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On a notion of category depending on a functional, Part II: An application to Hamiltonian systems

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ABSTRACT

We apply the notion of limit relative category depending on a functional f introduced in Beauchemin and Frigon (2009) [16] to establish the existence of multiple periodic solutions to a Hamiltonian system with periodic nonlinearity and superquadratic growth in some spatial variables.

1. Introduction

In this paper, we study the following Hamiltonian system

$$J\dot{u} + A(t)u + \nabla H(t, u) = 0, \tag{1.1}$$

where J is the standard symplectic matrix, $A(t)$ is a symmetric $2N \times 2N$ matrix continuous, A and H are 2π-periodic in t.

In past years, several authors studied the existence of periodic solutions to (1.1); see for instance [1–14].

Some of them, starting with Rabinowitz [12], established the existence of a nontrivial periodic solution in the case where H is superquadratic. In particular, in [6,7,9], they used the fact that the local linking condition was satisfied.

Multiplicity results were obtained by adding a periodicity condition. The case where $\nabla H(t, u) = h(t) + \nabla \hat{H}(t, u)$ and $\hat{H}$ is periodic with respect to the spectrum of $J\dot{u} + Au$ was treated in [2,4,5,10]. More precisely, in [2,4], it was also assumed that $\nabla \hat{H}$ and $\hat{H}_{uu}$ are bounded. The Hamiltonian did not have to be of class C^2 in [5,10] but $\nabla \hat{H}$ was asymptotic to 0 at infinity. Their proofs rely on the notion of limit relative category introduced by Fournier, Lupo, Ramos and Willem [5].

Liu [8] and Daouas [3] obtained multiplicity results in the case where $A = 0$, ∇H is bounded and H is periodic in some spatial variables.

A pioneering contribution is due to Conley and Zehnder [15]. Performing a Lyapunov–Schmidt reduction and using the Conley index theory, they obtained the existence of $2N + 1$ solutions in the case where $A \equiv 0$ and H is periodic of class C^2.

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They also considered the case where $A \equiv 0$, H is periodic in q, and $H(t,p,q) = \langle a,p\rangle + \langle p, b(x)p\rangle/2$ for $\|p\| \geq r > 0$ with a a constant and $b \in L(\mathbb{R}^N)$. They established the existence of $N+1$ periodic solutions. In [14], Szulkin considered the case where A has the form

$$A(t) = \begin{pmatrix} B(t) & 0 \\ 0 & 0 \end{pmatrix} \tag{1.2}$$

with $B(t) \equiv B$ non-singular. Assuming that H is periodic in q, ∇H is bounded, and $H(t,p,q) \to \infty$ (or $-\infty$) uniformly in t as $\|p\| \to \infty$, he showed that (1.1) as at least $N+1$ distinct periodic solutions. His proof relied on a new notion of relative category.

In this paper, we combine periodic conditions with superquadratic growth to obtain an existence and a multiplicity result. More precisely, we present a result establishing the existence of $N+1$ distinct periodic solutions to (1.1). The matrix A has the form (1.2) with $B(t)$ non-singular. We assume also that H is periodic in q and H has a superquadratic growth in p. Our existence results rely on the notion of limit relative category depending on a functional f (also called limit relative f-category) introduced by the authors [16], (see also [17]). It is worth to mention that the limit relative f-category is always larger or equal to the limit relative category introduced in [5]. This limit relative f-category takes advantage of linking-type situation, and in particular, general linking pairs in the sense of [18].

Our paper is organized as follows. In Section 2, we recall the notion of limit relative f-category and results related to it, obtained in [16]. In Section 3, we state our main theorem (Theorem 3.2). Technical results are obtained in the next section, while in Section 5, an equivalent formulation of the problem is presented. Finally the proof of our main theorem is given in the last section.

2. Category depending on a functional

The proof of our main theorem will rely on results on the category depending on a functional obtained by the authors in [16] (see also [17]). We recall some of them for the sake of completeness.

2.1. The notion of f-category

Let X be a normal topological space, and $f : X \to \mathbb{R}$ continuous.

Definition 2.1. Let A be a subset of X and $\varepsilon > 0$, we say that A is (f,ε)-*contractible in* X if there exist $\hat{x} \in X$ and a continuous deformation $\eta : A \times [0,1] \to X$ such that for all $x \in A$,

(a) $\eta(x,0) = x$;
(b) $\eta(x,1) = \hat{x}$;
(c) $f(\eta(x,t)) \leq f(x) + \varepsilon$ for all $t \in [0,1]$.

Using this notion of (f,ε)-contractibility, the authors introduced a notion of category depending on the functional f.

Definition 2.2. Let $B \subset X$. The f-*category of* B in X is defined by

$$f\text{-cat}_X(B) = \sup_{\varepsilon>0} n_\varepsilon^f(B,X),$$

where $n_\varepsilon^f(B,X)$ is the smallest $n \in \mathbb{N}$ such that there exist closed subsets $A_1,\ldots,A_n$ satisfying:

(a) $B \subset \bigcup_{i=1}^n A_i$;
(b) A_i is (f,ε)-contractible in X, for all $i = 1,\ldots,n$.

If such an n does not exist, we set $n_\varepsilon^f(B,X) = \infty$, and if $B = \emptyset$, we set $n_\varepsilon^f(B,X) = 0$.

The f-category satisfies the usual properties of the Lusternik–Schnirelman category. However

$$f\text{-cat}_X(B) \geq \text{cat}_X(B) \quad \forall B \subset X.$$

In general, this inequality is strict.

Observe that $f\text{-cat}_X(B) = \infty$ if $\inf f(B) = -\infty$. One sees that, as for the classical category, the f-category is not appropriate to treat unbounded functional. For this reason, the authors introduced a notion of relative category depending on the functional f. Denote

$$\mathcal{N}_f = \{\eta : X \times [0,1] \to X : \eta \text{ is continuous}, \eta(x,0) = x \text{ and}, f(\eta(x,t)) \leq f(x)\ \forall x \in X, t \in [0,1]\}.$$

Definition 2.3. Let Y be a closed subset of X, and $B \subset X$. We define the f-*category of* B *relative to* Y *in* X by

$$f\text{-cat}_{X,Y}(B) = \sup_{\varepsilon>0} n_\varepsilon^f(B,X,Y),$$

where $n_\varepsilon^f(B,X,Y)$ is the smallest $n \in \mathbb{N} \cup \{0\}$ such that there exist closed subsets $A_0,\ldots,A_n$ satisfying:

(a) $B \subset \bigcup_{i=0}^n A_i$;

(b) A_i is (f, ε)-contractible in X, for all $i = 1, \dots, n$;
(c) there exists $\eta_0 \in \mathcal{N}_f$ such that $\eta_0(Y, t) \subset Y$, and $\eta_0(A_0, 1) \subset Y$.

If such an n does not exist, we set $n^f_\varepsilon(B, X, Y) = \infty$.

Again, most of the usual properties of the relative category are preserved by the relative f-category. Moreover

$$f\text{-cat}_{X,Y}(B) \geq \text{cat}_{X,Y}(B) \quad \forall B \subset X.$$

In general, this inequality is strict.

The relative f-category is related to the general notion of linking introduced in [18].

Definition 2.4. Let $A \subset B \subset X$, $P \subset Q \subset X$. We say that (B, A) *links* (Q, P) if $B \cap Q \neq \emptyset$, $A \cap Q = \emptyset$, $B \cap P = \emptyset$ and for every $\eta \in \mathcal{N}(A)$, one of the following conditions is satisfied:

(a) $\eta(B, 1) \cap Q \neq \emptyset$;
(b) $\eta(B,]0, 1[) \cap P \neq \emptyset$;

where

$$\mathcal{N}(A) = \{\eta : X \times [0, 1] \to X \text{ continuous } : \eta(x, t) = x \,\forall (x, t) \in X \times \{0\} \cup A \times [0, 1]\}.$$

Observe that it is not assumed that A and P are nonempty. In the classical notion of linking, $A = \partial B$ and $P = \emptyset$. In what follows, we use the convention: $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$, and $\text{dist}(\emptyset, Q) = \infty$.

Theorem 2.5. *Assume that (B, A) links (Q, P), A is closed, and there exists $a \in \mathbb{R}$ such that*

$$\sup f(A) \leq a \leq \inf f(Q) \leq \sup f(B) < \inf f(P).$$

(i) *If* $\sup f(A) \leq a < \inf f(Q)$, *then*

$$f\text{-cat}_{X,f^a}(B) \geq 1 \quad \textit{and} \quad f\text{-cat}_{X,A}(B) \geq 1.$$

(ii) *If* $\sup f(A) = \inf f(Q)$ *and* $A \cap \overline{Q} = \emptyset$, *then* $f\text{-cat}_{X,A}(B) \geq 1$.

2.2. The f-category and critical points

In this section X is a metric space. As before $f : X \to \mathbb{R}$ is continuous.

A subset K of X is given and called the set of *critical points of f*. We assume that K satisfies

$$K \cap f^{-1}(C) \text{ is compact for every compact set } C \subset \mathbb{R}. \tag{2.1}$$

For $c \in \mathbb{R}$ and $I \subset \mathbb{R}$, we denote

$$K_c = K \cap f^{-1}(c), \qquad K_I = K \cap f^{-1}(I), \quad \text{and} \quad f^c = \{x \in X : f(x) \leq c\}.$$

The metric space X is assumed to satisfy a condition of contractibility. This notion can be found for example in Borsuk [19].

Definition 2.6. We say that a metric space X is *locally contractible* if for every $x \in X$ and every $\mathcal{U}$ neighborhood of x, there exists a closed neighborhood $\mathcal{V} \subset \mathcal{U}$ of x contractible in $\mathcal{U}$; that is there exist $\hat{x} \in \mathcal{U}$ and a continuous deformation $h : \mathcal{V} \times [0, 1] \to \mathcal{U}$ such that $h(v, 0) = v$ and $h(v, 1) = \hat{x}$ for every $v \in \mathcal{V}$.

It is well known that deformation lemmas play a key role in critical point theory. In this abstract context, f has to satisfy a suitable deformation property.

Definition 2.7. Let $c \in \mathbb{R}$. We say that f *satisfies* $\mathcal{D}(X, K_c)$ if for every $\hat{\varepsilon}, \rho > 0$ and every neighborhood $\mathcal{U}$ of K_c, there exist $\varepsilon \in\ ]0, \hat{\varepsilon}[$ and a continuous deformation $\eta : X \times [0, 1] \to X$ such that

(a) $\eta(x, t) = x$ for every $(x, t) \in X \times \{0\} \cup f^{c-\hat{\varepsilon}} \times [0, 1]$;
(b) $f(\eta(x, t)) \leq f(x)$ for every $(x, t) \in X \times [0, 1]$;
(c) $\eta(f^{c+\varepsilon} \setminus \mathcal{U}, 1) \subset f^{c-\varepsilon}$;
(d) $\text{dist}(x, \eta(x, t)) \leq \rho$ for every $(x, t) \in X \times [0, 1]$.

The following result establishes that $f\text{-cat}_X(X)$ is a lower bound to the cardinality of the set of critical points of f when f is bounded from below.

Theorem 2.8. *Assume (2.1) and assume that X is locally contractible, and f satisfies $\mathcal{D}(X, K_c)$ for every $c \in \mathbb{R}$. If f is bounded from below, then f has at least $f\text{-cat}_X(X)$ critical points.*

Combining the relative f-category and the notion of linking, the following result was obtained.

Theorem 2.9. *Assume (2.1), X is locally contractible, and assume that (B, A) links (Q, P) where B and Q are closed. If*

$$\sup f(A) \leq a \leq \inf f(Q) \leq \sup f(B) = b \leq \inf f(P),$$

$a, b \in \mathbb{R}$, $\operatorname{dist}(A, Q) > 0$ if $\sup f(A) = \inf f(Q)$, $\operatorname{dist}(B, P) > 0$ if $\sup f(B) = \inf f(P)$, and f satisfies $\mathcal{D}(X, K_c)$ for every $c \in [a, b]$, then

$$\operatorname{card}(K_{[a,b]}) \geq f\text{-}\operatorname{cat}_{X,f^a}(B).$$

2.3. Limit relative f-category

In order to establish the existence of critical points of indefinite functional a notion of limit relative f-category was introduced by the authors.

In this section, X is a metric space and $\{X_k\}$ is a sequence of closed subspaces of X such that

$$X = \overline{\bigcup_{k\in\mathbb{N}} X_k} \quad \text{and} \quad X_1 \subset X_2 \subset \cdots.$$

For $B \subset X$, we denote $B_k = B \cap X_k$.

As before $f : X \to \mathbb{R}$ is continuous and $K \subset X$ satisfies (2.1). For $k \in \mathbb{N}$, we denote $f_k = f\,|_{X_k}$.

Definition 2.10. Let $B \subset X$ and Y closed in X. The *limit f-category of B relative to Y in X* is defined by

$$f\text{-}\operatorname{cat}^{\infty}_{X,Y}(B) = \sup_{\varepsilon>0} \limsup_{k\to\infty} n^{f_k}_{\varepsilon}(B_k, X_k, Y_k),$$

where $n^{f_k}_{\varepsilon}(B_k, X_k, Y_k)$ is defined in Definition 2.3.

The limit relative f-category satisfies most of the properties of the limit relative category introduced by Fournier, Lupo, Ramos and Willem [5], and

$$f\text{-}\operatorname{cat}^{\infty}_{X,Y}(B) \geq \operatorname{cat}^{\infty}_{X,Y}(B).$$

To relate this notion with critical points of f, a contractibility condition is imposed on the space, and a deformation property is imposed on f.

Definition 2.11. We say that the metric space X is $*$-*locally contractible* if for every $x \in X$ and every $\mathcal{U}$ neighborhood of x, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x such that there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, $\mathcal{V}_k$ is contractible in $\mathcal{U}_k$.

Definition 2.12. Let $c \in \mathbb{R}$. We say that f satisfies $\mathcal{D}^{\star}(X, K_c)$ if for every $\hat{\varepsilon} > 0$ and every neighborhood $\mathcal{U}$ of K_c, there exist $\varepsilon \in\ ]0, \hat{\varepsilon}[$ and $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there exists a continuous deformation $\eta_k : X_k \times [0, 1] \to X_k$ such that

(a) $\eta_k(x, 0) = x$ for every $(x, t) \in X_k \times \{0\} \cup f_k^{c-\hat{\varepsilon}} \times [0, 1]$;
(b) $f(\eta_k(x, t)) \leq f(x)$ for every $(x, t) \in X_k \times [0, 1]$;
(c) $\eta_k(f_k^{c+\varepsilon} \setminus \mathcal{U}, 1) \subset f_k^{c-\varepsilon}$.

Theorem 2.13. *Assume (2.1), X is $*$-locally contractible. Let $A \subset B \subset X$, $P \subset Q \subset X$ be such that B and Q are closed, and (B_k, A_k) links (Q_k, P_k) in X_k for k sufficiently large. If*

$$\sup f(A) \leq a < \inf f(Q) \leq \sup f(B) = b < \inf f(P),$$

and f satisfies $\mathcal{D}^{*}(X, K_c)$ for every $c \in [a, b]$, then

$$\operatorname{card}(K_{[a,b]}) \geq f\text{-}\operatorname{cat}^{\infty}_{X,f^a}(B) \geq 1.$$

2.3.1. Particular case

Here is a particular case in which a lower estimate of the limit relative f-category is given.

Let $E = E^1 \oplus E^2$ be a Banach space, and M a compact differentiable manifold. For $i = 1, 2$, assume that there exist $E^i_1 \subset E^i_2 \subset \cdots$, a sequence of closed subspaces of E^i such that

$$E^i = \overline{\bigcup_{k\in\mathbb{N}} E^i_k},$$

with $0 < \dim(E^1_k) < \infty$. Denote

$$X = E \times M \quad \text{and} \quad X_k = E_k \times M, \quad \text{where } E_k = E^1_k \oplus E^2_k.$$

For $r_i > 0$, set

$$B^i = \{x \in E^i : \|x\| \le r_i\},$$
$$S^i = \{x \in E^i : \|x\| = r_i\}.$$

Theorem 2.14. *Let E, M, and X be as above. Let $f : X \to \mathbb{R}$ be continuous and K satisfying (2.1). Assume that there exist $r_1, r_2 > 0$ and $a, b \in \mathbb{R}$ such that*

$$\sup f(S^1 \times M) \le a < \inf f(B^2 \times M) \le \sup f(B^1 \times M) \le b < \inf f(S^2 \times M).$$

If f satisfies $\mathcal{D}^*(X, K_c)$ for every $c \in [a, b]$, then

$$\mathrm{card}(K_{[a,b]}) \ge f\text{-}\mathrm{cat}^{\infty}_{X,f^a}(B_1 \times M) \ge \limsup_{k\to\infty} \mathrm{cuplength}(B^1_k \times M, S^1_k \times M) + 1 \ge \mathrm{cuplength}(M) + 1.$$

The reader is referred to [20] for the definition of relative cuplength.

Now, we recall a deformation lemma which insures that the deformation properties $\mathcal{D}^*(X, K_c)$ is satisfied.

Definition 2.15. Let X be as above, $c \in \mathbb{R}$, and $f : X \to \mathbb{R}$ of class C^1. We say that f satisfies the *Palais–Smale-star condition at level c*, denoted $(PS)^*_c$, if every sequence (x_{n_j}) such that $n_j \to \infty$, $x_{n_j} \in X_{n_j}$, $f(x_{n_j}) \to c$ and $f'_{n_j}(x_{n_j}) \to 0$, has a convergent subsequence.

Lemma 2.16. *Let X be as above, and $c \in \mathbb{R}$. Assume that $f : X \to \mathbb{R}$ is C^1 and satisfies $(PS)^*_c$. If we consider K as the set of critical points of f, then condition (2.1) is verified and f satisfies $\mathcal{D}^*(X, K_c)$.*

The reader is referred to [5] for a proof of this result.

3. Main theorem

Let us recall that we want to establish the existence of 2π-periodic solutions of the following problem

$$J\dot{u}(t) + A(t)u(t) + \nabla H(t, u(t)) = 0, \tag{1.1}$$

where $u(t) = (u_p(t), u_q(t)) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard symplectic matrix, $A(t)$ is an $2N \times 2N$ matrix of the form

$$A(t) = \begin{pmatrix} B(t) & 0 \\ 0 & 0 \end{pmatrix}$$

with $B(t)$ a symmetric $N \times N$ matrix, and

$$\nabla H(t, p, q) = (\nabla_p H(t, p, q), \nabla_q H(t, p, q)),$$

for $(p, q) \in \mathbb{R}^N \times \mathbb{R}^N$.

Since we look for 2π-periodic solutions of (1.1), it is equivalent to look for solutions defined on the one-dimensional unit sphere S. So, in the following, we identify $[0, 2\pi]$ to S. Every function $u \in L^2(S, \mathbb{R}^{2N})$ can be written by a Fourier series

$$\sum_{k\in\mathbb{Z}} \hat{u}(k)\mathrm{e}^{\mathrm{i}kt}, \quad \text{where } \hat{u}(k) = \frac{1}{2\pi}\int_S u(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t.$$

Define the fractionary Sobolev space

$$H^{1/2}(S, \mathbb{R}^{2N}) = \left\{u \in L^2(S, \mathbb{R}^{2N}) : \sum_{k\in\mathbb{Z}} (1 + k^2)^{1/2}|\hat{u}(k)|^2 < \infty\right\}.$$

This space with the inner product

$$\langle u, v\rangle_{1/2} = \sum_{k\in\mathbb{Z}} (1 + k^2)^{1/2}\hat{u}(k)\overline{\hat{v}(k)}$$

is a Hilbert space. Let us denote the norm induced by this inner product by

$$\|u\|_{1/2} = \left(\sum_{k\in\mathbb{Z}} (1 + k^2)^{1/2}|\hat{u}(k)|^2\right)^{1/2}.$$

By the usual Rellich Theorem, this fractionary Sobolev space, is compactly embedded in $L^{\alpha}(S, \mathbb{R}^N)$.

Theorem 3.1. *For all $\alpha \in [1, \infty)$, $H^{1/2}(S, \mathbb{R}^{2N})$ is compactly embedded in $L^\alpha(S, \mathbb{R}^{2N})$. In particular, there exists a constant $\gamma_\alpha > 0$ such that*

$$\|u\|_{L^\alpha} \le \gamma_\alpha \|u\|_{1/2} \quad \forall u \in H^{1/2}(S, \mathbb{R}^{2N}).$$

We say that $u \in H^{1/2}(S, \mathbb{R}^{2N})$ is a *weak solution* of the problem (1.1) if

$$\int_S J\dot{u} \cdot v + Au \cdot v + \nabla H(t, u) \cdot v \, \mathrm{d}t = 0 \quad \forall v \in H^{1/2}(S, \mathbb{R}^{2N}). \tag{3.1}$$

We will assume the following conditions.

(H1) $B(t)$ is an invertible self-adjoint $N \times N$ matrix, and $t \mapsto B(t)$ is continuous and 2π-periodic;
(H2) $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is continuous, continuously differentiable with respect to $z = (p, q) \in \mathbb{R}^{2N}$, and 2π-periodic in t and q;
(H3) there are constants $a_1, a_2 > 0$ and $r > 1$ such that

$$\|\nabla H(t, p, q)\|^r \le a_1 + a_2 p \cdot \nabla_p H(t, p, q);$$

(H4) there are constants $R > 0$ and $\mu > 2$ such that for all (p, q) with $\|p\| \ge R$,

$$0 < \mu H(t, p, q) \le p \cdot \nabla_p H(t, p, q);$$

(H5) $\nabla_q H(t, p, q)$ is bounded;
(H6) $H(t, z + (0, q)) - H(t, 0, q)$ is $o(\|z\|^2)$ uniformly in t and in q as $z \to 0$.

Let us state our main theorem.

Theorem 3.2. *Suppose the conditions* (H1)–(H6) *are satisfied. Then* (1.1) *has at least one weak solution $u \in H^{1/2}(S, \mathbb{R}^{2N})$.*
Moreover, there exists $\beta_0 > 0$ such that if

$$\sup_{q \in \mathbb{R}^N} \left\{ \int_S H(t, 0, q) \, \mathrm{d}t \right\} - \inf_{q \in \mathbb{R}^N} \left\{ \int_S H(t, 0, q) \, \mathrm{d}t \right\} < \beta_0, \tag{3.2}$$

then (1.1) *has at least $N + 1$ distinct orbits of weak solutions.*

Remark 3.3. Observe that (H2) implies that

$$\sup_{q \in \mathbb{R}^N} \left\{ \int_S H(t, 0, q) \, \mathrm{d}t \right\} = \max_{q \in [0, 2\pi]^N} \left\{ \int_S H(t, 0, q) \, \mathrm{d}t \right\},$$
$$\inf_{q \in \mathbb{R}^N} \left\{ \int_S H(t, 0, q) \, \mathrm{d}t \right\} = \min_{q \in [0, 2\pi]^N} \left\{ \int_S H(t, 0, q) \, \mathrm{d}t \right\}.$$

Notice also that if

$$\max\{H(C)\} - \min\{H(C)\} < \frac{\beta_0}{2\pi}, \quad \text{for } C = S \times \{0\} \times [0, 2\pi]^N,$$

then (3.2) is satisfied.

4. Technical results

In order to prove our main Theorem 3.2, we need to establish some technical results concerning the properties of the function H. In the following, the necessary constants in the proofs will be noted $c_1, c_2, \ldots$. The numeration will restart at c_1 at the beginning of every proof.

Arguing in a classical manner, it is easy to prove the following lemma.

Lemma 4.1. *Assume* (H2) *and* (H3). *Then there are constants a_3, a_4 such that*

$$\|\nabla H(t, p, q)\| \le a_3 + a_4 \|p\|^s$$

for all $(t, p, q) \in \mathbb{R}^{2N+1}$ where $s = 1/(r - 1)$.

We obtain more precision on s if in addition (H4) is satisfied.

Lemma 4.2. *If H satisfies* (H2)–(H4), *then $s = 1/(r - 1) > 1$.*

Proof. From assumption (H4), we obtain the existence of positive constants c_1, c_2 such that

$$H(t, p, 0) \geq c_1 \|p\|^{\mu} - c_2 \quad \forall (t, p) \in \mathbb{R}^{N+1}.$$

This inequality combined with the fact that H is periodic in t gives the existence of a constant c_3 such that

$$H(t, p, 0) - H(t, 0, 0) \geq c_1 \|p\|^{\mu} - c_3 \quad \forall (t, p) \in \mathbb{R}^{N+1}. \tag{4.1}$$

On the other hand, Lemma 4.1 implies that for all $(t, p) \in \mathbb{R}^{N+1}$,

$$\begin{aligned} H(t, p, 0) - H(t, 0, 0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha} H(t, \alpha p, 0) \mathrm{d}\alpha \\ &= \int_0^1 p \cdot \nabla_p H(t, \alpha p, 0) \mathrm{d}\alpha \\ &\leq a_3 \|p\| + a_4 \|p\|^{s+1}. \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$c_1 \|p\|^{\mu} - c_3 \leq a_3 \|p\| + a_4 \|p\|^{s+1}.$$

Since this inequality holds for all p and $c_1 > 0$, we deduce that $s + 1 \geq \mu > 2$. □

Lemma 4.3. *Suppose H satisfies* (H2) *and* (H4). *Then there exists a constant $k > 0$ such that*

$$\int_S H(t, u(t)) \,\mathrm{d}t \leq k + \frac{1}{\mu} \int_S u_p(t) \cdot \nabla_p H(t, u(t)) \,\mathrm{d}t$$

for all $u = (u_p, u_q) \in H^{1/2}(S, \mathbb{R}^{2N})$.

Proof. The periodicity in t and q implies that there exist c_1 and c_2 such that

$$\left|p \cdot \nabla_p H(t, p, q)\right| \leq \frac{c_1}{2\pi} \quad \text{and} \quad |H(t, p, q)| \leq \frac{c_2}{2\pi} \quad \forall (t, p, q) \in \mathbb{R} \times [-R, R] \times \mathbb{R}^N,$$

where R is the constant given in (H4).

Let $u = (u_p, u_q) \in H^{1/2}(S, \mathbb{R}^{2N})$. Set $\mathcal{A} = \left\{t \in S : \|u_p(t)\| \leq R\right\}$. The previous inequalities imply that

$$\left| \int_{\mathcal{A}} u_p(t) \cdot \nabla_p H(t, u(t)) \,\mathrm{d}t \right| \leq c_1, \quad \text{and} \quad \left| \int_{\mathcal{A}} H(t, u(t)) \,\mathrm{d}t \right| \leq c_2.$$

By (H4),

$$\int_{\mathcal{A}^c} u_p(t) \cdot \nabla_p H(t, u(t)) \,\mathrm{d}t \geq \int_{\mathcal{A}^c} \mu H(t, u(t)) \,\mathrm{d}t.$$

Therefore, we obtain for every $u \in H^{1/2}(S, \mathbb{R}^{2N})$,

$$\begin{aligned} \int_S H(t, u(t)) \,\mathrm{d}t &\leq c_2 + \frac{1}{\mu} \int_{\mathcal{A}^c} u_p(t) \cdot \nabla_p H(t, u(t)) \,\mathrm{d}t \\ &= c_2 + \frac{1}{\mu} \int_S u_p(t) \cdot \nabla_p H(t, u(t)) \,\mathrm{d}t - \frac{1}{\mu} \int_{\mathcal{A}} u_p(t) \cdot \nabla_p H(t, u(t)) \,\mathrm{d}t \\ &\leq k + \frac{1}{\mu} \int_S u_p(t) \cdot \nabla_p H(t, u(t)) \,\mathrm{d}t, \end{aligned}$$

where $k = c_2 - c_1/\mu$. □

Lemma 4.4. *Suppose* (H2), (H3) *and* (H6) *are satisfied. Then for all $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that*

$$|H(t, z + (0, q)) - H(t, 0, q)| \leq \varepsilon \|z\|^2 + c_\varepsilon \|z\|^{s+1}$$

for all $t \in \mathbb{R}$, $z \in \mathbb{R}^{2N}$ and $q \in \mathbb{R}^N$.

Proof. Let $\varepsilon > 0$. By (H6), there is a constant R_ε such that

$$|H(t, z + (0, q)) - H(t, 0, q)| < \varepsilon \|z\|^2, \quad \forall \|z\| < R_\varepsilon. \tag{4.3}$$

Lemma 4.1 implies that

$$\begin{aligned}|H(t,z+(0,q))-H(t,0,q)| &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha}H(t,\alpha z+(0,q))\mathrm{d}\alpha \\ &= \left|\int_0^1 z\cdot\nabla(H(t,\alpha z+(0,q)))\mathrm{d}\alpha\right| \\ &\le \int_0^1 a_3\|z\| + a_4|\alpha|^s\|z\|^{s+1}\mathrm{d}\alpha \\ &= a_3\|z\| + \frac{a_4}{s+1}\|z\|^{s+1}.\end{aligned}$$

Therefore, there exists $c_\varepsilon > 0$ such that

$$|H(t,z+(0,q))-H(t,0,q)| \le c_\varepsilon\|z\|^{s+1}, \quad \forall\|z\| > R_\varepsilon. \tag{4.4}$$

The conclusion follows from (4.3) and (4.4). □

5. Equivalent problem formulation and $(PS)_c^*$ condition

Consider the bilinear operator $a : H^{1/2}(S,\mathbb{R}^{2N}) \times H^{1/2}(S,\mathbb{R}^{2N}) \to \mathbb{R}$ defined by

$$a(u,v) = \int_S J\dot{u}\cdot v + Au\cdot v\,\mathrm{d}t.$$

Since (H1) is satisfied, we can denote the eigenvalues associated with a by

$$\cdots \le \lambda_{-2} \le \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \le \lambda_2 \le, \cdots,$$

with $\lambda_n \to \infty$ and $\lambda_{-n} \to -\infty$. This permits to obtain a decomposition of the space

$$H^{1/2}(S,\mathbb{R}^{2N}) = X^+ \oplus X^0 \oplus X^-,$$

where X^+ (resp. X^-) is the space generated by eigenvectors associated with positive (resp. negative) eigenvalues, and X^0 is the eigenspace associated with $\lambda_0 = 0$. Observe that $\dim X^+ = \dim X^- = \infty$ and $\dim X^0 < \infty$. More precisely

$$X^0 = \{u^0 \in H^{1/2}(S,\mathbb{R}^{2N}) : u^0(t) \equiv (0,q) \in \{0\}\times\mathbb{R}^N\}.$$

Let us define the following inner product inducing an equivalent norm on $H^{1/2}(S,\mathbb{R}^{2N})$ by:

$$\langle\langle u,v\rangle\rangle_{1/2} = a(u^+,v^+) - a(u^-,v^-) + \langle u^0,v^0\rangle_{1/2}, \tag{5.1}$$

where $u = u^+ + u^0 + u^- \in X^+ \oplus X^0 \oplus X^-$. Observe that with this new scalar product the decomposition $X^+ \oplus X^0 \oplus X^-$ is orthogonal, and with this new norm (still denoted $\|\cdot\|_{1/2}$), we have

$$a(u,u) = \int_S (J\dot{u}+Au)\cdot u\,\mathrm{d}t = \|u^+\|_{1/2}^2 - \|u^-\|_{1/2}^2,$$

see [6] for more details.

Let us introduce the equivalence relation on X^0 defined by

$$x^0 \sim \hat{x}^0 \Leftrightarrow x^0 - \hat{x}^0 = (0,q) \in \{0\}\times 2\pi\mathbb{Z}^N.$$

Denote $[x^0]$ the equivalence class of x^0, and observe that

$$V = \{[x^0] : x^0 \in X^0\} \approx \mathbb{R}^N/2\pi\mathbb{Z}^N \approx \mathbb{T}^N,$$

where $\mathbb{T}^N$ denotes the N dimensional torus.

Consider the functional $\phi : H^{1/2}(S,\mathbb{R}^{2N}) \to \mathbb{R}$ defined by

$$\phi(u) = \frac{1}{2}\int_S J\dot{u}\cdot u + Au\cdot u\,\mathrm{d}t + \int_S H(t,u)\,\mathrm{d}t. \tag{5.2}$$

One can show that ϕ is C^1 and

$$\langle\phi'(u),v\rangle = \int_S J\dot{u}\cdot v + Au\cdot v + \nabla H(t,u)\cdot v\,\mathrm{d}t \tag{5.3}$$

for all $u, v \in H^{1/2}(S,\mathbb{R}^{2N})$. So weak solutions of (1.1) (see (3.1)) are critical points of the functional ϕ. By the periodicity of A and H assumed in (H1) and (H2) respectively, we can easily show that for all $u \in H^{1/2}(S,\mathbb{R}^{2N})$,

$$\phi(u) = \phi(u+x^0) \quad\text{and}\quad \phi'(u) = \phi'(u+x^0) \quad \forall x^0 \text{ such that } [x^0] = [0]. \tag{5.4}$$

Therefore, to each $\bar{u}$ weak solution of (1.1) corresponds in fact an orbit of weak solutions $\{\bar{u}+x^0 : [x^0]=0\}$. Hence, the functional ϕ does not satisfy the Palais–Smale-star condition. To avoid this problem, we introduce another functional.

Let $X = X^+ \oplus X^-$. Define $f : X \times V \to \mathbb{R}$ by

$$f(x,[x^0]) = \phi(x+x^0).$$

It follows from (5.4) that f is well defined. Moreover, f is C^1, and critical points of f are in correspondence with critical orbits of ϕ, (see [21,22] for more details).

For $n \in \mathbb{N}$, choose $e_{\pm n}$ an eigenvector corresponding to $\lambda_{\pm n}$ and such that $\{\dots, e_{-1}, e_1, e_2, \dots\}$ is an orthonormal set. Denote

$$\begin{aligned}
X_n^+ &= \operatorname{span}\{e_1,\dots,e_n\},\\
X_n^- &= \operatorname{span}\{e_{-1},\dots,e_{-n}\},\\
X_n &= X_n^+ \oplus X_n^-,\\
f_n &= f\,|_{X_n\times V},\\
\phi_n &= \phi\,|_{X_n\oplus X^0}.
\end{aligned}$$

Theorem 5.1. *Assume* (H1)–(H6), *then f satisfies $(PS)_c^*$ for all $c \in \mathbb{R}$.*

Proof. Consider $\{(x_n, v_n)\} \in X_n \times V$ a sequence such that

$$f(x_n,v_n) \to c \quad \text{and} \quad f_n'(x_n,v_n) \to 0. \tag{5.5}$$

Choose $x_n^0 \in \{0\}\times[0,2\pi]^N \subset X^0$ such that $v_n = [x_n^0]$, and denote $u_n = x_n + x_n^0 \in H^{1/2}(S,\mathbb{R}^{2N})$. To show that $\{(x_n,v_n)\}$ has a convergent subsequence, it is equivalent to show that $\{u_n\}$ has a convergent subsequence.

The sequence $\{x_n^0\}$ is bounded in the finite-dimensional space X^0. So, it has a subsequence still denoted $\{x_n^0\}$ converging to $x^0 \in X^0$.

Since $f(x_n,v_n) = \phi(u_n)$ and $f_n'(x_n,v_n) = \phi_n'(u_n)$, it follows from (5.2), (5.3) and (5.5) that for n large enough,

$$\begin{aligned}
1-2c+\|u_n\|_{1/2} &\ge \langle \phi_n'(u_n),u_n\rangle - 2\phi(u_n)\\
&= \int_S \nabla H(t,u_n)\cdot u_n - 2H(t,u_n)\,\mathrm{d}t.
\end{aligned} \tag{5.6}$$

By (H5) and Theorem 3.1, there exists a constant c_1 such that

$$\left|\int_S u_{n,q}\cdot\nabla_q H(t,u_n)\,\mathrm{d}t\right| \le c_1\|u_{n,q}\|_{1/2}. \tag{5.7}$$

On the other hand, (H3) and Lemma 4.3 imply that

$$\begin{aligned}
\int_S u_{n,p}\cdot\nabla_p H(t,u_n) - 2H(t,u_n)\,\mathrm{d}t &\ge \left(1-\frac{2}{\mu}\right)\int_S u_{n,p}\cdot\nabla_p H(t,u_n)\,\mathrm{d}t - 2k\\
&\ge c_2\|\nabla H(t,u_n)\|_{L^r}^r - c_3.
\end{aligned} \tag{5.8}$$

Combining (5.6)–(5.8) gives

$$1-2c+\|u_n\|_{1/2} \ge c_2\|\nabla H(t,u_n)\|_{L^r}^r - c_3 - c_1\|u_{n,q}\|_{1/2}.$$

Thus,

$$\|\nabla H(t,u_n)\|_{L^r} \le c_4(1+\|u_n\|_{1/2}^{1/r}). \tag{5.9}$$

On the other hand, it follows from (5.5), (5.9) and Theorem 3.1 that for n large enough,

$$\begin{aligned}
a(x_n^+,x_n^+) &= a(u_n,x_n^+)\\
&= \langle \phi_n'(u_n),x_n^+\rangle - \int_S x_n^+\cdot\nabla H(t,u_n)\,\mathrm{d}t\\
&\le \|x_n^+\|_{1/2} + \|x_n^+\|_{L^{r'}}\|\nabla H(t,u_n)\|_{L^r}\\
&\le \|x_n^+\|_{1/2} + c_5\|x_n^+\|_{1/2}(1+\|u_n\|_{1/2}^{1/r})\\
&\le c_6 + c_7\|u_n\|_{1/2}^{1+1/r},
\end{aligned} \tag{5.10}$$

where r' is the conjugate exponent of r. Similarly, we obtain

$$-a(x_n^-, x_n^-) \le c_6 + c_7 \|u_n\|_{1/2}^{1+1/r}. \tag{5.11}$$

We deduce from (5.1), (5.10), (5.11) and the fact that the sequence $\{x_n^0\}$ is bounded, that

$$\|u_n\|_{1/2}^2 \le c_8 + 2c_7 \|u_n\|_{1/2}^{1+1/r}.$$

Hence, since $r > 1$, the sequence $\{u_n\}$ is bounded in the Hilbert space $H^{1/2}(S, \mathbb{R}^{2N})$. Thus, $\{u_n\}$ has a subsequence, still noted $\{u_n\}$, weakly converging in $H^{1/2}(S, \mathbb{R}^{2N})$ and converging strongly in $L^2(S, \mathbb{R}^{2N})$ and in $L^{2s}(S, \mathbb{R}^{2N})$ by Theorem 3.1. Let us denote the limit $u = x^+ + x^- + x^0$. Therefore, we deduce using in addition (5.1) and (5.5), that

$$\begin{aligned} \pm\|x_n^\pm - x^\pm\|_{1/2}^2 &= a(x_n^\pm - x^\pm, x_n^\pm - x^\pm) \\ &= \langle \phi'(u_n), x_n^\pm - x^\pm \rangle - \langle \phi'(u), x_n^\pm - x^\pm \rangle - \int_S \Big(\nabla H(t, u_n) - \nabla H(t, u) \Big) \cdot \Big(x_n^\pm - x^\pm \Big)\, \mathrm{d}t \\ &\to 0. \end{aligned}$$

Thus $u_n \to u$ strongly in $H^{1/2}(S, \mathbb{R}^{2N})$ and hence $\{(x_n, v_n)\}$ converges in $X \times V$. □

6. Proof of the main theorem

In order to prove Theorem 3.2, we will first have to verify that f satisfies some appropriate inequalities on linking pairs. Let r^+ and r^- be positive constants to be fixed later. Denote

$$\begin{aligned} B^+ &= \{x \in X^+ : \|x\|_{1/2} \le r^+\}, \\ S^+ &= \{x \in X^+ : \|x\|_{1/2} = r^+\}, \\ B^- &= \{x \in X^- : \|x\|_{1/2} \le r^-\}, \\ S^- &= \{x \in X^- : \|x\|_{1/2} = r^-\}. \end{aligned}$$

Theorem 6.1. *Assume* (H1)–(H3) *and* (H6) *are satisfied. There exist $v_0 \in V$ and two positive constants r^+ and r^- such that*

$$\sup f(S^- \times V) < \inf f(B^+ \times \{v_0\}) \le \sup f(B^- \times V) < \inf f(S^+ \times \{v_0\}). \tag{6.1}$$

Moreover, there exists $\beta_0 > 0$ such that if

$$\max\{f(0, V)\} - \min\{f(0, V)\} < \beta_0, \tag{6.2}$$

then

$$\sup f(S^- \times V) < \inf f(B^+ \times V) \le \sup f(B^- \times V) < \inf f(S^+ \times V). \tag{6.3}$$

Proof. Let $(x^-, v) \in X^- \times V$, and choose $x^0 \in X^0$ such that $v = [x^0]$. By Theorem 3.1 and Lemma 4.4,

$$\begin{aligned} f(x^-, v) &= \frac{1}{2} a(x^-, x^-) + \int_S H(t, x^- + x^0)\, \mathrm{d}t \\ &= \frac{-1}{2}\|x^-\|_{1/2}^2 + \int_S H(t, x^- + x^0) - H(t, x^0)\, \mathrm{d}t + \int_S H(t, x^0)\, \mathrm{d}t \\ &\le \frac{-1}{2}\|x^-\|_{1/2}^2 + \int_S \varepsilon \|x^-\|^2 + c_\varepsilon \|x^-\|^{s+1}\, \mathrm{d}t + f(0, v) \\ &\le \left(\frac{-1}{2} + \varepsilon\gamma_2 \right) \|x^-\|_{1/2}^2 + c_\varepsilon \gamma_{s+1} \|x^-\|_{1/2}^{s+1} + f(0, v). \end{aligned} \tag{6.4}$$

Similarly, we show that for all $(x^+, v) \in X^+ \times V$,

$$f(x^+, v) \ge \left(\frac{1}{2} - \varepsilon\gamma_2 \right) \|x^+\|_{1/2}^2 - c_\varepsilon \gamma_{s+1} \|x^+\|_{1/2}^{s+1} + f(0, v). \tag{6.5}$$

Since $s > 1$, (6.4) and (6.5) imply that for $\varepsilon > 0$ small enough, there exist positive constants r^+, r^-, k^+ and k^- such that

$$\begin{aligned} & f(x^-, v) \le -k^- + f(0, v) \quad \forall (x^-, v) \in S^- \times V, \\ & f(x^-, v) \le f(0, v) \quad \forall (x^-, v) \in B^- \times V, \\ & f(x^+, v) \ge k^+ + f(0, v) \quad \forall (x^+, v) \in S^+ \times V, \\ & f(x^+, v) \ge f(0, v) \quad \forall (x^+, v) \in B^+ \times V. \end{aligned} \tag{6.6}$$

Set $v_0 \in V$ such that

$$f(0, v_0) = \max_{v\in V} f(0, v).$$

This equality combined with (6.6) imply that

$$f(x^+, v_0) \geq f(0, v_0) > f(0, v_0) - k^- \geq f(x^-, v) \quad \forall x^+ \in B^+, \forall x^- \in S^-, \forall v \in V,$$
$$f(x^+, v_0) \geq k^+ + f(0, v_0) > f(0, v_0) \geq f(x^-, v) \quad \forall x^+ \in S^+, \forall x^- \in B^-, \forall v \in V.$$

Thus

$$\sup f(S^- \times V) < \inf f(B^+ \times \{v_0\}) \leq \sup f(B^- \times V) < \inf f(S^+ \times \{v_0\}).$$

Now, set $\beta_0 = \min\{k^-, k^+\}$. If (6.2) is satisfied and from (6.6), we have

$$\begin{aligned} f(x^+, v_1) &\geq \inf f(0, V) > \sup f(0, V) - \beta_0 \\ &\geq k^- + f(x^-, v_2) - \beta_0 \\ &\geq f(x^-, v_2) \quad \forall x^+ \in B^+, \forall x^- \in S^-, \forall v_1, v_2 \in V, \end{aligned}$$

and

$$\begin{aligned} f(x^-, v_1) &\leq \sup f(0, V) < \inf f(0, V) + \beta_0 \\ &\leq f(x^+, v_2) - k^+ + \beta_0 \\ &\leq f(x^+, v_2) \quad \forall x^- \in B^-, \forall x^+ \in S^+, \forall v_1, v_2 \in V. \end{aligned}$$

Thus,

$$\sup f(S^- \times V) < \inf f(B^+ \times V) \leq \sup f(B^- \times V) < \inf f(S^+ \times V). \quad \square$$

We are now ready to prove the main theorem.

Proof of Theorem 3.2. It follows from Lemma 2.16 and Theorem 5.1 that f satisfies $\mathcal{D}^\star(X, K_c)$ for all $c \in \mathbb{R}$.

Observe that for all $n \in \mathbb{N}$,

$$\left((B^- \cap X_n) \times V, (S^- \cap X_n) \times V\right) \quad \text{links} \quad \left((B^+ \cap X_n) \times \{v_0\}, (S^+ \cap X_n) \times \{v_0\}\right),$$

where v_0 is given in Theorem 6.1. Indeed, let $\eta = (\eta_{X_n}, \eta_v) : X_n \times V \times [0, 1] \to X_n \times V$ be continuous and such that

$$\eta(x, v, t) = (x, v) \quad \text{for every } (x, v, t) \in (X_n \times V \times \{0\}) \cup ((S^- \cap X_n) \times V \times [0, 1]).$$

Define $H : (B^- \cap X_n) \times V \times [0, 1] \to X_n^- \times V$ by

$$H(x, v, t) = \begin{cases} \left(P_{X_n^-}(\eta_{X_n}(x, v, 2t - 1)), \eta_V(x, v, 2t - 1)\right), & \text{if } t \in \left]\frac{1}{2}, 1\right], \\ \left(2tP_{X_n^-}(x) + (1 - 2t)x, v\right), & \text{if } t \in \left[0, \frac{1}{2}\right], \end{cases}$$

where $P_{X_n^-}$ is the projection on X_n^-. By topological degree theory (see [23] or [24]), there exists a continuum $\mathcal{C} \subset (B^- \cap X_n) \times V \times [0, 1]$ intersecting $(B^- \cap X_n) \times V \times \{t\}$ for every $t \in [0, 1]$ such that $H(x, v, t) = (0, v_0)$ for every $(x, v, t) \in \mathcal{C}$. Therefore, $\{\eta_{X_n}(x, v, 2t - 1) : (x, v, t) \in \mathcal{C}, t \in [\frac{1}{2}, 1]\}$ is a connected subset of X_n^+ such that $\{\eta_{X_n}(x, v, 0) : (x, v, \frac{1}{2}) \in \mathcal{C}\} = \{0\}$. So, if $\eta((B^- \cap X_n) \times V, 1) \cap (B_n^+ \times \{v_0\}) = \emptyset$, there exists $t \in \,]\frac{1}{2}, 1[$ and $(x, v, t) \in \mathcal{C}$ such that $\eta(x, v, 2t - 1) \in (S^+ \cap X_n) \times \{v_0\}$.

This linking pairs combined with Theorem 6.1 permit to apply Theorem 2.13 to deduce that f has a critical point and hence (1.1) has a 2π-periodic solution.

On the other hand, if (3.2) is satisfied with β_0 obtained in Theorem 6.1. Theorem 2.14 implies that f has at least cuplength$(V) + 1$ critical points. It is well known that cuplength(V) = cuplength$(\mathbb{T}^N) = N$ (see [25, p. 161]). Therefore, (1.1) has at least $N + 1$ distinct orbits of weak solutions. $\square$

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