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Differential inclusions and implicit equations on closed subsets of $\mathbb{R}^n$

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Abstract. The initial value problem and periodic boundary value problem are studied for first order differential inclusions of lower semicontinuous type and upper semicontinuous type, and next, for implicit differential equations which are not continuously solvable for the derivative. The existence of solutions remaining in a certain closed subset K of $\mathbb{R}^n$ is proved under a rather mild assumption on K.

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1. Introduction

The aim of this paper is twofold. We first study the existence of solutions $x : [0,T] \to K \subset \mathbb{R}^n$ (so called *viable solutions*, c.f.[19]) to the initial value problem

$$x' \in \varphi(t,x), \text{ a.e. } t, \; x(0) = x_0 \in K, \tag{I}$$

where $\varphi : [0,T] \times K \to 2^{\mathbb{R}^n}$ is a multivalued function and $K \subset \mathbb{R}^n$ is a so-called *proximate retract* (introduced in [3, 18] under the name *set with property* ρ). We postpone the definitions to the next section but let us comment now that any closed convex subset of $\mathbb{R}^n$, as well as any C^2 submanifold of $\mathbb{R}^n$ is a proximate retract, so the class of such sets is quite large. Similar problems were studied in the viability theory by many authors, e.g. [5, 6, 13, 19] under specific assumptions on K such as compactness and/or convexity. In [3, 18], the problem (I) is studied for compact proximate retracts, and the periodic boundary value problem

$$x' \in \varphi(t,x), \text{ a.e. } t, \; x(0) = x(T), \tag{P}$$

is studied under the additional assumption that the Euler characteristic $\chi(K)$ of K is nonzero. Those results of [3, 18] which concern the initial value problem are extended here to the noncompact case and those concerning the periodic problem are used to derive further conclusions.

We next apply results concerning (I) and (P) to implicit differential equations of the form

$$x' = f(t,x,x'), \text{ a.e. } t, \; x(0) = x_0 \in K \tag{I'}$$

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and, respectively,

$$x' = f(t, x, x'), \text{ a.e. } t, \; x(0) = x(T), \tag{P'}$$

where $f : [0,T] \times K \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. In general, the equation $y = f(t,x,y)$ is not globally and continuously solvable for y, so one cannot reduce the problem to the quasi-linear equation $x' = g(t,x)$. Our way of applying (I) or (P) is to choose $\varphi(t,x)$ to be either the set of fixed points of $f(t,x,\cdot)$ or a certain lower semicontinuous multivalued selection of it; depending on conditions on f. In the case $K = \mathbb{R}^n$, implicit equations such as (I′) or (P′), or similar ones of higher order, were studied by the use of k-set contractions, condensing, or A-proper mappings (see e.g. [16]), however, very strong Lipschitz-type conditions were imposed. The method of reducing the problem to a differential inclusion, which allows to impose less restrictions on f, has been used in [2, 10, 12, 14, 17] and most recently in [15], when $K = \mathbb{R}^n$. To the authors' best knowledge, viable solutions of implicit differential equations are studied here for the first time.

The paper is organized as follows. In Section 2, basic definitions and preliminary results concerning proximate retracts and multivalued maps are presented. In Section 3 we state and prove theorems concerning differential inclusions, and in Section 4, those concerning implicit differential equations.

2. Definitions and preliminary results

Let K be a nonempty closed subset of $\mathbb{R}^n$ and $u \in \mathbb{R}^n$. We define $\operatorname{dist}(u,K) = \inf\{\| u - x \| : x \in K\}$. We recall from [1, 3] that a subset $T_K(x) \subset \mathbb{R}^n, x \in K$, defined by

$$T_K(x) = \{y \in \mathbb{R}^n : \liminf_{t \to 0^+} \frac{1}{t} \operatorname{dist}(x + ty, K) = 0\}$$

is called the *Bouligand tangent cone to* K *at* x. A nonempty closed subset K of $\mathbb{R}^n$ is called a *proximate retract* if there exists an open neighbourhood U of K in $\mathbb{R}^n$ and a continuous map $r : U \to K$ (called *metric retraction*) such that the following two conditions are verified :

i) $r(x) = x$ for all $x \in K$;

ii) $\| r(u) - u \| = \operatorname{dist}(u, K)$ for all $u \in U$.

It is easy to prove that, for a given K and U, if $r : U \to K$ exists then it is unique. Since one can take a sufficiently small U (for example by restricting U to $U \cap \{u \in \mathbb{R}^n : \operatorname{dist}(u,K) < \delta\}, \delta > 0$, we may assume that $\| r(u) - u \| \leq \delta$ for a given $\delta > 0$ and $u \in U$.

Lemma 2.1. *Let K be a proximate retract. Then $T_K(r(x)) \subset \{y \in \mathbb{R}^n : (y, x - r(x)) \leq 0\}$ for all $x \in U$, where $(\cdot,\cdot)$ stands for the euclidean scalar product.*

Proof. Assume that $x \in U$ and $y \in \mathbb{R}^n$ are such that $(y, x - r(x)) > 0$. We verify that

$$\lim_{t \to 0} \frac{\operatorname{dist}(r(x) + ty, \mathbb{R}^n - B(x, \| x - r(x) \|))}{t} > 0$$

where $B(x, s)$ is the open ball centered at x of radius s. However, $\operatorname{dist}(r(x) + ty, K) \leq \operatorname{dist}(r(x) + ty, \mathbb{R}^n - B(x, \| x - r(x) \|))$, hence $y \notin T_K(r(x))$. □

Lemma 2.2. *Let K be a proximate retract, $r : U \to K$ a metric retraction, and $N > 0$ such that $K \cap \overline{B}(0, N) \neq \emptyset$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ there exist subsets $K \subset K_\varepsilon \subset U_\varepsilon \subset U$ of $\mathbb{R}^n$, K_ε closed and U_ε open, and a continuous function $r_\varepsilon : U_\varepsilon \to K_\varepsilon$ such that the following conditions are verified*

i) $\bigcap_{\varepsilon \leq \varepsilon_0} \quad K_\varepsilon = K$;
ii) $\| r_\varepsilon(u) - u \| = \operatorname{dist}(u, K_\varepsilon)$ *for all* $u \in U_\varepsilon \cap \overline{B}(0, N)$;
iii) $\{y \in \mathbb{R}^n : (y, x - r(x)) \leq 0\} \subset T_{K_\varepsilon}(x)$ *for all* $x \in K_\varepsilon \cap \overline{B}(0, N)$.

Proof. Let $\varepsilon_0 < \dfrac{1}{2}$ be such that $(K \cap \overline{B}(0, N + 1)) + B(0, 2\varepsilon_0) \subset U$. Let $\varepsilon \leq \varepsilon_0$, and let $\delta_\varepsilon : K \to [0, 1]$ be defined by

$$\delta_\varepsilon(x) = \begin{cases} \max\left\{\delta \in [0,1] : x \in \overline{B}(0, N + 1 + (1 - \delta)\varepsilon\right\} & \text{if } x \in \overline{B}(0, N + 1 + \varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

We define $K_\varepsilon = \{x + \delta_\varepsilon(x) y : x \in K \text{ and } y \in \overline{B}(0, N+1+\varepsilon)\}, U_\varepsilon = (K + \overline{B}(0, 2\varepsilon)) \cap U$,
and

$$r_\varepsilon(u) = \begin{cases} u & \text{if } u \in r(u) + \delta_\varepsilon(r(u))\overline{B}(0, \varepsilon) \\ r(u) + \varepsilon \delta_\varepsilon(r(u)) \dfrac{(u - r(u)}{\| u - r(u) \|} & \text{otherwise} \end{cases}$$

A routine verification shows that $K_\varepsilon, U_\varepsilon$ and r_ε have the required properties. □

We will now discuss preliminaries related to multivalued maps. Let $\varphi : [0, T] \times K \to 2^{\mathbb{R}^n}$ be a set-valued function with nonempty compact values, where K is a proximate retract. We say that φ is *integrably bounded on bounded sets* (abbreviation *ibbs*) if for every $k > 0$ there exists $h_k \in L^1([0, T], \mathbb{R}^+)$ such that $\| \varphi(t, x) \| \leq h_k(t)$ for a.e. t and all x with $\| x \| \leq k$. We say that φ is of *lsc-type* if it is lower semicontinuous in x for a.e. t and $\mathcal{L} \otimes \mathcal{B}$ measurable or, alternatively, if it is continuous in x for a.e. t and measurable in t for all x. We say that φ is of *usc-type* if it is upper semicontinuous in x for a.e. t, measurable in t for all x, and it has convex values. We say that φ is *tangent to* K if $\varphi(t, x) \cap T_K(x) \neq \emptyset$ for all $x \in K$ and a.e. t. It is *strongly tangent to* K if $\varphi(t, x) \subset T_K(x)$ for all $x \in K$ and a.e. t.

In the next section we shall use a construction presented below. Let U be a fixed open neighbourhood of K and $\lambda : \mathbb{R}^n \to [0, 1]$ an *Urysohn function* for K and U, i.e. $\lambda(x) = 1$ if $x \in K, \lambda(x) = 0$ if $x \notin U$, and λ is continuous. Let $r : U \to K$

be a metric retraction. We define the extension of $\varphi, \widetilde{\varphi} : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$, by the formula

$$\widetilde{\varphi}(t,x) = \begin{cases} \lambda(x)\varphi(t, r(x)) & \text{if } x \in U \\ \{0\} & \text{if } x \notin U. \end{cases} \tag{2.1}$$

Proposition 2.3. *If φ is, respectively, ibbs, usc-type, or lsc-type, then so is $\widetilde{\varphi}$.*

Lemma 2.4. *Assume that $\varphi(t,x)$ is strongly tangent to K. If $x : [0,T] \to \mathbb{R}^n$ is an absolutely continuous function such that $x'(t) \in \widetilde{\varphi}(t, x(t))$, for a.e. t, and $x(0) \in K$, then $x(t) \in K$ for all t.*

Proof. Let $d : [0,T] \to \mathbb{R}^+$ be defined by $d(t) = \text{dist}(x(t), K), t \in [0,T]$. We want to show that $d(t) = 0$ for all $t \in [0,T]$. By our assumption, $d(0) = 0$. It is easily verified that d is absolutely continuous. Therefore it is sufficient to show that $d'(t) \leq 0$, for a.e. t, which implies that d is nonincreasing. Let $t_0 \in (0,T]$ and $0 < h \leq T - t_0$ be such that $x'(t_0) \in \widetilde{\varphi}(t_0, x(t_0))$. We may assume that $x(t_0) \in U$ since otherwise $x'(t_0) = 0$. It is easily verified that

$$d(t_0 + h) \leq \text{ dist } (r(x(t_0)) + hx'(t_0), K) + \| x(t_0 + h) - x(t_0) - hx'(t_0) \| + d(t_0)$$

which implies that $\liminf_{h \to 0+} \frac{1}{h} (d(t_0 + h) - d(t_0)) \leq 0$ and the conclusion follows. □

Remark 2.5. The assumption that K is a proximate retract can be weaken in the last lemma as follows. Let K be a nonempty closed subset of $\mathbb{R}^n$ such that $K \cap B(0,M) \neq \emptyset$ for some $M > 0$ and such that, for a given $U \supset K$ and continuous $r : U \to K$, the condition $\| r(u) - u \| = \text{ dist}(u, K)$ is verified by all $u \in U \cap \overline{B}(0,M)$. If $x : [0,T] \to \mathbb{R}^n$ is as in Lemma 2.4 and, in addition, $\| x(t) \| < M$ for all t, then the conclusion of Lemma 2.4 holds.

3. Differential inclusions on proximate retracts

We study the problem (I) announced in the introduction, where K is a proximate retract and $\varphi : [0,T] \times K \to 2^{\mathbb{R}^n}$ is multivalued with nonempty compact values.

Theorem 3.1. *Let φ be ibbs, either usc-type or lsc-type, and strongly tangent to K. Assume that the following a priori bound condition is verified :*

(C) *There exists a constant $M > \| x_0 \|$ such that any possible solution of* (I) *defined on $[0, t_0], 0 < t_0 \leq T$, must satisfy $\| x(t) \| < M$ for all $0 \leq t \leq t_0$.*

Then the problem (I) *has a solution defined on $[0,T]$.*

Proof. By the definition of $\widetilde{\varphi}$ and Lemma 2.4, any solution of the problem

$$x'(t) \in \widetilde{\varphi}(t, x(t)), \text{ a.e. } t, \ x(0) = x_0 \in K \tag{$\widetilde{\text{I}}$}$$

is also a solution of (I). Let $\varepsilon > 0$ and let $\mu_\varepsilon : \mathbb{R}^n \to [0,1]$ be an Urysohn function for $\overline{B}(0, M)$ and $B(0, M+\varepsilon)$, i.e. $\mu_\varepsilon(x) = 1$ if $\| x \| \leq M$ and $\mu_\varepsilon(x) = 0$ if $\| x \| \geq M + \varepsilon$. We define $\widetilde{\varphi}_\varepsilon : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ by $\widetilde{\varphi}_\varepsilon(t,x) = \mu_\varepsilon(x)\widetilde{\varphi}(t,x)$, and we condiser the problem

$$x'(t) \in \widetilde{\varphi}_\varepsilon(t, x(t)), \text{ a.e. } t, \; x(0) = x_0. \qquad (\tilde{\mathrm{I}}_\varepsilon)$$

If x is a solution of $(\tilde{\mathrm{I}}_\varepsilon)$ defined on $[0,T]$, then it is a solution of $(\tilde{\mathrm{I}})$ and, consequently, of (I) : Indeed, if x verifies $\| x \|_0 \leq M$, then this is a direct consequence of the definition of $\widetilde{\varphi}_\varepsilon$. If not, then there is a smallest $t_0 \in [0,T]$ such that $\| x(t_0) \| < M$. Then $x_{|[0,T]}$ is a solution of (I), and (C) implies that $\| x(t_0) \| < M$, a contradiction.

It remains to show that $(\tilde{\mathrm{I}}_\varepsilon)$ has a solution on $[0,T]$. By Proposition 2.3, the semicontinuity and boundeness assumptions carry over from φ to $\widetilde{\varphi}$ and $\widetilde{\varphi}_\varepsilon$. Define the map $F_\varepsilon : C([0,T], \mathbb{R}^n) \to 2^{L^1([0,T],\mathbb{R}^n)}$ by

$$F_\varepsilon(x) = \{v \in L^1([0,t], \mathbb{R}^n) : v(t) \in \widetilde{\varphi}_\varepsilon(t, x(t)) \text{ a.e. } t\},$$

and the operator $S : L^1([0,T], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$ by

$$Sv(t) = x_0 + \int_0^t v(s)ds.$$

Then $(\tilde{\mathrm{I}}_\varepsilon)$ is equivalent to the fixed point problem for the mapping $S \circ F_\varepsilon$.

Case 1: φ is of usc-type.

Then $S \circ F_\varepsilon$ is a convex-valued compact u.s.c. map and the conclusion follows from the Ky-Fan Fixed Point Theorem (c.f. [9], Th. 5.11.3).

Case 2: φ is of lsc-type.

Then F_ε has a continuous selection $f_\varepsilon : C([0,T], \mathbb{R}^n) \to L^1([0,T], \mathbb{R}^n)$ by the Bressan-Colombo Selection Theorem (c.f. [4]) and the conclusion follows from the Schauder Fixed Point Theorem (c.f. [9]) applied for $S \circ f_\varepsilon$. □

Theorem 3.2. *Let φ be ibbs, usc-type, and tangent to K. Assume that the condition* (C) *of Theorem* 3.1 *is verified. Then the set $S(\varphi, X_0)$ of solutions to* (I) *defined on* $[0,T]$ *is an R_δ-set.*

Proof. Step 1: Assume that $\varphi(t,x) \subset T_K(x)$.

By Lemma 2.4., it is sufficient to show that the set $S(\widetilde{\varphi}, x_0)$ of solutions to $(\tilde{\mathrm{I}})$ is an R_δ-set. Let $\varepsilon > 0$, and let μ_ε and $\widetilde{\varphi}_\varepsilon$ be as in the proof of Theorem 3.1. By the result of [7] the set $S(\widetilde{\varphi}_\varepsilon, x_0)$ of solutions to $(\tilde{\mathrm{I}}_\varepsilon)$ is an R_δ-set. As in the previuous proof, (C) implies that $S(\widetilde{\varphi}_\varepsilon, x_0) = S(\widetilde{\varphi}, x_0)$.

Step 2: $\varphi(t,x) \cap T_K(x) \neq \emptyset$.

We take $N = M + 1$, and let $\varepsilon, K_\varepsilon, r_\varepsilon : U_\varepsilon \to K_\varepsilon$ be given for $\varepsilon \leq \varepsilon_0$ as in Lemma 2.2. Define $\psi_\varepsilon : [0,T] \times K_\varepsilon \to \mathbb{R}^n$ by

$$\psi_\varepsilon(t,x) = \mu_\varepsilon(x)\varphi(t, r(x)) \cap \{y \in \mathbb{R}^n : (y, x - r(x)) \leq 0\}$$

Then ψ_ε is ibbs and usc-type. By Lemmas 2.1 and 2.2, $\emptyset \neq \psi_\varepsilon(t,x) \subset T_{K_\varepsilon}(x)$ for all $x \in K_\varepsilon$. We consider the problems

$$x'(t) \in \psi_\varepsilon(t, x(t)), \text{ a.e. } t, \ x(0) = x_0, \tag{J_ε}$$

$$x'(t) \in \widetilde{\psi}_\varepsilon(t, x(t)), \text{ a.e. } t, \ x(0) = x_0, \tag{$\tilde{\mathrm{J}}_\varepsilon$}$$

where $\widetilde{\psi}_\varepsilon$ is defined for ψ_ε as in (2.1) by using r_ε. By Lemma 2.2 and Remark 2.6 we get $S(\psi_\varepsilon, x_0) = S(\widetilde{\psi}_\varepsilon, x_0)$. By arguing as in the first step, we conclude that $S\psi_\varepsilon, x_0)$ is an R_δ-set. Let $\varepsilon_n \leq \varepsilon_0, n = 1, 2, \ldots$, be a sequence converging to 0. The condition (C), Lemma 2.1 and Lemma 2.2 give $S(\varphi, x_0) = \bigcap_n S(\psi_{\varepsilon_n}, x_0)$, hence $S(\varphi, x_0)$ is an R_δ-set. □

Corollary 3.3. *Let φ be ibbs, either usc-type or lsc-type, and strongly tangent to K. Assume that there exists $\alpha \in L^1([0,T], \mathbb{R}^+)$ and a Borel measurable $\beta : \mathbb{R}^+ \to (0, \infty)$ such that $(x, \varphi(t,x)) \leq \alpha(t) \parallel x \parallel \beta(\parallel x \parallel)$, for all $x \in K$ and a.e. $t \in [0,T]$. If*

$$\int_0^T \alpha(t)dt \leq \int_{\|x_0\|}^\infty \frac{ds}{\beta(s)},$$

then the problem (I) *has a solution on* $[0,T]$. *Moreover, if φ is of usc-type, it is sufficient to assume that φ is tangent to K and we get the additional conclusion that the set $S(\varphi, x_0)$ of solutions to* (I) *defined on* $[0,T]$ *is an R_δ-set.*

Proof. The conclusion immediately follows from Theorems 3.1 and 3.2 if we show that (C) is satisfied. Choose $M > 0$ such that

$$\int_0^T \alpha(t)dt < \int_{\|x_0\|}^M \frac{ds}{\beta(s)} \tag{3.1}$$

Let x be a solution of (I). Assume that there is a $t_0 \in (0,T]$ such that $\parallel x(t_0) \parallel \geq M$. Then there is $t_1 < t_0$ such that $\parallel x(t_1) \parallel = \parallel x_0 \parallel < \parallel x(t) \parallel \leq \parallel x(t_0) \parallel$ for all $t \in (t_1, t_0)$. Then

$$\parallel x(t) \parallel' = \frac{(x(t), x'(t))}{\parallel x(t) \parallel} \leq \alpha(t)\beta(\parallel x(t) \parallel) \text{ a.e. } t \in (t_1, t_0).$$

By the change of variable formula, we get

$$\int_{\|x_0\|}^{\|x(t_0)\|} \frac{ds}{\beta(s)} = \int_{t_1}^{t_0} \frac{\parallel x(t)' \parallel dt}{\beta(\parallel x(t) \parallel)} \leq \int_{t_1}^{t_0} \alpha(t)dt,$$

which contradicts (3.1). □

Remark 3.4. If the assumption ibbs is replaced, in any of the above theorems, by the global one : There exists $h \in L^1([0,T], \mathbb{R}^+)$ such that $\parallel \varphi(t,x) \parallel \leq h(t)$ for a.e. t and all $x \in K$; then the condition (C) is automatically fulfilled. This is, for example, the case when K is compact.

4. Implicit differential equations

Given a continuous function $f : [0,T] \times K \times \mathbb{R}^n \to \mathbb{R}^n$, where K is as before a proximate retract, one can define a multivalued function $\sigma : [0,T] \times K \to 2^{\mathbb{R}^n}$, associated with f, by the formula

$$\sigma(t,x) = \{y \in \mathbb{R}^n : y = f(t,x,y)\} = Fixf(t,x,\cdot). \tag{4.1}$$

Evidently, the implicit differential equation $x' = f(t,x,x')$, a.e. t, is equivalent to the differential inclusion $x' \in \sigma(t,x)$, a.e. t. However, this is not yet a big deal, since σ may take empty values, it is almost never of lsc-type and rarely usc-type. We discuss below situations where results of the previous section can still be applied by more delicate arguments. In what follows *ind* stands for the *fixed point index of a map* and *dim* for the *topological (covering) dimension* (c.f. [2] and its references)

Theorem 4.1. *Suppose that f satisfies the following conditions*

(a) *for every (t,x), dim $\sigma(t,x) = 0$ and there exists a bounded open set $U_{t,x}$ such that $\sigma(t,x) \cap \partial U_{t,x} = \emptyset$ and ind$(f(t,x,\cdot), U_{t,x}) \neq 0$;*
(b) *σ is ibbs;*
(c) *σ is strongly tangent to K;*
(d) *there exists $M > \| x_0 \|$ such that every possible solution of* (I′) *must satisfy $\| x(t) \| < M$ for all t;*

Then the problem (I′) *has an absolutely continuous solution on* $[0,T]$.

Proof. Conditions (a) and (b), and the selection theorem in [2] imply that σ has a multivalued lower semicontinuous selection $\varphi, \varphi(t,x) \subset \sigma(t,x)$ for all (t,x), with nonempty compact values. For that φ, any solution of (I) is also a solution of (I′). Therefore the conclusion follows from Theorem 3.1. □

Corollary 4.2. *Suppose that f satisfies the conditions* (a), (b), (c) *and the following*

(d') *there exists $\alpha \in L^1([0,T],\mathbb{R}^+)$ and a Borel measurable $\beta : \mathbb{R}^+ \to (0,\infty)$ such that $\| f(t,x,y) \| \leq \alpha(t)\beta(\| x \|)$, for all (t,x,y) such that $y = f(t,x,y)$.*

If

$$\int_0^T \alpha(t)dt < \int_{\|x_0\|}^{\infty} \frac{ds}{\beta(s)}$$

then the problem (I′) *has an absolutely continuous solution on* $[0,T]$.

Corollary 4.3. *Let $K \subset \mathbb{C} = \mathbb{R}^2$ be a proximate retract and let $a : [0,T] \times K \to \mathbb{C}^{m+1}$ be a continuous function with complex coordinates $a_0, a_1, \ldots a_m$. Suppose* *that there is* $g \in L^1([0,T], \mathbb{R}^+)$ *and a constant* $0 < A < \dfrac{1}{T}$ *such that*

$$\left| \frac{ma_j(t,z)}{a_m(t,z)} \right|^{\frac{1}{m-j}} \leq g(t) + A \mid z \mid, \tag{4.2}$$

for all $j = 0, .1, \ldots, m-1, z \in K,$ *and* $t \in [0,T]$. *Suppose that* $w \in T_K(z)$ *for all* $z \in K$ *and* $w \in \mathbb{C}$ *such that* $\sum_{j=0}^{m} a_j(t,z)w^j = 0$ *for some* $t \in [0,T]$. *Then the problem*

$$\sum_{j=0}^{m} a_j(t, z(t))(z'(t))^j = 0, \ \text{a.e. } t, z(0) = z_0 \in K \tag{4.3}$$

has an absolutely continuous solution $z(t)$ *on* $[0,T]$.

Proof. The problem (4.3) is equivalent to (I′) with $f : [0,T] \times K \times \mathbb{C} \to \mathbb{C}$ defined by $f(t,z,w) = w + \sum_{j=0}^{m} a_j(t,z)w^j$. We verify the conditions (a) to (d) of Theorem 4.1. Condition (a) holds since any complex polynomial has finitely many roots and its topological degree on an open set containing all the roots is equal to its algebraic degree m. Condition (b) follows from (4.2), (d) from (4.2) and the choice of A, and (c) from the last assumption of our corollary. □

Theorem 4.4. *Suppose that* f *satisfies the following conditions*

- (i) $y \to f(t,x,y)$ *is nonexpansive for all* $(t,x) \in [0,T] \times K$ (*i.e.* $\| f(t,x,y_1) - f(t,x,y_2) \| \leq \| y_1 - y_2 \|$)*;*
- (ii) $\sigma(t,x) \neq \emptyset$ *for all* (t,x)*;*
- (iii) σ *is ibbs;*
- (iv) σ *is tangent to* K*;*
- (v) *there exists* $M > \| x_0 \|$ *such that every possible solution of* (I′) *must satisfy* $\| x(t) \| < M$ *for all* t.

Then the set of solutions to (I′) *is a nonempty* R_δ*-set. Moreover if* K *is compact, the Euler characteristic* $\chi(K)$ *is nonzero, and* (v) *is verified by all possible solutions of* (P′)*, then the periodic problem* (P′) *has a solution.*

Proof. Since the set of fixed points of a nonexpansive mapping is convex, σ has nonempty convex values which are bounded due to (iii). By the closed graph property, σ is u.s.c., so the first conclusion follows from Theorem 3.2 applied to $\varphi = \sigma$. The second conclusion follows from Theorem 4.5 in [18]. □

Remarks.

1) As in Theorem 4.5 of [18], an additional conclusion concerning periodic solutions (i.e. $x(t+T) = x(t)$ for all $t \geq 0$) of the implicit equation $x' = f(t,x,x')$, a.e. $t \geq 0$, can be obtained if $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the condition:
   For any $(t,x,y) \in [0,\infty) \times K \times \mathbb{R}^n$, $\ y = f(t,x,y)$ implies $y = f(t+T,x,y)$.

2) The condition (ii) of Theorem 4.4 is verified, for exemple, if f is bounded in y for each given (t, x). For a variety of fixed point theorems on nonexpansive mappings with boundary behaviour conditions implying (ii), we refer to [9].
3) We assumed that f is continuous for the simplicity of the presentation. Our theorems remain true with weaker Carathéodory-type assumptions on f such as those imposed in [10] and [12].

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