

SYSTEMS OF SINGULAR POISSON EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. We establish the existence of nontrivial solutions to systems of singular Poisson equations in unbounded domains, under some invariance conditions and singular subcritical growth. The proofs rely on a concentration-compactness argument and on a generalized linking theorem due to Krysewski and Szulkin.

1. INTRODUCTION

In this paper, we study the existence of solutions to the following systems of singular elliptic equations on \mathbb{R}^N :

$$\begin{cases} -\operatorname{div}(\nabla u|x|^{-2a}) = H_v(x, u, v), \\ -\operatorname{div}(\nabla v|x|^{-2a}) = H_u(x, u, v); \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(\nabla u|x|^{-2a}) = \beta|x|^{-2(1+a)}v + H_v(x, u, v), \\ -\operatorname{div}(\nabla v|x|^{-2a}) = \eta|x|^{-2(1+a)}u + H_u(x, u, v); \end{cases}$$

where $N \geq 3$, $a \in [0, (N-2)/2)$, and $\beta, \eta \in [0, S(a, a+1))$, with $S(a, a+1)$ the best constant in the Caffarelli, Kohn, and Nirenberg inequality [2]. This type of problem comes from systems of anisotropic semilinear Schrödinger equations.

The first problem has been studied in the case $a = 0$ and on bounded domains by Felmer and Wang [7], Husolf, Mitidieri and van der Vorst [9],

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and by Husholf and van der Vorst [8]. Recently, we [5] treated these problems (also with $a = 0$) for $H(x, u, v) := (|u|^{2^*} + |v|^{2^*})/2^*$ and in an unbounded domain. We established the existence of a nontrivial solution to the first problem in a domain $\Omega \subset \mathbb{R}^N \setminus \mathbb{Z}^N$ invariant by \mathbb{Z}^N translations; and to the second one in an unbounded cylindrical domain and for $\beta, \eta \in (0, \lambda_1)$ with λ_1 the best constant in the Poincaré inequality.

To our knowledge, in the literature, there are no results establishing the existence of nontrivial solutions to these singular problems ($a \neq 0$) in unbounded domains. The aim of this paper is to establish the first results in this direction.

In this paper, for a fixed $R > 1$, we will assume that H has a singular subcritical growth and satisfies an invariance condition. More precisely, we will assume the following hypotheses:

- (A0) The function $H \in C^1((\mathbb{R}^N \setminus \{0\}) \times \mathbb{R} \times \mathbb{R})$ is such that $H(x, 0, 0) = 0$.
- (A1) There exists $R > 1$ such that for each $m \in \mathbb{Z}$,

$$R^{mN} H(x, u, v) = H(R^{-m}x, R^{m\xi}u, R^{m\xi}v),$$

where $\xi := \frac{(N-2a-2)}{2}$.

- (A2) There exist a constant $k > 0$ and $b, \hat{b} \in [a, a + 1[$ such that

$$\left(|H_u(x, u, v)| + |H_v(x, u, v)| \right) \left(|u| + |v| \right) \leq k \left(|x|^{-bp} |u|^p + |x|^{-\hat{b}\hat{p}} |v|^{\hat{p}} \right),$$

where $p := \frac{2N}{N-2+2(b-a)}$ and $\hat{p} := \frac{2N}{N-2+2(\hat{b}-a)}$.

- (A3) There exist

$$\alpha \in \left[\frac{2N}{N+2}(p-1), 2(p-1) \right), \quad \text{and} \quad \hat{\alpha} \in \left[\frac{2N}{N+2}(\hat{p}-1), 2(\hat{p}-1) \right)$$

such that, for every $u, v \neq 0$,

$$\frac{1}{\alpha} H_u(x, u, v)u + \frac{1}{\hat{\alpha}} H_v(x, u, v)v \geq H(x, u, v) > 0.$$

- (A4) There exists $\sigma > 0$ such that

$$\inf\{H(x, u, v) : (x, u, v) \in S\} = \sigma,$$

where $S := \{(x, u, v) \in B(0, 1) \times \mathbb{R}^2 : |x|^{-d}|u|^\alpha + |x|^{-\hat{d}}|v|^{\hat{\alpha}} = 1\}$ with $d := N - \xi\alpha$ and $\hat{d} := N - \xi\hat{\alpha}$.

We will establish the existence of a nontrivial solution (u, v) such that $u, v = 0$ on $\{x \in \mathbb{R}^N : |x| = R^m, m \in \mathbb{Z}\}$.

Our proofs will rely on variational methods. Indeed, these problems have a variational formulation of the form

$$\varphi(u, v) := A(u, v) - \psi(u, v),$$

with the quadratic part A being a strongly indefinite operator. Therefore, classical min-max results can not be applied. Here our results will follow from a generalized linking theorem due to Krysewski and Szulkin [10] for suitable functionals defined on a Hilbert space $X = Y \oplus Z$ where both subspaces Y and Z could be infinite dimensional.

It is worthwhile to notice that the invariance condition imposed on H will imply that the functional φ will be invariant by dilatations of the form

$$(u_m(x), v_m(x)) := \left(R^{m\xi} u(R^m x), R^{m\xi} v(R^m x) \right).$$

Therefore, the Palais-Smale condition will not be satisfied at any critical levels. A concentration-compactness argument will permit us to overcome this difficulty.

The paper is organized as follows. Firstly, we present some preliminary results, then we study the functional φ . Our main results are established in section 4. Finally, we discuss the existence of radial and nonradial solutions.

2. PRELIMINARIES

Let $N \geq 3$, $a \in [0, (N - 2)/2)$ and $R \geq 1$; we consider the space $D_{a,R}^{1,2}(\mathbb{R}^N)$ (respectively $D_a^{1,2}(\mathbb{R}^N)$) the completion of $D_R(\mathbb{R}^N)$ (respectively $D(\mathbb{R}^N)$) with respect to the inner product:

$$(u, v) := \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v \, dx,$$

where

$$\begin{aligned} D(\mathbb{R}^N) &:= \left\{ u \in C^\infty(\mathbb{R}^N) : u \text{ has compact support} \right\}, \\ D_R(\mathbb{R}^N) &:= \left\{ u \in D(\mathbb{R}^N) : u \text{ has compact support in } \bigcup_{m \in \mathbb{Z}} A_m \right\}, \end{aligned}$$

with

$$A_m := \left\{ x \in \mathbb{R}^N : R^m < |x| < R^{m+1} \right\}, \quad m \in \mathbb{Z}.$$

Let us recall the following result.

Lemma 2.1 (Wang and Willem [13]). *Let $N \geq 3$ and $0 \leq a < (N - 2)/2$. If $u_n \rightharpoonup u$ in $D_a^{1,2}(\mathbb{R}^N)$, then $|x|^{-a} u_n \rightarrow |x|^{-a} u$ in $L_{loc}^2(\mathbb{R}^N)$.*

Let $b \in [a, a + 1]$. We denote

$$L_b^p(\mathbb{R}^N) := \{u : \mathbb{R}^N \rightarrow \mathbb{R} : u \text{ is measurable and } \|u\|_{p,b} < \infty\},$$

where

$$\|u\|_{p,b} := \left(\int_{\mathbb{R}^N} |x|^{-bp} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

It was established by Caffarelli, Kohn and Nirenberg [2] that if

$$p = p(a, b) := \frac{2N}{N - 2 + 2(b - a)}, \tag{2.1}$$

$D_a^{1,2}(\mathbb{R}^N)$ can be continuously imbedded in $L_b^p(\mathbb{R}^N)$. This result is called the Caffarelli-Kohn-Nirenberg inequality (noted CKN inequality) with the best positive constant

$$S(a, b) := \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ \|u\|_{p,b} = 1}} \left| |x|^{-a} \nabla u \right|_2^2 > 0. \tag{2.2}$$

The following result gives sufficient conditions to ensure the convergence to 0 in $L_b^p(\mathbb{R}^N)$ of a bounded sequence of $D_{a,R}^{1,2}(\mathbb{R}^N)$. See Colin [C1] or [C2] for similar results on a cylindrical domain.

Lemma 2.2. *Let $\{u_n\}$ be a bounded sequence in $D_{a,R}^{1,2}(\mathbb{R}^N)$. If $b < a + 1$, and*

$$\sup_{m \in \mathbb{Z}} \int_{A_m} \left(|x|^{-b} |u_n| \right)^p \rightarrow 0, \quad \text{when } n \rightarrow \infty,$$

then $u_n \rightarrow 0$ in $L_b^p(\mathbb{R}^N)$.

Proof. For each $m \in \mathbb{Z}$ and $u \in D_R(\mathbb{R}^N)$, the CKN inequality (2.2) leads to

$$\int_{A_m} \left(|x|^{-b} |u| \right)^p dx \leq S(a, b)^{-1} \left(\int_{A_m} |x|^{-2a} |\nabla u|^2 dx \right)^{p/2}$$

since $u \in H_0^1(A_m)$ for all $m \in \mathbb{Z}$. By a density argument, the previous inequality holds for each $u \in D_{a,R}^{1,2}(\mathbb{R}^N)$. Consequently, for any $\mu \in (0, 1)$, we have

$$\begin{aligned} & \int_{A_m} \left(|x|^{-b} |u| \right)^p dx \\ & \leq S(a, b)^{-\mu} \left(\int_{A_m} |x|^{-2a} |\nabla u|^2 dx \right)^{(p/2)\mu} \left(\int_{A_m} \left(|x|^{-b} |u| \right)^p dx \right)^{1-\mu}. \end{aligned}$$

Since $b \neq a + 1$, we have that $p > 2$, and we can choose $\mu = 2/p$. Therefore,

$$\int_{\mathbb{R}^N} (|x|^{-b}|u|)^p dx \leq S(a, b)^{-2/p} \left(\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx \right) \sup_{m \in \mathbb{Z}} \left(\int_{A_m} (|x|^{-b}|u|)^p dx \right)^{1-(2/p)}.$$

The conclusion follows. □

Now, we recall for the sake of completeness two results which will be used in what follows. The first one is a corollary of a generalized linking theorem due to Krysewski and Szulkin [10] (see also [14]) that will play a crucial role in the proof of our main existence results.

Let $X = Y \oplus Z$ be a Hilbert space with Y a separable subspace of X which could be infinite dimensional and $Z = Y^\perp$. Let $P : X \rightarrow Y, Q : X \rightarrow Z$ be the orthogonal projections. Let $\rho > r > 0$, and $z \in Z$ such that $\|z\| = 1$. Define

$$\begin{aligned} M &:= \{u = y + \lambda z : \|u\| \leq \rho, \lambda \geq 0, y \in Y\}, \\ M_0 &:= \{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}, \\ N &:= \{u \in Z : \|u\| = r\}. \end{aligned}$$

Theorem 2.3 (Krysewski-Szulkin, 1998). *Let $\psi \in C^1(X, \mathbb{R})$ be weakly sequentially lower semicontinuous, bounded below and such that ψ' is weakly sequentially continuous. If*

$$\phi(u) := \frac{\|Qu\|^2}{2} - \frac{\|Pu\|^2}{2} - \psi(u)$$

satisfies

$$m_0 := \inf_N \phi > 0 = \sup_{M_0} \phi, \quad m_1 := \sup_M \phi < \infty, \tag{2.3}$$

then there exists $c \in [m_0, m_1]$ and a sequence $\{u_n\} \subset X$ such that

$$\phi(u_n) \rightarrow c, \quad \phi'(u_n) \rightarrow 0.$$

We refer the reader to [14] for a proof of the following result.

Theorem 2.4 (Principle of symmetric criticality, Palais, 1979). *Let X be a Hilbert space, and G a topological group such that its action on X is isometric. If $\phi \in C^1(X, \mathbb{R})$ is invariant and if u is a critical point for the restriction of ϕ to $\text{Fix } G$, then u is a critical point of ϕ .*

3. STUDY OF THE FUNCTIONAL φ

Let us denote by $X := D_{a,R}^{1,2}(\mathbb{R}^N) \times D_{a,R}^{1,2}(\mathbb{R}^N)$ the Hilbert space endowed with the scalar product:

$$\langle (u, v), (u_1, v_1) \rangle := \int_{\mathbb{R}^N} |x|^{-2a} (\nabla u \cdot \nabla u_1 + \nabla v \cdot \nabla v_1) \, dx.$$

We can write $X = Y \oplus Z$ with $Y := \{(-v, v) \in X\}$ and $Z := \{(u, u) \in X\}$. Let us denote by P (respectively Q) the projection of X onto Y (respectively Z).

We define the functional $\varphi : X \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(u, v) &:= \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v - H(x, u, v) \, dx \\ &= \frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v), \end{aligned}$$

where

$$\psi(u, v) := \int_{\mathbb{R}^N} H(x, u, v) \, dx,$$

and $H \in C^1((\mathbb{R}^N \setminus \{0\}) \times \mathbb{R} \times \mathbb{R})$ satisfies assumptions (A0)–(A4) given in the introduction.

In this section, we study the functional φ . We start with some observations on the map H .

Remark 3.1. Assumptions (A0) and (A2) imply the following growth conditions:

$$\begin{aligned} |H(x, u, v)| &= \left| \int_0^1 \frac{d}{dt} H(x, tu, tv) \, dt \right| \\ &\leq \int_0^1 |H_u(x, tu, tv)u + H_v(x, tu, tv)v| \, dt \\ &\leq k \int_0^1 |x|^{-bp} |u|^p + |x|^{-\hat{b}\hat{p}} |v|^{\hat{p}} \, dt = k(|x|^{-bp} |u|^p + |x|^{-\hat{b}\hat{p}} |v|^{\hat{p}}); \end{aligned} \tag{3.1}$$

$$\max \left\{ |H_u(x, u, v)|, |H_v(x, u, v)| \right\} \leq k \left(|x|^{-bp} |u|^{p-1} + |x|^{-\hat{b}\hat{p}} |v|^{\hat{p}-1} \right). \tag{3.2}$$

As in classical results, Assumption (A4) permits us to deduce that H is bounded below by an appropriate function for x in the closed unit ball.

Lemma 3.2. Assumptions (A0), (A3), and (A4) imply that

$$H(x, u, v) \geq \sigma(|x|^{-d} |u|^\alpha + |x|^{-\hat{d}} |v|^{\hat{\alpha}}) - \sigma \tag{3.3}$$

for all $|x| \leq 1$.

Proof. For every $(x, u, v) \in S$, we define $h : [1, \infty[\rightarrow \mathbb{R}$ by

$$h(t) := H(x, t^{1/\alpha}u, t^{1/\hat{\alpha}}v).$$

In integrating $h'(s)/h(s)$ from 1 to t , we deduce by (A3) that $h(t) \geq h(1)t$; i.e.,

$$H(x, t^{1/\alpha}u, t^{1/\hat{\alpha}}v) \geq tH(x, u, v) \geq \sigma t. \tag{3.4}$$

Now, consider $(x, u, v) \in B(0, 1) \times \mathbb{R}^2$ such that

$$t := |x|^{-d}|u|^\alpha + |x|^{-\hat{d}}|v|^{\hat{\alpha}} \geq 1.$$

So, $(x, u_1, v_1) \in S$ for $u_1 := t^{-1/\alpha}u$, and $v_1 := t^{-1/\hat{\alpha}}v$. From (3.4), we deduce that

$$H(x, u, v) = H(x, t^{1/\alpha}u_1, t^{1/\hat{\alpha}}v_1) \geq \sigma t = \sigma \left(|x|^{-d}|u|^\alpha + |x|^{-\hat{d}}|v|^{\hat{\alpha}} \right).$$

Finally, on the set $\{(x, u, v) \in B(0, 1) \times \mathbb{R}^2 : |x|^{-d}|u|^\alpha + |x|^{-\hat{d}}|v|^{\hat{\alpha}} \leq 1\}$,

$$H(x, u, v) \geq 0 \geq \sigma \left(|x|^{-d}|u|^\alpha + |x|^{-\hat{d}}|v|^{\hat{\alpha}} \right) - \sigma,$$

the desired conclusion is obtained. □

Remark 3.3. Assumption (A4) is satisfied if we assume (A0), (A1), (A3), and

$$\liminf_{m \rightarrow \infty} c_m > 0,$$

where, for every $m \in \mathbb{N} \cup \{0\}$,

$$c_m := \inf\{R^m H(x, R^{-Nm/\alpha}u, R^{-Nm/\hat{\alpha}}v) : (x, u, v) \in S_0\},$$

with $S_0 := \{(x, u, v) \in S : x \in (1/R, 1]\}$, and S is defined in (A4).

Indeed, for $(x, u, v) \in S$,

$$(x_0, u_0, v_0) := (R^m x, R^{md/\alpha}u, R^{m\hat{d}/\hat{\alpha}}v) \in S_0.$$

Hence,

$$\begin{aligned} H(x, u, v) &= H(R^{-m}x_0, R^{-md/\alpha}u_0, R^{-m\hat{d}/\hat{\alpha}}v_0) \\ &= R^{mN}H(x_0, R^{-mN/\alpha}u_0, R^{-mN/\hat{\alpha}}v_0) \geq R^N \tilde{c}, \end{aligned}$$

where $0 < \tilde{c} \leq c_m$ for every $m \in \mathbb{N} \cup \{0\}$. Such a \tilde{c} exists since $c_m > 0$ by (A3).

The functional φ and its derivative satisfies some nice continuity properties as is shown in the two following lemmas.

Lemma 3.4. *The function φ is C^1 . Moreover, for every $(u, v), (w, z) \in X$,*

$$\begin{aligned} \langle \varphi'(u, v), (w, z) \rangle &= \int_{\mathbb{R}^N} |x|^{-2a} (\nabla u \cdot \nabla z + \nabla v \cdot \nabla w) \\ &\quad - (H_u(x, u, v)w + H_v(x, u, v)z) \, dx. \end{aligned}$$

Proof. Let $(u, v), (w, z) \in X$. For $x \in \mathbb{R}^N \setminus \{0\}$, and $|t| \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$\begin{aligned} &\frac{H(x, u(x) + tw(x), v(x) + tz(x)) - H(x, u(x), v(x))}{t} \\ &= H_u(x, u(x) + \lambda tw(x), v(x) + \lambda tz(x))w(x) \\ &\quad + H_v(x, u(x) + \lambda tw(x), v(x) + \lambda tz(x))z(x). \end{aligned}$$

Assumption (A2) and (3.2) imply that

$$\begin{aligned} &\frac{|H(x, u(x) + tw(x), v(x) + tz(x)) - H(x, u(x), v(x))|}{|t|} \\ &\leq \left(|x|^{-bp} |u(x) + \lambda tw(x)|^{p-1} + |x|^{-\hat{b}\hat{p}} |v(x) + \lambda tz(x)|^{\hat{p}-1} \right) (|w(x)| + |z(x)|) \\ &\leq \left(|x|^{-b} (|u(x)| + |w(x)|) \right)^{p-1} |x|^{-b} (|w(x)| + |z(x)|) \\ &\quad + \left(|x|^{-\hat{b}} (|v(x)| + |z(x)|) \right)^{\hat{p}-1} |x|^{-\hat{b}} (|w(x)| + |z(x)|). \end{aligned}$$

The term on the right-hand side is in L^1 by the Hölder inequality, since $w, z \in L_b^p(\mathbb{R}^N) \cap L_{\hat{b}}^{\hat{p}}(\mathbb{R}^N)$. The Lebesgue dominated convergence theorem implies that

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathbb{R}^N} H(x, u + tw, v + tz) - H(x, u, v) \, dx \\ = \int_{\mathbb{R}^N} H_u(x, u, v)w + H_v(x, u, v)z \, dx. \end{aligned}$$

Now, assume that $(u_n, v_n) \rightarrow (u, v)$ in X . From (A0) and Lemma 2.1, we deduce that

$$H_u(x, u_n(x), v_n(x)) - H_u(x, u(x), v(x)) \rightarrow 0,$$

and

$$H_v(x, u_n(x), v_n(x)) - H_v(x, u(x), v(x)) \rightarrow 0,$$

almost everywhere in \mathbb{R}^N . Also, (A2), (3.2) and the continuous imbeddings $D_{a,R}^{1,2}(\mathbb{R}^N) \hookrightarrow L_b^p(\mathbb{R}^N)$ and $D_{a,R}^{1,2}(\mathbb{R}^N) \hookrightarrow L_{\hat{b}}^{\hat{p}}(\mathbb{R}^N)$ imply that

$$|x|^b \left(H_u(x, u_n(x), v_n(x)) - H_u(x, u(x), v(x)) \right)$$

and

$$|x|^{\hat{b}} \left(H_v(x, u_n(x), v_n(x)) - H_v(x, u(x), v(x)) \right)$$

are bounded in $L^{p/(p-1)}$ and $L^{\hat{p}/(\hat{p}-1)}$ respectively. So,

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} H_u(x, u_n, v_n)w + H_v(x, u_n, v_n)z - H_u(x, u, v)w - H_v(x, u, v)z \, dx \right| \\ &= \left| \int_{\mathbb{R}^N} |x|^b \left(H_u(x, u_n, v_n) - H_u(x, u, v) \right) |x|^{-b} w \right. \\ &\quad \left. + |x|^{\hat{b}} \left(H_v(x, u_n, v_n) - H_v(x, u, v) \right) |x|^{-\hat{b}} z \, dx \right| \\ &\leq \|w\|_{L^{p,b}} \left(\int_{\mathbb{R}^N} \left| |x|^b \left(H_u(x, u_n, v_n) - H_u(x, u, v) \right) \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\quad + \|z\|_{L^{\hat{p},\hat{b}}} \left(\int_{\mathbb{R}^N} \left| |x|^{\hat{b}} \left(H_v(x, u_n, v_n) - H_v(x, u, v) \right) \right|^{\frac{\hat{p}}{\hat{p}-1}} dx \right)^{\frac{\hat{p}-1}{\hat{p}}} \\ &\leq C \left(\left(\int_{\mathbb{R}^N} \left| |x|^b \left(H_u(x, u_n, v_n) - H_u(x, u, v) \right) \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^N} \left| |x|^{\hat{b}} \left(H_v(x, u_n, v_n) - H_v(x, u, v) \right) \right|^{\frac{\hat{p}}{\hat{p}-1}} dx \right)^{\frac{\hat{p}-1}{\hat{p}}} \right), \end{aligned}$$

for every $(w, z) \in X$ such that $\|(w, z)\| \leq 1$. This shows that the Gâteaux derivative of ψ is continuous and hence ψ is C^1 .

On the other hand,

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathbb{R}^N} |x|^{-2a} \nabla(u + tw) \cdot \nabla(v + tz) - \nabla u \cdot \nabla v \, dx = \langle (u, v), (z, w) \rangle.$$

This Gâteaux derivative is obviously continuous; so φ is C^1 and

$$\begin{aligned} \langle \varphi'(u, v), (w, z) \rangle &= \int_{\mathbb{R}^N} |x|^{-2a} (\nabla u \cdot \nabla z + \nabla v \cdot \nabla w) \\ &\quad - (H_u(x, u, v)w - H_v(x, u, v)z) \, dx. \end{aligned}$$

□

Lemma 3.5. *The maps ψ and ψ' are weakly sequentially continuous.*

Proof. Suppose that $(u_n, v_n) \rightharpoonup (u, v)$ in X . So, $\{u_n\}$ and $\{v_n\}$ are bounded in $D_{a,R}^{1,2}(\mathbb{R}^N)$. Arguing as in the proof of the previous lemma, and using (A0), (A2), and Lemma 2.1, we deduce that

$$\psi(u_n, v_n) \rightarrow \psi(u, v),$$

and for every $w, z \in D_R(\mathbb{R}^N)$,

$$\langle \psi'(u_n, v_n), (w, z) \rangle \rightarrow \langle \psi'(u, v), (w, z) \rangle.$$

Moreover, $\{\psi'(u_n, v_n)\}$ is bounded in X , so $\psi'(u_n, v_n) \rightharpoonup \psi'(u, v)$. □

Now, we want to show that φ satisfies appropriate inequalities relative to a linking situation.

Lemma 3.6. *Let $z \in D_R(\mathbb{R}^N)$ be such that $\|(z, z)\| = 1$ with compact support in $B(0, 1) \setminus \{0\}$. Then there exists $r > 0$ such that*

$$m_0 := \inf_{\substack{(u,u) \in Z \\ \|(u,u)\|=r}} \varphi(u, u) > 0. \tag{3.5}$$

Moreover, there exists $\rho > r$ such that

$$\max_{M_0} \varphi = 0 \quad \text{and} \quad \sup_M \varphi =: m_1 < \infty, \tag{3.6}$$

where

$$M := \{u = y + \lambda z : \|u\| \leq \rho, \lambda \geq 0, y \in Y\}, \tag{3.7}$$

$$M_0 := \{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}. \tag{3.8}$$

Proof. Inequality (3.1) and the CKN inequality (2.2) imply directly (3.5) since there exists $K > 0$ such that, for $(u, u) \in Z$,

$$\varphi(u, u) \geq \frac{\|(u, u)\|^2}{2} - K \left(\|(u, u)\|^p + \|(u, u)\|^{\hat{p}} \right). \tag{3.9}$$

Observe that, on Y , we have

$$\varphi(-v, v) = \frac{-\|(-v, v)\|^2}{2} - \int_{\mathbb{R}^N} H(x, -v, v) dx \leq 0.$$

On the other hand, by (A3) and Lemma 3.2, on $Y \oplus \mathbb{R}(z, z)$, we have

$$\begin{aligned} & \varphi((-v, v) + \lambda(z, z)) \\ &= -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2}\|(z, z)\|^2 - \int_{\mathbb{R}^N} H(x, -v + \lambda z, v + \lambda z) dx \end{aligned}$$

$$\begin{aligned} &\leq -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2}\|(z, z)\|^2 - \int_{\text{supp}(z)} H(x, -v + \lambda z, v + \lambda z) dx \\ &\leq -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2} \\ &\quad - \sigma \int_{\text{supp}(z)} |x|^{-d} | -v + \lambda z|^\alpha + |x|^{-\hat{d}} |v + \lambda z|^{\hat{\alpha}} - 1 dx \\ &\leq -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2} - c_1 \int_{\text{supp}(z)} | -v + \lambda z|^\alpha + |v + \lambda z|^{\hat{\alpha}} dx - c_2. \end{aligned}$$

Denote by W the closure of $Y \oplus \mathbb{R}(z, z)$ in $L^\alpha(\text{supp}(z)) \times L^{\hat{\alpha}}(\text{supp}(z))$. Since there exists a continuous projection of W onto $\mathbb{R}(z, z)$ and all the norms are equivalent on the latter space, it follows that

$$\varphi((-v, v) + \lambda(z, z)) \leq -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2} - c_3(\lambda^\alpha + \lambda^{\hat{\alpha}}) - c_2,$$

with $c_2, c_3 > 0$. Therefore, we deduce that

$$\varphi(w) \rightarrow -\infty \quad \text{whenever } \|w\| \rightarrow \infty, \text{ and } w \in Y \oplus \mathbb{R}(z, z).$$

So, for some $\rho > r$, $\max_{M_0} \varphi = 0$. Finally, the CKN inequality (2.2) and (3.1) imply that φ maps bounded sets into bounded sets, hence $\sup_M \varphi < \infty$. \square

Lemma 3.7. *There exists $c \in [m_0, m_1]$ and a bounded sequence $\{(u_n, v_n)\}$ in X such that*

$$\varphi(u_n, v_n) \rightarrow c > 0, \quad \varphi'(u_n, v_n) \rightarrow 0. \tag{3.10}$$

Proof. It follows from Theorem 2.3 and Lemmas 3.4–3.6 that there exist $c \in [m_0, m_1]$ and a sequence $\{(u_n, v_n)\}$ in X satisfying (3.10). Moreover, since φ is invariant under a rescaling of the form (4.2), we may assume that the sequence $\{(u_n, v_n)\}$ is such that

$$\|(u_n, v_n)\| \leq \left(\int_{|x| \leq 1} |x|^{-2a} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \right)^{1/2} + 1. \tag{3.11}$$

Let us define

$$(s_n, t_n) := \frac{\alpha \hat{\alpha}}{\alpha + \hat{\alpha}} \left(\frac{1}{\alpha} u_n, \frac{1}{\hat{\alpha}} v_n \right).$$

Assumption (A3), Lemma 3.2, and (3.10) imply that, for n large enough,

$$\begin{aligned} c + 1 + \|(s_n, t_n)\| &\geq \varphi(u_n, v_n) - \langle \varphi'(u_n, v_n), (s_n, t_n) \rangle \\ &= \frac{\alpha \hat{\alpha}}{\alpha + \hat{\alpha}} \int_{\mathbb{R}^N} \frac{1}{\alpha} H_u(x, u_n, v_n) u_n + \frac{1}{\hat{\alpha}} H_v(x, u_n, v_n) v_n dx - \int_{\mathbb{R}^N} H(x, u_n, v_n) dx \\ &\geq \left(\frac{\alpha \hat{\alpha}}{\alpha + \hat{\alpha}} - 1 \right) \int_{\mathbb{R}^N} H(x, u_n, v_n) dx \end{aligned}$$

$$\geq \sigma \left(\frac{\alpha \hat{\alpha}}{\alpha + \hat{\alpha}} - 1 \right) \int_{|x| \leq 1} |x|^{-d} |u_n|^\alpha + |x|^{-\hat{d}} |v_n|^{\hat{\alpha}} - c_1 \, dx.$$

Hence,

$$C_1 + C_2 \|(u_n, v_n)\| \geq \int_{|x| \leq 1} |x|^{-d} |u_n|^\alpha + |x|^{-\hat{d}} |v_n|^{\hat{\alpha}} \, dx. \tag{3.12}$$

On the other hand, it follows from (3.10) that for n large enough,

$$\|(u_n, v_n)\|^2 \leq \left| \int_{\mathbb{R}^N} H_u(x, u_n, v_n)v_n + H_v(x, u_n, v_n)u_n \, dx \right| + \varepsilon \|(v_n, u_n)\|. \tag{3.13}$$

Using Assumption (A2), the CKN inequality (2.2), and (3.11), we deduce the existence of a constant $C_3 > 0$ such that

$$\int_{|x| \geq 1} |H_u(x, u_n, v_n)v_n| + |H_v(x, u_n, v_n)u_n| \, dx \leq C_3. \tag{3.14}$$

Again, by (A2), we have

$$\begin{aligned} & \int_{|x| \leq 1} |H_u(x, u_n, v_n)v_n| + |H_v(x, u_n, v_n)u_n| \, dx \\ & \leq k \int_{|x| \leq 1} |x|^{-bp} |u_n|^{p-1} |v_n| + |x|^{-\hat{b}\hat{p}} |v_n|^{\hat{p}-1} |u_n| \, dx \\ & = k \int_{|x| \leq 1} |x|^{-bp+\theta} |u_n|^{p-1} |x|^{-\theta} |v_n| + |x|^{-\hat{b}\hat{p}+\hat{\theta}} |v_n|^{\hat{p}-1} |x|^{-\hat{\theta}} |u_n| \, dx, \end{aligned}$$

where $\theta := N \left(\frac{\alpha-p+1}{\alpha} \right) - \xi$ and $\hat{\theta} := N \left(\frac{\hat{\alpha}-\hat{p}+1}{\hat{\alpha}} \right) - \xi$. So, since

$$d = (bp - \theta) \frac{\alpha}{p - 1} \quad \text{and} \quad \hat{d} = (\hat{b}\hat{p} - \hat{\theta}) \frac{\hat{\alpha}}{\hat{p} - 1},$$

the Hölder inequality yields

$$\begin{aligned} & \int_{|x| \leq 1} |H_u(x, u_n, v_n)v_n| + |H_v(x, u_n, v_n)u_n| \, dx \\ & \leq k \left[\left(\int_{|x| \leq 1} |x|^{-d} |u_n|^\alpha \, dx \right)^{\frac{p-1}{\alpha}} \left(\int_{|x| \leq 1} (|x|^{-\theta} |v_n|)^{\frac{\alpha}{\alpha-p+1}} \, dx \right)^{\frac{\alpha-p+1}{\alpha}} \right. \\ & \quad \left. + \left(\int_{|x| \leq 1} |x|^{-\hat{d}} |v_n|^{\hat{\alpha}} \, dx \right)^{\frac{\hat{p}-1}{\hat{\alpha}}} \left(\int_{|x| \leq 1} (|x|^{-\hat{\theta}} |u_n|)^{\frac{\hat{\alpha}}{\hat{\alpha}-\hat{p}+1}} \, dx \right)^{\frac{\hat{\alpha}-\hat{p}+1}{\hat{\alpha}}} \right]. \end{aligned}$$

Observe that $\theta, \hat{\theta} \in [a, a + 1)$, and

$$\frac{\alpha}{\alpha - p + 1} = \frac{2N}{N - 2 + 2(\theta - a)} \quad \text{and} \quad \frac{\hat{\alpha}}{\hat{\alpha} - \hat{p} + 1} = \frac{2N}{N - 2 + 2(\hat{\theta} - a)}.$$

Therefore, the CKN inequality (2.2) implies that

$$\begin{aligned} & \int_{|x| \leq 1} |H_u(x, u_n, v_n)v_n| + |H_v(x, u_n, v_n)u_n| \, dx \\ & \leq k_1 \left[\left(\int_{|x| \leq 1} |x|^{-d} |u_n|^\alpha \, dx \right)^{\frac{p-1}{\alpha}} \|(0, v_n)\| + \left(\int_{|x| \leq 1} |x|^{-\hat{d}} |v_n|^{\hat{\alpha}} \, dx \right)^{\frac{\hat{p}-1}{\hat{\alpha}}} \|(u_n, 0)\| \right] \\ & \leq k_1 \left[\left(\int_{|x| \leq 1} |x|^{-d} |u_n|^\alpha \, dx \right)^{\frac{p-1}{\alpha}} + \left(\int_{|x| \leq 1} |x|^{-\hat{d}} |v_n|^{\hat{\alpha}} \, dx \right)^{\frac{\hat{p}-1}{\hat{\alpha}}} \right] \|(u_n, v_n)\|. \end{aligned}$$

Consequently, by (3.13), (3.14), since $\|(u_n, v_n)\| = \|(v_n, u_n)\|$,

$$\begin{aligned} \|(u_n, v_n)\|^2 & \leq k_1 \left[\left(\int_{|x| \leq 1} |x|^{-d} |u_n|^\alpha \, dx \right)^{\frac{p-1}{\alpha}} \right. \\ & \quad \left. + \left(\int_{|x| \leq 1} |x|^{-\hat{d}} |v_n|^{\hat{\alpha}} \, dx \right)^{\frac{\hat{p}-1}{\hat{\alpha}}} + \tilde{\varepsilon} \right] \|(u_n, v_n)\| + C_3. \end{aligned} \tag{3.15}$$

Now, it follows from (3.12) and (3.15) that

$$\begin{aligned} \|(u_n, v_n)\|^2 & \leq k_1 \left[\left(C_1 + C_2 \|(u_n, v_n)\| \right)^{\frac{p-1}{\alpha}} \right. \\ & \quad \left. + \left(C_1 + C_2 \|(u_n, v_n)\| \right)^{\frac{\hat{p}-1}{\hat{\alpha}}} + \tilde{\varepsilon} \right] \|(u_n, v_n)\| + C_3. \end{aligned}$$

Finally, since $(p-1)/\alpha < 1$ and $(\hat{p}-1)/\hat{\alpha} < 1$, we conclude that the sequence $\{(u_n, v_n)\}$ is bounded. \square

4. MAIN RESULTS

We first consider the following system of semilinear singular Poisson equations:

$$\begin{cases} -\operatorname{div}(\nabla u |x|^{-2a}) = H_v(x, u, v), \\ -\operatorname{div}(\nabla v |x|^{-2a}) = H_u(x, u, v), \end{cases} \quad u, v \in D_{a,R}^{1,2}(\mathbb{R}^N). \tag{4.1}$$

This problem allows a variational formulation and we look for (weak) solutions of (4.1) which are critical points of the functional φ studied in the previous section and defined by

$$\varphi(u, v) := \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v - H(x, u, v) \, dx.$$

Notice that if H satisfies the invariance assumption (A1), then, for any $(u, v) \in X$ and for any $m \in \mathbb{Z}$, the dilated functions

$$\tilde{u}_m(x) := R^{m\xi} u(R^m x) \quad \text{and} \quad \tilde{v}_m(x) := R^{m\xi} v(R^m x) \tag{4.2}$$

satisfy

$$||x|^{-a}\nabla\tilde{u}_m|_2 = ||x|^{-a}\nabla u|_2, \quad ||x|^{-a}\nabla\tilde{v}_m|_2 = ||x|^{-a}\nabla v|_2,$$

and

$$\int_{\mathbb{R}^N} H(x, \tilde{u}_m, \tilde{v}_m) dx = \int_{\mathbb{R}^N} H(x, u, v) dx.$$

Consequently, the functional φ is invariant by dilation, and hence the Palais-Smale condition fails at each critical level c .

Here is our first main result.

Theorem 4.1. *Under the assumptions (A0)–(A4), the problem (4.1) has a nontrivial solution.*

Proof. Lemma 3.7 guarantees the existence of a bounded sequence $\{(u_n, v_n)\}$ in X such that

$$\varphi(u_n, v_n) \rightarrow c > 0, \quad \text{and} \quad \varphi'(u_n, v_n) \rightarrow 0. \tag{4.3}$$

Now, let us assume that

$$\begin{aligned} \delta_1 &:= \limsup_{n \rightarrow \infty} \sup_{m \in \mathbb{Z}} \int_{A_m} |x|^{-bp} |u_n|^p = 0, \\ \delta_2 &:= \limsup_{n \rightarrow \infty} \sup_{m \in \mathbb{Z}} \int_{A_m} |x|^{-\hat{b}\hat{p}} |v_n|^{\hat{p}} = 0. \end{aligned}$$

Lemma 2.2 implies that

$$u_n \rightarrow 0 \text{ in } L_b^p(\mathbb{R}^N), \quad \text{and} \quad v_n \rightarrow 0 \text{ in } L_{\hat{b}}^{\hat{p}}(\mathbb{R}^N). \tag{4.4}$$

As in the proof of Lemma 3.7, we fix

$$(s_n, t_n) = \frac{\alpha\hat{\alpha}}{\alpha + \hat{\alpha}} \left(\frac{1}{\alpha} u_n, \frac{1}{\hat{\alpha}} v_n \right).$$

Combining (A2), (A3), (4.3), and (4.4), we deduce that

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \varphi(u_n, v_n) - \langle \varphi'(u_n, v_n), (s_n, t_n) \rangle \\ &= \lim_{n \rightarrow \infty} \frac{\alpha\hat{\alpha}}{\alpha + \hat{\alpha}} \int_{\mathbb{R}^N} \frac{1}{\alpha} H_u(x, u_n, v_n) u_n + \frac{1}{\hat{\alpha}} H_v(x, u_n, v_n) v_n dx \\ &\quad - \int_{\mathbb{R}^N} H(x, u_n, v_n) dx \\ &\leq \lim_{n \rightarrow \infty} \frac{\alpha\hat{\alpha}}{\alpha + \hat{\alpha}} \int_{\mathbb{R}^N} \frac{1}{\alpha} H_u(x, u_n, v_n) u_n + \frac{1}{\hat{\alpha}} H_v(x, u_n, v_n) v_n dx \\ &\leq \lim_{n \rightarrow \infty} C \int_{\mathbb{R}^N} |x|^{-bp} |u_n|^p + |x|^{-\hat{b}\hat{p}} |v_n|^{\hat{p}} dx = 0, \end{aligned}$$

which leads to a contradiction since $c > 0$.

Therefore, we must have $\delta := \max\{\delta_1, \delta_2\} > 0$. Taking a subsequence if necessary, we deduce the existence of $m_n \in \mathbb{Z}$ such that

$$\int_{A_{m_n}} |x|^{-bp} |u_n|^p + |x|^{-\hat{b}\hat{p}} |v_n|^{\hat{p}} dx > \frac{\delta}{2}.$$

The sequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ defined by

$$\tilde{u}_n(x) := R^{-m_n \xi} u_n(R^{-m_n} x) \quad \text{and} \quad \tilde{v}_n(x) := R^{-m_n \xi} v_n(R^{-m_n} x)$$

is such that

$$\int_{A_0} |x|^{-bp} |\tilde{u}_n|^p + |x|^{\hat{b}\hat{p}} |\tilde{v}_n|^{\hat{p}} dx > \frac{\delta}{2}, \tag{4.5}$$

it satisfies (4.3), and is bounded by dilation invariance. Taking again a subsequence if needed, there exists $(u, v) \in X$ such that

$$(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v) \quad \text{in } X.$$

Lemma 2.1 implies that $|x|^{-a} \tilde{u}_n \rightarrow |x|^{-a} u$, $|x|^{-a} \tilde{v}_n \rightarrow |x|^{-a} v$, in $L^2_{loc}(\mathbb{R}^N)$. This, combined with (4.5), insures that $(u, v) \neq 0$. Finally, the weakly sequential continuity of φ' gives

$$\|\varphi'(u, v)\| \leq \liminf_{n \rightarrow \infty} \|\varphi'(\tilde{u}_n, \tilde{v}_n)\| = 0.$$

Consequently (u, v) is a nontrivial solution of (4.1). □

Corollary 4.2. *For every $\lambda, \gamma > 0$, the problem*

$$\begin{cases} -\operatorname{div}(\nabla u |x|^{-2a}) = \lambda |x|^{-\hat{b}\hat{p}} |v|^{\hat{p}-2} v, \\ -\operatorname{div}(\nabla v |x|^{-2a}) = \gamma |x|^{-bp} |u|^{p-2} u, \end{cases} \quad u, v \in D^{1,2}_{a,R}(\mathbb{R}^N) \tag{4.6}$$

has a nontrivial solution.

Now, we consider another system which is a slight perturbation of the preceding problem (4.1):

$$\begin{cases} -\operatorname{div}(\nabla u |x|^{-2a}) = \beta |x|^{-2(1+a)} v + H_v(x, u, v), \\ -\operatorname{div}(\nabla v |x|^{-2a}) = \eta |x|^{-2(1+a)} u + H_u(x, u, v), \end{cases} \quad u, v \in D^{1,2}_{a,R}(\mathbb{R}^N). \tag{4.7}$$

Again, the problem (4.7) has a variational formulation since the critical points of φ_1 are solutions of (4.7), where

$$\begin{aligned} \varphi_1(u, v) &:= \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v - \frac{1}{2} |x|^{-2(1+a)} (\eta u^2 + \beta v^2) - H(x, u, v) dx \\ &= \frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v) - \Phi(u, v) = \varphi(u, v) - \Phi(u, v), \end{aligned}$$

and

$$\Phi(u, v) := \int_{\mathbb{R}^N} |x|^{-2(1+a)} \left(\eta \frac{u^2}{2} + \beta \frac{v^2}{2} \right) dx,$$

and where φ and ψ are given above in this article. Here is our second main result.

Theorem 4.3. *Let $\beta, \eta \in (0, S(a, a + 1))$ and H satisfy (A0)–(A4) with $\alpha = \hat{\alpha} > 2$. Then the problem (4.7) has a nontrivial solution.*

Proof. Arguing as in Lemmas 3.4 and 3.5 we can show that φ_1 is C^1 , weakly sequentially lower semicontinuous, Φ' is also weakly sequentially continuous, and

$$\begin{aligned} \langle \varphi'_1(u, v), (w, z) \rangle &= \int_{\mathbb{R}^N} |x|^{-2a} (\nabla u \cdot \nabla z + \nabla v \cdot \nabla w) dx \\ &\quad - \int_{\mathbb{R}^N} |x|^{-2(a+1)} (\eta uw + \beta vz) - H_u(x, u, v)w - H_v(x, u, v)z dx. \end{aligned}$$

Set $\zeta := (\beta + \eta)/2$, then for $(u, u) \in Z$ it follows from the CKN inequality (2.2), and (3.9), that

$$\varphi_1(u, u) \geq \left(\frac{1}{2} - \frac{\zeta}{2S(a, a + 1)} \right) \|(u, u)\|^2 - K(\|(u, u)\|^p + \|(u, u)\|^{\hat{p}}).$$

Therefore, there exists $r > 0$ such that

$$m_0 := \inf_{\substack{(u,u) \in Z \\ \|(u,u)\|=r}} \varphi_1(u, u) > 0.$$

Again, we fix $z \in D_R(\mathbb{R}^N)$ with compact support in $B(0, 1) \setminus \{0\}$ and such that $\|(z, z)\| = 1$. The fact that for $(u, v) \in X$ $\varphi_1(u, v) \leq \varphi(u, v)$, and the arguments used in the proof of Lemma 3.6 lead to the existence of $\rho > r$ such that $\max_{M_0} \varphi_1 = 0$ and $m_1 := \sup_M \varphi_1 < \infty$, where M and M_0 are given respectively by (3.7) and (3.8). Theorem 2.3 guarantees the existence of a sequence $\{(u_n, v_n)\} \subseteq X$ such that

$$\varphi_1(u_n, v_n) \rightarrow c \in [m_0, m_1], \quad \varphi'_1(u_n, v_n) \rightarrow 0.$$

To show that the sequence $\{(u_n, v_n)\}$ is bounded, we proceed as in the proof of Lemma 3.7 in fixing $(s_n, t_n) := (u_n, v_n)/2$. To conclude, we argue as in the proof of Theorem 4.1. \square

Corollary 4.4. *For every $\lambda, \gamma > 0$, $\beta, \eta \in (0, S(a, a + 1))$, the problem*

$$\begin{cases} -\operatorname{div}(\nabla u |x|^{-2a}) = \beta |x|^{-2(1+a)} v + \lambda |x|^{-\hat{b}\hat{p}} |v|^{\hat{p}-2} v, \\ -\operatorname{div}(\nabla v |x|^{-2a}) = \eta |x|^{-2(1+a)} u + \gamma |x|^{-bp} |u|^{p-2} u, \end{cases} \quad u, v \in D_{a,R}^{1,2}(\mathbb{R}^N) \tag{4.8}$$

has a nontrivial solution.

5. RADIAL AND NONRADIAL SOLUTIONS

In this paragraph, we want to observe that if H is invariant under some group action, then we can deduce more information about a solution of (4.1) or (4.7). In particular, we will see that the problems (4.6) and (4.8) have a nontrivial radial solution and a nontrivial nonradial solution.

Theorem 5.1. *Assume (A0)–(A4), and*

$$(A5) \quad H(gx, u, v) = H(x, u, v), \quad \forall g \in O(N), \forall x \in \mathbb{R}^N \setminus \{0\}, \forall (u, v) \in \mathbb{R}^2.$$

Then the problems (4.1) and (4.7) have a nontrivial radially symmetric solution.

Proof. Let us define the action of the group $O(N)$ on X by

$$g(u(x), v(x)) := (u(g^{-1}x), v(g^{-1}x)), \quad g \in O(N).$$

We consider the following space

$$X_{O(N)} := \text{Fix}(O(N)) = \{(u, v) \in X : g(u, v) = (u, v) \quad \forall g \in G\},$$

whose elements are the pairs of radially symmetric functions. Replacing X by $X_{O(N)}$ in the above proofs, Theorem 4.1 (respectively Theorem 4.3) give us a nontrivial critical point (u, v) of φ (respectively φ_1) restricted to $X_{O(N)}$. But by the principle of symmetric criticality (Theorem 2.4), (u, v) is a nontrivial and radially symmetric critical point of φ (respectively φ_1). \square

Similarly, if H satisfies another invariance condition, then we can deduce the existence of a nonradial solution.

Theorem 5.2. *Assume (A0)–(A4), and*

$$(A6) \quad H(g_0(x), -u, -v) = H(x, u, v), \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \forall (u, v) \in \mathbb{R}^2, \text{ and } g_0 \text{ be the involution on } \mathbb{R}^N \text{ defined by}$$

$$g_0(x_1, x_2, x_3, \dots, x_n) = (x_2, x_1, x_3, \dots, x_n).$$

Then the problems (4.1) and (4.7) have a nonradially symmetric solution.

Proof. As in the proof of the previous theorem, we deduce the existence of $(u, v) \in X_G$ a solution of (4.1) or (4.7), where

$$X_G := \text{Fix}(G) = \{(u, v) \in X : g(u, v) = (u, v) \quad \forall g \in G\},$$

with the action of the group $G = \{1, g_0\}$ on X defined by

$$g(u(x), v(x)) = \begin{cases} (u(x), v(x)), & \text{if } g = 1, \\ (-u(g_0(x)), -v(g_0(x))), & \text{if } g = g_0. \end{cases}$$

It is also clear that $(0, 0)$ is the only radial function of X_G . \square

As a corollary, we obtain the existence of a nontrivial radial and a nontrivial nonradial solution to problems (4.6) and (4.8).

Corollary 5.3. *For every $\lambda, \gamma > 0$ (respectively and $\beta, \eta \in (0, S(a, a + 1))$), the problem (4.6) (respectively (4.8)) has a nontrivial radial and a nontrivial nonradial solutions.*

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