $k \geq 1 + \operatorname{card} K_b + \operatorname{card}(K_a \cap (X \setminus Q))$;*
(v) *if $a = \inf f(Q) = b = \inf f(P)$, then $k \geq 1 + \operatorname{card}(K_a \cap (X \setminus Q)) + \operatorname{card}(K_a \cap P)$.*

Proof. Theorems 3.16(iii) and 5.11 ensure (5.10). This inequality combined with Theorem 4.2 lead to statement (i).

(ii) Let $2\hat{\varepsilon} = \min\{\operatorname{dist}(B, P), \operatorname{dist}(B, K_b), b - \inf f(Q)\} > 0$. Take $\mathcal{U} = B(K_b, \hat{\varepsilon})$ and $\rho = \hat{\varepsilon}$, and let $\varepsilon \in \,]0, \hat{\varepsilon}[$ and η a continuous deformation given by Definition 5.3. Let λ be an Urysohn function such that $\lambda(f^{b-2\hat{\varepsilon}}) = 0$ and $\lambda(\overline{X \setminus f^{b-\hat{\varepsilon}}}) = 1$. Set $\hat{\eta}(x, t) = \eta(x, \lambda(x)t)$. It is easy to verify that $(\hat{\eta}(B, 1), A)$ links (Q, P),

$$\sup f(A) \leq a \leq \inf f(Q) \leq \sup f(\hat{\eta}(B, 1)) \leq b - \varepsilon < \inf f(P).$$

Again Theorems 4.2 and 5.11 combined with Theorem 3.16 give

$$\operatorname{card}(K_{[a,b-\varepsilon]}) \geq f\text{-cat}_{X,f^a}(\hat{\eta}(B, 1)) \geq 1.$$

(iii) Assume that $k = \operatorname{card}(K_a \cap (X \setminus Q))$, i.e. $K_{[a,b]} = K_a \cap (X \setminus Q)$. Let $3\tilde{\varepsilon} = \min\{\operatorname{dist}(A, Q), \operatorname{dist}(Q, K_a)\} > 0$. Take $\mathcal{U} = B(K_a, \tilde{\varepsilon})$ and $\rho = \tilde{\varepsilon}$. Combining Definition 5.3 with Lemma 5.4, and using an Urysohn function λ such that $\lambda(A) = 0$, $\lambda(X \setminus B(A, \tilde{\varepsilon})) = 1$, we obtain $\bar{\varepsilon} \in \,]0, \tilde{\varepsilon}[$ and $\tilde{\eta} \in \mathcal{N}(A) \cap \mathcal{N}_f$ such that

$$\tilde{\eta}(B, 1) \subset f^{a-\bar{\varepsilon}} \cup B(K_a, 2\tilde{\varepsilon}) \cup B(A, 2\tilde{\varepsilon}).$$

Since (B, A) links (Q, P) and since $b < \inf f(P)$, there exists $\bar{x} \in B$ such that $\tilde{\eta}(\bar{x}, 1) \in Q$; contradiction since $Q \cap (f^{a-\bar{\varepsilon}} \cup B(K_a, 2\tilde{\varepsilon}) \cup B(A, 2\tilde{\varepsilon})) = \emptyset$.

(iv) Argue as in (iii) with $\hat{\eta}(B, 1)$ instead of B, where $\hat{\eta}$ is obtained in (ii).

(v) Assume $K_a = (K_a \cap (X \setminus Q)) \cup (K_a \cap P)$. Let $2\delta = \operatorname{dist}(B, P)$ and

$$3\tilde{\varepsilon} = \min\{\delta,\ \operatorname{dist}(A, Q),\ \operatorname{dist}(K_a, Q \cap B(B, \delta))\} > 0.$$

Take $\mathcal{U} = B(K_a, \tilde{\varepsilon})$ and $\rho = \tilde{\varepsilon}$. By Definition 5.3, and using an Urysohn function λ such that $\lambda(A) = 0$, $\lambda(X \setminus B(A, \tilde{\varepsilon})) = 1$, we obtain $\bar{\varepsilon} \in \,]0, \tilde{\varepsilon}[$ and $\tilde{\eta} \in \mathcal{N}(A) \cap \mathcal{N}_f$ such that

$$\tilde{\eta}(B \times [0, 1]) \subset B(B, \tilde{\varepsilon}) \quad \text{and} \quad \tilde{\eta}(B, 1) \subset f^{a-\bar{\varepsilon}} \cup B(K_a, 2\tilde{\varepsilon}) \cup B(A, 2\tilde{\varepsilon}).$$

Since (B, A) links (Q, P) and since $\tilde{\eta}(B \times [0, 1]) \cap P = \emptyset$, there exists $\bar{x} \in B$ such that $\tilde{\eta}(\bar{x}, 1) \in Q \cap B(B, \tilde{\varepsilon})$; contradiction since $Q \cap B(B, \tilde{\varepsilon}) \cap (f^{a-\bar{\varepsilon}} \cup B(K_a, 2\tilde{\varepsilon}) \cup B(A, 2\tilde{\varepsilon})) = \emptyset$. □

The following result gives a lower bound to the number of critical points of f in presence of linking of type splitting spheres. It is a direct consequence of Theorems 4.7 and 5.11.

Theorem 5.14. *Let $E = E_1 \oplus E_2$ be a Banach space with $0 < \dim(E_1) < \infty$, and M a compact manifold. Let $f : E \times M \to \mathbb{R}$ be continuous and K satisfying (5.1). Assume that there exist $r_1, r_2 > 0$ and $a, b \in \mathbb{R}$ such that*

$$\sup f(S_1 \times M) \leq a < \inf f(B_2 \times M) \leq \sup f(B_1 \times M) \leq b < \inf f(S_2 \times M).$$

If f satisfies $\mathcal{D}(E \times M, K_c)$ for every $c \in [a, b]$, then

$$\begin{aligned}\operatorname{card}(K_{[a,b]}) &\geq f\text{-cat}_{E\times M,f^a}(B_1 \times M)\\ &\geq \operatorname{cuplength}(B_1 \times M, S_1 \times M) + 1 \geq \operatorname{cuplength}(M) + 1.\end{aligned}$$

5.4. Particular cases

The results of the previous subsection are satisfied if for example X is a C^2-Finsler manifold, f is C^1 and satisfies the Palais–Smale condition $(\mathrm{PS})_c$. Indeed, it is well known that K the set of critical points of f satisfies (5.1), and f satisfies $\mathcal{D}(X, K_c)$ and $\mathcal{D}_2(X, K_c)$; see for example [26,3,27].

The previous results also hold if X is a locally contractible complete metric space, f is continuous, $K = \{x \in X : |\mathrm{d}f|(x) = 0\}$, where $|df|$ means the weak slope of f introduced by Degiovanni and Marzocchi [13], and f satisfies $(\mathrm{PS})_c$, the Palais–Smale condition in the sense of the weak slope; see [14,28]. It is worth mentioning that results can also be obtained for lower semi-continuous functionals; see [14].

6. Limit *f*-category

We finished the last section with a result giving a lower bound to the number of critical points of a functional f defined on $E\times M$ where $E = E_1\oplus E_2$ is a Banach space with $0 < \dim(E_1) < \infty$. It is well known that many solutions to partial differential equations, or Hamiltonian systems are critical points of suitable f defined on $E \times M$ with $\dim(E_1) = \dim(E_2) = \infty$. Fournier, Lupo, Ramos and Willem [9] introduced the limit relative category to treat this type of problem. In order to obtain a better lower bound to the number of critical points, we want to take into account the functional.

In this section, X is a topological space and $\{X_k\}$ is a sequence of closed subspaces of X such that

$$X = \overline{\bigcup_{k\in\mathbb{N}} X_k} \quad \text{and} \quad X_1 \subset X_2 \subset \cdots .$$

For $B \subset X$, we denote $B_k = B \cap X_k$.

We consider a functional $f : X \to \mathbb{R}$ and we denote $f_k = f|_{X_k}$. We define

$$\mathcal{N}_f^* = \{\{\eta_k : X_k \times [0, 1] \to X_k\} : \text{ for every } k \in \mathbb{N},\ \eta_k \text{ is continuous}, \eta_k(x, 0) = x,$$
$$f(\eta_k(x, t)) \le f(x)\ \forall x \in X_k,\ t \in [0, 1]\} .$$

6.1. Definition and properties

Here are the definitions of the limit f-category, the limit truncated f-category and the relative limit f-category.

Definition 6.1. Let $B \subset X$.

(a) We define the *limit f-category of B in X* by

$$f\text{-cat}_X^\infty(B) = \sup_{\varepsilon>0} \limsup_{k\to\infty} n_\varepsilon^{f_k}(B_k, X_k),$$

where $n_\varepsilon^{f_k}(B_k, X_k)$ is defined in Definition 3.2.

(b) We define the *limit truncated f-category of B in X* by

$$f\text{-catt}_X^\infty(B) = \sup_{\varepsilon>0} \limsup_{k\to\infty} t_\varepsilon^{f_k}(B_k, X_k),$$

where $t_\varepsilon^{f_k}(B_k, X_k)$ is defined in Definition 3.9.

(c) Let Y be a closed subset of X. We define the *limit f-category of B relative to Y in X* by

$$f\text{-cat}_{X,Y}^\infty(B) = \sup_{\varepsilon>0} \limsup_{k\to\infty} n_\varepsilon^{f_k}(B_k, X_k, Y_k),$$

where $n_\varepsilon^{f_k}(B_k, X_k, Y_k)$ is defined in Definition 3.14.

The limit f-category (resp. limit truncated f-category, limit relative f-category) satisfies analogous properties to the f-category (resp. truncated f-category, relative f-category).

Theorem 6.2. *Let Y be a closed subset of X, $A, B \subset X$ the following properties are satisfied:*

(i) $f\text{-cat}_X^\infty(B) \ge \text{cat}_X^\infty(B)$ *and* $f\text{-cat}_{X,Y}^\infty(B) \ge \text{cat}_{X,Y}^\infty(B)$;

(ii) $f\text{-cat}_{X,Y}^\infty(B) \le f\text{-cat}_X^\infty(B)$;

(iii) *if* $A \subset B$, *then* $f\text{-cat}_X^\infty(A) \le f\text{-cat}_X^\infty(B)$, $f\text{-catt}_X^\infty(A) \le f\text{-catt}_X^\infty(B)$, *and* $f\text{-cat}_{X,Y}^\infty(A) \le f\text{-cat}_{X,Y}^\infty(B)$;

(iv) $f\text{-cat}_X^\infty(A \cup B) \le f\text{-cat}_X^\infty(A) + f\text{-cat}_X^\infty(B)$, $f\text{-catt}_X^\infty(A \cup B) \le f\text{-catt}_X^\infty(A) + f\text{-cat}_X^\infty(B)$, *and* $f\text{-cat}_{X,Y}^\infty(A \cup B) \le f\text{-cat}_{X,Y}^\infty(A) + f\text{-cat}_X^\infty(B)$;

(v) *if Y is such that for every* $\{\eta_k\} \in \mathcal{N}_f^*$ *for which there exists* $k_0 \in \mathbb{N}$ *such that for every* $k \ge k_0$, $\eta_k(A_k, 1) \subset B_k$ *(resp. and* $\eta_k(Y_k, t) \subset Y_k$*), then* $f\text{-cat}_X^\infty(A) \le f\text{-cat}_X^\infty(B)$, $f\text{-catt}_X^\infty(A) \le f\text{-catt}_X^\infty(B)$, *(resp.* $f\text{-cat}_{X,Y}^\infty(A) \le f\text{-cat}_{X,Y}^\infty(B)$*)*;

(vi) *if Y is such that for every* $\{\eta_k\} \in \mathcal{N}_f^*$ *for which there exists* $k_0 \in \mathbb{N}$ *such that for every* $k \ge k_0$, *one has* $\eta_k(Y_k, t) \subset Y_k$; *and if Z is a closed set such that there exists* $\{\hat\eta_k\} \in \mathcal{N}_f^*$ *for which there exists* $k_1 \in \mathbb{N}$ *such that for every* $k \ge k_1$, $\hat\eta_k(Z_k, 1) \subset Y_k$; *then*

$$f\text{-cat}_{X,Y}^\infty(B) \le f\text{-cat}_{X,Z}^\infty(B).$$

Proof. Observe that if $f\text{-cat}_X^\infty(B) = j \in \mathbb{N}$ (resp. $f\text{-catt}_X^\infty(B) = j$, $f\text{-cat}_{X,Y}^\infty(B) = j$), there exist $\hat{\varepsilon}$ and an increasing sequence $\{k_m\}$ such that for every $\varepsilon \in \,]0, \hat{\varepsilon}]$

$$n_\varepsilon^{f_{k_m}}(B_{k_m}, X_{k_m}) \to j \quad \text{as } m \to \infty$$
$$(\text{resp. } t_\varepsilon^{f_{k_m}}(B_{k_m}, X_{k_m}) \to j,\ n_\varepsilon^{f_{k_m}}(B_{k_m}, X_{k_m}, Y_{k_m}) \to j \text{ as } m \to \infty),$$

and there exists $k_\varepsilon > 0$ such that

$$n_\varepsilon^{f_k}(B_k, X_k) \le j \quad \text{for every } k \ge k_\varepsilon$$
$$(\text{resp. } t_\varepsilon^{f_k}(B_k, X_k) \le j,\ n_\varepsilon^{f_k}(B_k, X_k, Y_k) \le j \text{ for every } k \ge k_\varepsilon).$$

Using this fact and arguing as in the proofs of Theorems 3.5, 3.12 and 3.16, we prove the result. □

As we did for the relative f-category, we can compare

$$f\text{-cat}_{X,Y}^\infty(B) \quad \text{with } f\text{-cat}_{\hat{X},\hat{Y}}^\infty(B \cap \hat{X})$$

for a closed pair $(\hat{X}, \hat{Y}) \subset (X, Y)$. Arguing as in the proof of Theorem 3.19, we obtain the following result.

Theorem 6.3. *Let Y and $\hat{Y} \subset \hat{X}$ be closed subsets of X, and let $B \subset X$.*

(i) *If there exists $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$, $(\hat{X}_k, \hat{Y}_k)$ is a retract of (X_k, Y_k) for some retraction $r_k : X_k \to \hat{X}_k$ such that $f(r_k(x)) \le f(x)$ for every $x \in X_k$, then*

$$f\text{-cat}_{X,Y}^\infty(B) \ge f\text{-cat}_{\hat{X},\hat{Y}}^\infty(B \cap \hat{X}).$$

(ii) *If $\{\eta_k\} \in \mathcal{N}_f^*$ is such that there exists $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$, $(\hat{X}_k, \hat{Y}_k)$ is a deformation retract of (X_k, Y_k) with the deformation η_k, i.e $\eta_k(Y_k \times [0, 1]) \subset Y_k$, $(\eta_k(X_k, 1), \eta_k(Y_k, 1)) = (\hat{X}_k, \hat{Y}_k)$, and $\eta_k(x, t) = x$ for every $x \in \hat{X}_k$, $t \in [0, 1]$, then*

$$f\text{-cat}_{X,Y}^\infty(B) = f\text{-cat}_{\hat{X},\hat{Y}}^\infty(B \cap \hat{X}).$$

6.2. *Limit f-category and critical points*

We assume that X is a metric space and $f : X \to \mathbb{R}$ is continuous. As in Section 5, we consider $K \subset X$ called the set of *critical points of f* which satisfies (5.1).

Definition 6.4. We say that the metric space X is $*$-*locally contractible* if for every $y \in X$ and every $\mathcal{U}$ neighborhood of y, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of y such that there exists $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$, $\mathcal{V}_k$ is contractible in $\mathcal{U}_k$.

In order to obtain critical points, we need a deformation property which is in some sense deformations property satisfied by $f|_{X_k}$ and uniform with respect to k for k sufficiently large.

Definition 6.5. Let $c \in \mathbb{R}$. We say that f satisfies $\mathcal{D}^\star(X, K_c)$ if for every $\hat{\varepsilon} > 0$ and every neighborhood $\mathcal{U}$ of K_c, there exist $\varepsilon \in \,]0, \hat{\varepsilon}[$ and $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ there exists a continuous deformation $\eta_k : X_k \times [0, 1] \to X_k$ such that

(a) $\eta_k(x, 0) = x$ for every $(x, t) \in X_k \times \{0\} \cup f_k^{c-\hat{\varepsilon}} \times [0, 1]$;
(b) $f(\eta_k(x, t)) \le f(x)$ for every $(x, t) \in X_k \times [0, 1]$;
(c) $\eta_k(f_k^{c+\varepsilon} \setminus \mathcal{U}, 1) \subset f_k^{c-\varepsilon}$.

This property permits to deduce a noncritical interval deformation lemma. The proof is analogous to the proof of Lemma 5.4.

Lemma 6.6. *Let $\hat{\varepsilon} > 0$, $a \in \mathbb{R}$, $b \in \,]a, \infty[$. Assume that and f satisfies $\mathcal{D}^\star(X, K_c)$ and $K_c = \emptyset$ for every $c \in [a, b]$. Then there exists $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ there exists a continuous deformation $\eta_k : X_k \times [0, 1] \to X_k$ such that*

(i) *$\eta_k(x, t) = x$ for every $(x, t) \in X_k \times \{0\} \cup f_k^{a-\hat{\varepsilon}} \times [0, 1]$;*
(ii) *$f(\eta_k(x, t)) \le f(x)$ for every $(x, t) \in X_k \times [0, 1]$;*
(iii) *$\eta_k(f_k^b, 1) \subset f_k^a$.*

A result analogous to Theorem 5.11 is true in this context.

Theorem 6.7. *Let $a < b$ be real constants. Assume (5.1) and assume that X is $*$-locally contractible, and f satisfies $\mathcal{D}^*(X, K_c)$ for every $c \in [a, b]$. Then*

$$\text{card}(K_{[a,b]}) \ge f\text{-cat}_{X,f^a}^\infty(f^b).$$

Proof. We assume that $\mathrm{card}(K_{[a,b]}) < \infty$ and $f\text{-cat}^{\infty}_{X,f^a}(f^b) \geq 1$, otherwise, the result is trivial. For $n \in \mathbb{N}$, set

$$\Gamma_n^a = \{B \subset f^b : f\text{-cat}^{\infty}_{X,f^a}(B) \geq n\},$$

and

$$c_n = \inf_{B \in \Gamma_n^a} \sup f(B).$$

Observe that $a \leq c_1 \leq c_n \leq b$ for every $n \leq f\text{-cat}^{\infty}_{X,f^a}(f^b)$ since $f\text{-cat}^{\infty}_{X,f^a}(B) = 0$ if $B \subset f^a$.

We claim that

$$\text{if } a = c_1 = \cdots = c_m, \quad \text{then card } K_a \geq m. \tag{6.1}$$

Indeed, otherwise, $\text{card } K_a < m$. There exists $\delta > 0$ such that $f(K) \cap [a, a+\delta] = \{a\}$. Let $B \in \Gamma_m^a$ be such that $\sup f(B) < a+\delta$. There exist $\bar{\varepsilon}$ and an increasing sequence $\{k_j\}$ such that for every $\varepsilon \in \ ]0, \bar{\varepsilon}]$

$$n_{\varepsilon}^{f_{k_j}}(B_{k_j}, X_{k_j} f_{k_j}^a) \geq m \quad \text{as } j \to \infty,$$

and there exists $k_\varepsilon > 0$ such that

$$n_{\varepsilon}^{f_k}(B_k, X_k, f_k^a) \leq f\text{-cat}^{\infty}_{X,f^a}(B) \quad \text{for every } k \geq k_\varepsilon.$$

Since X is $*$-locally contractible and f is continuous, it is easy to show that there exists $\bar{k} \in \mathbb{N}$ such that for every $y \in K_a$, there exists $\mathcal{V}^y$ a closed neighborhood of y such that $\mathcal{V}_k^y$ is $(f_k, \bar{\varepsilon})$-contractible for every $k \geq \bar{k}$. Denote

$$\mathcal{U}^a = \bigcup_{y \in K_a} \mathrm{int}(\mathcal{V}^y).$$

Let $\varepsilon \in \ ]0, \delta[$, $\hat{k} \geq \bar{k}$ and for every $k \geq \hat{k}$, η_k be the continuous deformation given by Definition 6.5, and $\hat{\eta}_k$ given by Lemma 6.6 such that

$$\eta_k(f_k^{a+\varepsilon} \setminus \mathcal{U}^a, 1) \subset f_k^{a-\varepsilon} \quad \text{and} \quad \hat{\eta}_k(f_k^{a+\delta}, 1) \subset f_k^{a+\varepsilon}.$$

Therefore, for every $k_j \geq \hat{k}$,

$$n_{\bar{\varepsilon}}^{f_{k_j}}(B_{k_j}, X_{k_j}, f_{k_j}^a) \leq n_{\bar{\varepsilon}}^{f_{k_j}}(f_{k_j}^{a+\delta}, X_{k_j}, f_{k_j}^a) \leq \text{card } K_a < m;$$

contradiction.

Arguing as above and as in the proof of Theorem 5.11, we can show that

$$\text{if } a < c = c_{k+1} = \cdots = c_{k+m} < b, \quad \text{then card } K_c \geq m, \tag{6.2}$$

and

$$\text{if } b = c_{k+1} = \cdots < \infty, \quad \text{then card } K_b \geq f\text{-cat}^{\infty}_{X,f^a}(f^b) - k. \tag{6.3}$$

To conclude, we combine (6.1)–(6.3). □

In this context, we can also deduce the existence of critical points in presence of linking.

Theorem 6.8. *Let $A \subset B \subset X$, $P \subset Q \subset X$ be such that B and Q are closed. Assume (5.1), X is $*$-locally contractible, and assume that (B_k, A_k) links (Q_k, P_k) in X_k for k sufficiently large. If*

$$\sup f(A) \leq a < \inf f(Q) \leq \sup f(B) = b < \inf f(P),$$

and f satisfies $\mathcal{D}^*(X, K_c)$ for every $c \in [a, b]$, then

$$\mathrm{card}(K_{[a,b]}) \geq f\text{-cat}^{\infty}_{X,f^a}(B) \geq 1.$$

Proof. It follows from Theorems 6.2 and 6.7 that the conclusion holds if we can show that $f\text{-cat}^{\infty}_{X,f^a}(B) \geq 1$.

Let $\delta > 0$ be such that $a + \delta < \inf f(Q)$. Assume that $f\text{-cat}^{\infty}_{X,f^a}(B) = 0$. There exists $\{\eta_k\} \in \mathcal{N}_f^*$ for which there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, $\eta_k(f_k^a, t) \subset f_k^a$ and $\eta_k(B_k, 1) \subset f_k^a$.

Consider $\lambda : X \to [0, 1]$ an Urysohn function such that $\lambda(f^a) = \{0\}$ and $\lambda(\overline{X \setminus f^{a+\delta}}) = \{1\}$. Define a deformation $\hat{\eta}_k$ by $\hat{\eta}_k(x, t) = \eta_k(x, t\lambda(x))$. It is easy to check that $\hat{\eta}_k \in \mathcal{N}(A_k)$ for the space X_k. Observe that $\hat{\eta}_k$ does not satisfy condition (b) of Definition 4.1. Indeed, $f(\hat{\eta}_k(x, t)) \leq f(x) \leq b < \inf f(P) \leq \inf f(P_k)$ for every $x \in B_k$.

Therefore, for $k \geq k_0$ such that (B_k, A_k) links (Q_k, P_k) in X_k, there exists $\hat{x}_k \in B_k$ such that $\hat{\eta}_k(\hat{x}_k, 1) \in Q_k$, and hence

$$a + \delta < \inf f(Q) \leq \inf f(Q_k) \leq f(\hat{\eta}_k(\hat{x}_k, 1)) \leq f(\hat{x}_k).$$

So, $\lambda(\hat{x}_k) = 1$. Thus $\hat{\eta}_k(\hat{x}_k, 1) = \eta_k(\hat{x}_k, 1) \in f_k^a$. Contradiction. □

We can extend Theorem 5.14 to the case where $E = E^1 \oplus E^2$ with $\dim(E^1) = \infty$.

Let $E = E^1 \oplus E^2$ be a Banach space, and M a compact manifold. For $i = 1, 2$, assume that there exist $E_1^i \subset E_2^i \subset \cdots$, a sequence of closed subspaces of E^i such that

$$E^i = \overline{\bigcup_{k\in\mathbb{N}} E_k^i},$$

with $0 < \dim(E_k^1) < \infty$. Denote

$$X = E \times M \quad \text{and} \quad X_k = E_k \times M, \quad \text{where } E_k = E_k^1 \oplus E_k^2.$$

For $r_i > 0$, set

$$B^i = \{x \in E^i : \|x\| \le r_i\},$$
$$S^i = \{x \in E^i : \|x\| = r_i\}.$$

Theorem 6.9. *Let E, M, and X be as above. Let $f : X \to \mathbb{R}$ be continuous and K satisfying (5.1). Assume that there exist $r_1, r_2 > 0$ and $a, b \in \mathbb{R}$ such that*

$$\sup f(S^1 \times M) \le a < \inf f(B^2 \times M) \le \sup f(B^1 \times M) \le b < \inf f(S^2 \times M).$$

If f satisfies $\mathcal{D}^*(X, K_c)$ for every $c \in [a, b]$, then

$$\begin{aligned}\text{card}(K_{[a,b]}) &\ge f\text{-cat}^{\infty}_{X,f^a}(B_1 \times M)\\ &\ge \limsup_{k\to\infty} \text{cuplength}(B_k^1 \times M, S_k^1 \times M) + 1 \ge \text{cuplength}(M) + 1.\end{aligned}$$

Proof. It follows from Theorems 6.2(iii) and 6.7 that

$$\text{card}(K_{[a,b]}) \ge f\text{-cat}^{\infty}_{X,f^a}(f^b) \ge f\text{-cat}^{\infty}_{X,f^a}(B_1 \times M).$$

The proof of Proposition 4.8 permits to see that

$$n_\varepsilon^{f_k}(B_k^1 \times M, X_k, f_k^a) \ge \text{cat}_{E_k\setminus S_k^2\times M, f_k^a}(B_k^1 \times M) \quad \forall \varepsilon > 0.$$

This inequality combined with Theorem 2.3 leads to

$$n_\varepsilon^{f_k}(B_k^1 \times M, X_k, f_k^a) \ge \text{cuplength}(B_k^1 \times M, S_k^1 \times M) + 1.$$

The conclusion follows from the definition of $f\text{-cat}^{\infty}_{X,f^a}(B_1 \times M)$ and Lemma 2.4. □

6.3. Particular cases

The results of the previous subsection are satisfied if for example X is a C^2-Finsler manifold, f is C^1 and satisfies the Palais–Smale-star condition $(\text{PS})_c^*$. Indeed, K, the set of critical points of f, satisfies (5.1), and f satisfies $\mathcal{D}^*(X, K_c)$; see [11].

The results of Section 6.2 also hold if X is a $*$-locally contractible, complete metric space, f is continuous, $K = \{x \in X : |df|(x) = 0\}$, where $|df|$ means the weak slope of f introduced in [13], and f satisfies $(\text{PS})_c^*$ the Palais–Smale-star condition in the sense of the weak slope; see [29] or [30].

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