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Systems of differential inclusions with maximal monotone terms ☆

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Abstract

In this paper, we establish the existence of solutions to systems of second order differential inclusions with maximal monotone terms. Our proofs rely on the theory of maximal monotone operators and the Schauder degree theory. A notion of solution-tube to these problems is introduced. This notion generalizes the notion of upper and lower solutions of second order differential equations.

Keywords: Systems of differential inclusions; Boundary value problems; Monotone operator; Fixed point theory

1. Introduction

In this paper, we establish existence results for the following system of second order differential inclusions:

$$\begin{aligned} &x''(t) \in B\big(x(t)\big) + f\big(t, x(t), x'(t)\big), \quad \text{a.e. } t \in [0,1], \\ &x \in \mathrm{BC}. \end{aligned} \tag{1.1}$$

Here $B : \mathrm{dom}(B) \subset \mathbb{R}^n \to \mathbb{R}^n$ is a multi-valued maximal monotone operator, $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a Carathéodory function, and BC denotes the periodic or the Sturm–Liouville boundary conditions:

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$$x(0) = x(1), \qquad x'(0) = x'(1); \tag{1.2}$$

$$A_0 x(0) - \beta_0 x'(0) = \theta_0, \qquad A_1 x(1) + \beta_1 x'(1) = \theta_1, \tag{1.3}$$

where for $i = 0, 1$, $\theta_i \in \mathbb{R}^n$, $\beta_i \in \{0, 1\}$, and A_i is an $n \times n$ matrix such that there exists $\alpha_i \geqslant 0$ satisfying $\langle A_i x, x \rangle \geqslant \alpha_i \|x\|^2$, and $\alpha_i + \beta_i > 0$.

In the literature, this problem was studied with topological methods in the case where $B = 0$ for example by Erbe and Palamides [3], Fabry and Habets [4], Frigon [5–7], Frigon and Montoki [8], Gaprindashvili [9], Granas, Guenther and Lee [11], and Mawhin [15].

Here, incorporating a multi-valued maximal monotone operator permits us to consider second order systems with convex potential (see, for example, [16]), as well as nonsmooth potential. A natural example is the case where B is the subdifferential of a convex function. Also, it permits us to include second order variational inequalities which are useful in mechanics and engineering, and which appear also in problems with constraints (see, for example, [1]).

In [13], Halidias and Papageorgiou obtained solutions to this problem under more general boundary conditions and with B a maximal monotone operator. Here, we establish the existence of solutions to this problem in considering more general or different assumptions to theirs but with the boundary condition BC (see also [17]). We assume that B is bounded on bounded sets or B satisfies an appropriate inequality as in [13] (see (H5)).

The proofs rely on degree theory. Therefore, we need to obtain a priori bounds on the solutions of a suitable family of problems. Different conditions are considered to obtain a priori bounds in C^1-norm or in $W^{1,2}$-norm. In particular, we extend the notion of solution-tube introduced in [5] to problem (1.1). This notion generalizes the notion of upper and lower solutions of differential equations. It generalizes also the condition introduced by Hartman [12] for systems of differential equations:

$$\exists M \in]0, \infty[\quad \text{such that} \quad \bigl\langle x, f(t, x, y)\bigr\rangle + \|y\|^2 \geqslant 0 \quad \text{for } \|x\| = M \text{ and } \langle x, y \rangle = 0.$$

Also, when f satisfies a Nagumo–Wintner type growth condition, it is well known that for systems of differential equations an extra assumption is needed in order to bound $\|x'\|$ in L^1-norm. We may assume a condition introduced by Hartman [12] (see (H10)). However, since this extra assumption is not needed in the particular case where our system is in fact a single differential inclusion, it is interesting to find a condition for systems of n differential inclusions which is trivially satisfied when $n = 1$. This is what permits our assumption (H11). It slightly generalizes a condition introduced in [6] (see also [8]).

2. Preliminaries

In what follows, we will use the following notations: $I = [0, 1]$, $C(I, \mathbb{R}^n)$ (respectively $C^1(I, \mathbb{R}^n)$) is the space of continuous (respectively continuously differentiable) functions endowed with the usual norm that we denote $\|\cdot\|_0$ (respectively $\|\cdot\|_1$). For $p \in [1, \infty]$, $L^p(I, \mathbb{R}^n)$ is the space of L^p-integrable functions with the usual norm $\|\cdot\|_{L^p}$; and for $k = 1, 2$, $W^{k,p}(I, \mathbb{R}^n)$ is the Sobolev space $\{x \in C^{k-1}[0, 1]\colon x^{(k-1)}$ is absolutely continuous and $x^{(k)} \in L^p(I, \mathbb{R}^n)\}$ endowed with the usual norm $\|\cdot\|_{k,p}$; $W_B^{k,p}(I, \mathbb{R}^n)$ is the subset of x in $W^{k,p}(I, \mathbb{R}^n)$ satisfying the boundary condition BC.

We say that $f : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, a single-valued map, is L^2-Carathéodory if $t \mapsto f(t, x, p)$ is measurable for all x, p; $(x, p) \mapsto f(t, x, p)$ is continuous for a.e. $t \in I$; for every $k > 0$, there exists $h_k \in L^2(I)$ such that $f(t, B(0, k), B(0, k)) \subset B(0, h_k(t))$ a.e. $t \in I$, where $B(0, r)$ is the closed ball of radius r centered at the origin.

Let H be a Hilbert space and $M : \mathrm{dom}(M) \subset H \to H$ a multi-valued maximal monotone operator. Let us recall that M is a *monotone operator*, if

$$\langle x^* - y^*, x - y\rangle \geqslant 0 \quad \forall x, y \in \mathrm{dom}(M),\ \forall x^* \in M(x),\ \forall y^* \in M(y);$$

and it is *maximal* if

$$\langle x^* - y^*, x - y\rangle \geqslant 0 \quad \forall y \in \mathrm{dom}(M),\ \forall y^* \in M(y) \quad \Longrightarrow \quad x \in \mathrm{dom}(M) \text{ and } x^* \in M(x).$$

This definition implies that M has closed, convex values, and $\mathrm{Gr}(M) := \{(x, x^*)\colon\ x^* \in Mx\}$ is closed in $(H, \mathcal{T}_s) \times (H, \mathcal{T}_w)$ and in $(H, \mathcal{T}_w) \times (H, \mathcal{T}_s)$, where $\mathcal{T}_s$ and $\mathcal{T}_w$ denote respectively the strong and the weak topologies of H.

We define for $\lambda > 0$,

$$J_\lambda = (\mathrm{id} + \lambda M)^{-1} \quad (\text{the } \textit{resolvent} \text{ of } M),$$
$$M_\lambda = \frac{1}{\lambda}(\mathrm{id} - J_\lambda) \quad (\text{the } \textit{Yosida approximation} \text{ of } M).$$

It is well known that $\mathrm{dom}(J_\lambda) = \mathrm{dom}(M_\lambda) = H$, J_λ and M_λ are single-valued, J_λ is non-expansive, M_λ is monotone and Lipschitzian with constant $1/\lambda$, and hence maximal monotone. Moreover, $M_\lambda(x) \in M(J_\lambda(x))$, and $|M_\lambda(x)| \leqslant |y|$ for all $y \in M(x)$.

We recall some results on monotone operators. For their proofs and more information on monotone operators the reader is referred to [2,14] or [18].

Lemma 2.1. *Let $M : \mathrm{dom}(M) \subset H \to H$ be a multi-valued monotone operator. Then the following statements are equivalent*:

(a) *M is maximal*;
(b) $\mathrm{id} + M$ *is surjective.*

Lemma 2.2. *Let $M : \mathrm{dom}(M) \subset H \to H$ and $N : \mathrm{dom}(N) \subset H \to H$ be multi-valued maximal monotone operators such that* $\mathrm{dom}(M) \cap \mathrm{dom}(N) \neq \emptyset$. *Then*

(a) *$M_\lambda + N$ is maximal for every $\lambda > 0$*;
(b) *$y \in \mathrm{Im}(\mathrm{id} + M + N)$ if and only if $\{M_\lambda(x_\lambda)\}$ is bounded as $\lambda \to 0^+$, where $y = (\mathrm{id} + M_\lambda + N)(x_\lambda)$.*

Lemma 2.3. *Let $M : \mathrm{dom}(M) \subset H \to H$ be a multi-valued maximal monotone operator and $N : H \to H$ a single-valued Lipschitzian monotone operator. Then $M + N$ is maximal monotone.*

We can associate to M the operator $\widehat{M} : \mathrm{dom}(\widehat{M}) \subset L^2(I, H) \to L^2(I, H)$ defined by

$$\widehat{M}(x) = \{y \in L^2(I, H)\colon\ y(t) \in M(x(t)) \text{ a.e. } t \in I\},$$

where

$$\mathrm{dom}(\widehat{M}) = \{x \in L^2(I, H)\colon\ x(t) \in \mathrm{dom}(M) \text{ a.e. } t \in I \text{ and } \exists y \in L^2(I, \mathbb{R}^n) \text{ such that } y(t) \in M(x(t)) \text{ a.e. } t \in I\}.$$

The operator $\widehat{M}$ is maximal monotone, as well as $\widehat{M_\lambda}$. In order to simplify the notation, in what follows we will write M (respectively M_λ) instead of $\widehat{M}$ (respectively $\widehat{M_\lambda}$) when there will be no confusion.

3. Existence results

Our goal is to establish existence results for the problem (1.1). By a solution we mean a function $x \in W_B^{2,2}(I, \mathbb{R}^n)$ satisfying (1.1).

We introduce the notion of solution-tube of the problem (1.1). This notion will play a fundamental role in our main result. It extends the notion of solution-tube introduced in [5] for systems of differential equations, which generalizes naturally to systems the well-known notion of upper and lower solutions.

Definition 3.1. Let $v \in W^{2,2}(I, \mathbb{R}^n)$ and $r \in W^{2,2}(I, \mathbb{R})$. We say that (v, r) is a *solution-tube of* (1.1) if there exists $b \in L^2(I, \mathbb{R}^n)$ such that

(i) $b(t) \in Bv(t)$ a.e. $t \in I$;
(ii) for a.e. $t \in I$ and every $(x, p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| = r(t)$ and $\langle x - v(t), p - v'(t)\rangle = r(t)r'(t)$,

$$\bigl\langle x - v(t), f(t, x, p) + b(t) - v''(t)\bigr\rangle + \bigl\|p - v'(t)\bigr\|^2 \geqslant r(t)r''(t) + r'(t)^2;$$

(iii) $v''(t) = b(t) + f(t, v(t), v'(t))$ a.e. on $\{t \in [0, 1]\colon\ r(t) = 0\}$;
(iv) if BC denotes (1.2),

$$r(0) = r(1), \qquad v(0) = v(1), \qquad \bigl\|v'(0) - v'(1)\bigr\| \leqslant r'(1) - r'(0);$$

and if BC denotes (1.3),

$$\bigl\|A_0 v(0) - \beta_0 v'(0) - \theta_0\bigr\| \leqslant \alpha_0 r(0) - \beta_0 r'(0),$$
$$\bigl\|A_1 v(1) + \beta_1 v'(1) - \theta_1\bigr\| \leqslant \alpha_1 r(1) + \beta_1 r'(1).$$

We denote

$$T(v, r) = \bigl\{x \in C\bigl(I, \mathbb{R}^n\bigr)\colon\ \bigl\|x(t) - v(t)\bigr\| \leqslant r(t)\ \forall t \in I\bigr\}.$$

Remark 3.2. If $B \equiv 0$, this definition coincides with the definition of solution-tube introduced in [5] for systems of differential equations.

Remark 3.3. If $\phi : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $B = \partial\phi$ the subdifferential of ϕ, and (v, r) is a solution-tube of (1.1), then

$$\begin{aligned}\phi(x) &+ \bigl\langle x - v(t), f(t, x, p)\bigr\rangle + \bigl\|p - v'(t)\bigr\|^2 \\ &\geqslant \phi\bigl(v(t)\bigr) + \bigl\langle x - v(t), v''(t)\bigr\rangle + r(t)r''(t) + r'(t)^2\end{aligned}$$

for a.e. $t \in I$ and every $(x, p) \in \mathbb{R}^{2n}$ such that

$$\bigl\|x - v(t)\bigr\| = r(t) \quad \text{and} \quad \bigl\langle x - v(t), p - v'(t)\bigr\rangle = r(t)r'(t).$$

In particular, if $f(t,x,p)=f(t,x)$, then

$$\phi(x)+\big\langle x-v(t), f(t,x)\big\rangle \geqslant \phi\big(v(t)\big)+\big\langle x-v(t), v''(t)\big\rangle + r(t)r''(t)$$

for a.e. $t\in I$ and every $x\in\mathbb{R}^n$ such that $\|x-v(t)\|=r(t)$.

Example 3.4. Consider the system

$$\begin{aligned}
&\begin{pmatrix} x'' \\ y'' \end{pmatrix} \in \partial\phi(x,y)+\begin{pmatrix} 3x+a_1(t,x,y)\sin((x-1)^2+y^2)+\cos t \\ 3y+a_2(t,x,y)\sin((x-1)^2+y^2)-\sin t \end{pmatrix},\\
&x(0)=x(1), \qquad x'(0)=x'(1),\\
&y(0)=y(1), \qquad y'(0)=y'(1),
\end{aligned} \tag{3.1}$$

where $\phi(x,y)=\|(x,y)\|$ and a_1, a_2 are Carathéodory functions. It is easy to check that (v,r) is a solution-tube of (3.1) with $v(t)=(1,0)$ and $r(t)=\sqrt{\pi}$.

Our results will rely on some of the following assumptions:

(H1) $f: I\times\mathbb{R}^{2n}\to\mathbb{R}^n$ is L^2-Carathéodory;

(H2) the operator $B:\mathrm{dom}(B)\subset\mathbb{R}^n\to\mathbb{R}^n$ is a multi-valued maximal monotone operator such that $\mathrm{dom}(\widehat{B})\cap W^{2,2}_B(I,\mathbb{R}^n)\neq\emptyset$, i.e., there exist $w\in W^{2,2}_B(I,\mathbb{R}^n)$ and $b_w\in L^2(I,\mathbb{R}^n)$ such that $w(t)\in\mathrm{dom}(B)$ and $b_w(t)\in B(w(t))$ a.e. $t\in I$;

(H3) there exists $(v,r)\in W^{2,2}(I,\mathbb{R}^n)\times W^{2,2}(I,[0,\infty[)$ a solution-tube of (1.1);

(H4) B is bounded on bounded sets;

(H5) $0\in B0$; moreover, if BC denotes (1.3), then

$$\beta_0\big\langle B_\lambda(x), A_0(x)\big\rangle+\beta_1\big\langle B_\lambda(y), A_1(y)\big\rangle\geqslant 0 \quad \forall x,y\in\mathbb{R}^n,\ \forall\lambda>0,$$

where B_λ is the Yosida approximation of B;

(H6) for every $m>0$, $l\in L^2(I,[0,\infty[)$, there exists $R\in L^2(I,[0,\infty[)$ such that for every $x\in\mathbb{R}^n$, $y\in L^2(I,\mathbb{R}^n)$ satisfying $\|x\|\leqslant m$, $\|y(t)\|\leqslant l(t)$ a.e. $t\in I$, one has

$$\big\|f\big(t,x,y(t)\big)\big\|\leqslant R(t) \quad \text{a.e. } t\in I;$$

(H7) there exist $k\geqslant 0$, $\mu\in[0,2[$ (respectively $k\in[0,1[$, $\mu=2$), $h\in L^1(I,[0,\infty[)$, and $z\in W^{2,2}(I,\mathbb{R}^n)\cap\mathrm{dom}(\widehat{B})$ such that

$$\big\langle x-z(t), f(t,x,y)\big\rangle\geqslant -k\big\|y-z'(t)\big\|^\mu-h(t)$$

a.e. $t\in I$ and for all $(x,y)\in\mathbb{R}^{2n}$ satisfying $\|x-v(t)\|\leqslant r(t)$; moreover, $z\in$ BC, or BC denotes (1.3) with $\beta_0=\beta_1=1$;

(H8) there exist $\gamma\in L^1(I,[0,\infty[)$ and $\phi:[0,\infty[\to[1,\infty[$ a Borel measurable function such that

$$\big\|f(t,x,p)\big\|\leqslant\gamma(t)\phi\big(\|p\|\big)$$

a.e. $t\in I$ and for all $(x,p)\in\mathbb{R}^{2n}$ such that $\|x-v(t)\|\leqslant r(t)$, and

$$\int_c^\infty \frac{ds}{\phi(s)}=\infty \quad \forall c\geqslant 0;$$

(H9) there exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I,[0,\infty[)$ such that

$$\left|\langle p, f(t,x,p)\rangle\right| \leqslant \phi\big(\|p\|\big)\big(\gamma(t)+\|p\|\big)$$

a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| \leqslant r(t)$, and

$$\int_c^\infty \frac{s\,ds}{\phi(s)+s} = \infty \quad \forall c \geqslant 0;$$

(H10) there exist $a \geqslant 0$ and $h \in L^1(I,[0,\infty[)$ such that

$$\big\| f(t,x,p)\big\| \leqslant a\big(\langle x, f(t,x,p)\rangle + \|p\|^2\big) + h(t)$$

a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| \leqslant r(t)$;

(H11) there exist $R > 0$, $\delta > 0$, $d \geqslant 0$, and $h \in L^1(I)$ such that

$$\big(\delta + d\|x\|\big)\sigma(t,x,p) + \frac{d\langle x,p\rangle^2}{\|x\|\|p\|} \geqslant \|p\| - h(t),$$

for a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| \leqslant r(t)$, $\|p\| \geqslant R$, where

$$\sigma(t,x,p) = \frac{\langle x, f(t,x,p)\rangle + \|p\|^2}{\|p\|} - \frac{\langle p, f(t,x,p)\rangle\langle x,p\rangle}{\|p\|^3}.$$

Remark 3.5. (1) Observe that (H11) is trivially satisfied in the scalar case ($n = 1$).
(2) Hypothesis (H7) with $z = 0$ becomes

(H7′) $0 \in \operatorname{dom}(B)$ and there exist $k \geqslant 0$, $\mu \in [0,2[$ (respectively $k \in [0,1[$, $\mu = 2$), and $h \in L^1(I,[0,\infty[)$ such that

$$\big\langle x, f(t,x,y)\big\rangle \geqslant -k\|y\|^\mu - h(t)$$

a.e. $t \in I$ and for all $(x,y) \in \mathbb{R}^{2n}$ satisfying $\|x - v(t)\| \leqslant r(t)$; moreover, $0 \in \mathrm{BC}$, or BC denotes (1.3) with $\beta_0 = \beta_1 = 1$.

This condition is more general than hypothesis H(f)(iii) imposed in [13].

The aim of this paper is to establish the following existence results. The first one generalizes Halidias and Papageorgiou's result [13] in the case of the boundary condition BC.

Theorem 3.6. *Assume* (H1)–(H3), (H6), (H7). *Assume also that* (H4) *or* (H5) *is satisfied. Then the problem* (1.1) *has a solution* $x \in W^{2,2}(I,\mathbb{R}^n) \cap T(v,r)$.

We can replace assumptions (H6) and (H7) if we assume that f satisfies some appropriate growth conditions.

Theorem 3.7. *Let* BC *denote* (1.3) *with* $\beta_0 + \beta_1 > 0$. *Assume that* (H1)–(H4) *and* (H8) *are satisfied, then the problem* (1.1) *has a solution* $x \in W^{2,2}(I,\mathbb{R}^n) \cap T(v,r)$.

Observe that if f is quadratic with respect to its last variable, it does not satisfy (H8). In order to permit a growth condition more general than (H8), an extra assumption has to be imposed.

Theorem 3.8. *Assume* (H1)–(H4), (H9). *Assume also that* (H10) *or* (H11) *is satisfied, then the problem* (1.1) *has a solution* $x \in W^{2,2}(I, \mathbb{R}^n) \cap T(v, r)$.

The previous theorem is also true if we replace (H9) by (H8).

Remark 3.9. We could have consider the problem

$$x'' \in \widehat{B}(x) + F(x),\\ x \in \mathrm{BC}, \tag{3.2}$$

where $F(x)(t) = f(t, x(t), x'(t))$ and $\widehat{B}$ (not necessarily defined from B) satisfies

(H2′) the operator $\widehat{B} : \mathrm{dom}(\widehat{B}) \subset L^2(I, \mathbb{R}^n) \to L^2(I, \mathbb{R}^n)$ is a multi-valued maximal monotone operator such that $\mathrm{dom}(\widehat{B}) \cap W_B^{2,2}(I, \mathbb{R}^n) \neq \emptyset$.

In this case, the previous results are true if we replace (H4) by

(H4′) $\widehat{B}$ maps bounded sets of $W^{1,2}(I, \mathbb{R}^n)$ in bounded sets of $L^2(I, \mathbb{R}^n)$.

4. A priori bounds

To prove our existence theorems, we will consider an appropriate family of problems for which we need to establish a priori bounds on the solutions.

Let (v, r) and b be given in (H3) and Definition 3.1. For $\lambda \in [0, 1]$, we define

$$f_\lambda : I \times \mathbb{R}^{2n} \to \mathbb{R}^n \quad \text{by } f_\lambda = \lambda \bar{f} + g_\lambda - (1 - \lambda)v,$$

where

$$\bar{f}(t, x, p) = \begin{cases} \frac{r(t)}{\|x - v(t)\|} f(t, \tilde{x}, \hat{p}) - \tilde{x}, & \text{if } \|x - v(t)\| > r(t), \\ f(t, x, p) - x, & \text{if } \|x - v(t)\| \leqslant r(t); \end{cases}$$

$$g_\lambda(t, x, p) = \begin{cases} \left(1 - \frac{\lambda r(t)}{\|x - v(t)\|}\right)\left(v''(t) - b(t) + \frac{r''(t)}{\|x - v(t)\|}(x - v(t))\right), & \text{if } \|x - v(t)\| > r(t), \\ (1 - \lambda)\left(v''(t) - b(t) + \frac{r''(t)}{r(t)}(x - v(t))\right), & \text{otherwise}; \end{cases}$$

with

$$\tilde{x} = v(t) + \frac{r(t)}{\|x - v(t)\|}\big(x - v(t)\big),$$
$$\hat{p} = p + \left(r'(t) - \frac{\langle x - v(t), p - v'(t)\rangle}{\|x - v(t)\|}\right)\left(\frac{x - v(t)}{\|x - v(t)\|}\right),$$

and where we mean $r''(t)(x - v(t))/r(t) = 0$ on $\{t \in [0, 1]:\ r(t) = 0\}$.

Remark 4.1. For $(x, p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| > 0$,

$$\big\|\tilde{x} - v(t)\big\| = r(t), \qquad \big\langle \tilde{x} - v(t), \hat{p} - v'(t)\big\rangle = r(t)r'(t),$$
$$\big\|\hat{p} - v'(t)\big\|^2 = \big\|p - v'(t)\big\|^2 + \big(r'(t)\big)^2 - \frac{\langle x(t) - v(t), p - v'(t)\rangle^2}{\|x(t) - v(t)\|^2};$$

and for $x \in C^1(I, \mathbb{R}^n)$,

$$\big(\widetilde{x(t)}, \widehat{x'(t)}\big) = \big(x(t), x'(t)\big) \quad \text{a.e. on } \big\{t \in I\colon \ \|x(t) - v(t)\| = r(t) > 0\big\}.$$

We consider for $\lambda \in [0, 1]$, the problem

$$\begin{aligned} &x''(t) - x(t) \in B\big(x(t)\big) + f_\lambda\big(t, x(t), x'(t)\big) \quad \text{a.e. } t \in [0, 1], \\ &x \in \mathrm{BC}. \end{aligned} \tag{P_λ}$$

A priori bounds can be obtained for the solutions of (P_λ).

Proposition 4.2. *Assume that* (H1)–(H3) *are satisfied. Then every solution $x \in W^{2,2}(I, \mathbb{R}^n)$ of (P_λ) with $\lambda \in [0, 1]$, is such that $x \in T(v, r)$.*

Proof. For $\lambda \in [0, 1]$, let x be a solution of (P_λ). There exists $b_x \in L^2(I, \mathbb{R}^n)$ such that $b_x(t) \in B(x(t))$ and $x''(t) - x(t) = b_x(t) + f_\lambda(t, x(t), x'(t))$ a.e. $t \in I$. Since B is a maximal monotone operator, by (H3) and Remark 4.1, we deduce that for almost every $t \in E := \{t \in I\colon \ \|x(t) - v(t)\| > r(t)\}$,

$$\begin{aligned}
&\frac{1}{\|x(t) - v(t)\|}\big(\big\langle x(t) - v(t), x''(t) - v''(t)\big\rangle + \big\|x'(t) - v'(t)\big\|^2\big) \\
&\quad - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} - \big\|x(t) - v(t)\big\| \\
&= \frac{1}{\|x(t) - v(t)\|}\bigg(\bigg\langle x(t) - v(t), \frac{\lambda r(t)}{\|x(t) - v(t)\|}\big(f\big(t, \tilde{x}(t), \hat{x}'(t)\big) + b(t) - v''(t)\big)\bigg\rangle \\
&\quad - \big\langle x(t) - v(t), \lambda\tilde{x}(t) + (1 - \lambda)v(t) - x(t)\big\rangle\bigg) \\
&\quad + \bigg(1 - \frac{\lambda r(t)}{\|x(t) - v(t)\|}\bigg)r''(t) + \frac{\langle x(t) - v(t), b_x(t) - b(t)\rangle}{\|x(t) - v(t)\|} \\
&\quad + \frac{\|x'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} - \big\|x(t) - v(t)\big\| \\
&\geqslant \frac{\lambda}{\|x(t) - v(t)\|}\big(\big\langle \tilde{x} - v(t), f\big(t, \tilde{x}(t), \hat{x}'(t)\big) + b(t) - v''(t)\big\rangle + \big\|\hat{x}'(t) - v'(t)\big\|^2\big) \\
&\quad + \frac{\|x'(t) - v'(t)\|^2 - \lambda\|\hat{x}'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} \\
&\quad - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} + \bigg(1 - \frac{\lambda r(t)}{\|x(t) - v(t)\|}\bigg)r''(t) - \lambda r(t) \\
&\geqslant \frac{\lambda}{\|x(t) - v(t)\|}\big(r(t)r''(t) + r'(t)^2\big) + \bigg(1 - \frac{\lambda r(t)}{\|x(t) - v(t)\|}\bigg)r''(t) \\
&\quad + \frac{1}{\|x(t) - v(t)\|}\big((1 - \lambda)\big\|\hat{x}'(t) - v'(t)\big\|^2 - r'(t)^2\big) - \lambda r(t) \\
&\geqslant r''(t) - r(t).
\end{aligned}$$

So, $w''(t) - w(t) \geqslant 0$ a.e. $t \in E$, where $w(t) := \|x(t) - v(t)\| - r(t)$. This implies that for all $]t_0, t_1[\subset E$,

$$0 \leqslant \int_{t_0}^{t_1} w''(t) - w(t)\, dt \leqslant w'(t_1) - w'(t_0). \tag{4.1}$$

Observe that if

$$\begin{aligned} w(t_0) = 0 \quad &\Longrightarrow \quad w'(t_0) \geqslant 0, \\ w(t_1) = 0 \quad &\Longrightarrow \quad w'(t_1) \leqslant 0. \end{aligned} \tag{4.2}$$

On the other hand, in the case where BC denotes (1.2), by (H3) we have

$$\begin{aligned} &\frac{d}{dt}\|x(t) - v(t)\|\bigg|_{t=1} - \frac{d}{dt}\|x(t) - v(t)\|\bigg|_{t=0} \\ &\quad = \frac{\langle x(1) - v(1), x'(1) - v'(1)\rangle}{\|x(1) - v(1)\|} - \frac{\langle x(0) - v(0), x'(0) - v'(0)\rangle}{\|x(0) - v(0)\|} \\ &\quad = \frac{\langle x(0) - v(0), v'(0) - v'(1)\rangle}{\|x(0) - v(0)\|} \\ &\quad \leqslant \big\|v'(1) - v'(0)\big\| \\ &\quad \leqslant r'(1) - r'(0). \end{aligned}$$

So,

$$w(0) = w(1) \quad \text{and} \quad w'(1) \leqslant w'(0). \tag{4.3}$$

In the case where BC denotes (1.3), again by (H3),

$$\begin{aligned} &\alpha_0\big\|x(0) - v(0)\big\| - \beta_0\frac{d}{dt}\big\|x(t) - v(t)\big\|\bigg|_{t=0} \\ &\quad = \alpha_0\big\|x(0) - v(0)\big\| - \frac{\beta_0\langle x(0) - v(0), x'(0) - v'(0)\rangle}{\|x(0) - v(0)\|} \\ &\quad \leqslant \frac{\langle x(0) - v(0), A_0(x(0) - v(0)) - \beta_0(x'(0) - v'(0))\rangle}{\|x(0) - v(0)\|} \\ &\quad = -\frac{\langle x(0) - v(0), A_0 v(0) - \beta_0 v'(0) - \theta_0\rangle}{\|x(0) - v(0)\|} \\ &\quad \leqslant \alpha_0 r(0) - \beta_0 r'(0). \end{aligned}$$

Similarly for $t = 1$, and hence

$$\alpha_0 w(0) - \beta_0 w'(0) \leqslant 0 \quad \text{and} \quad \alpha_1 w(1) + \beta_1 w'(1) \leqslant 0. \tag{4.4}$$

The conclusions follows from (4.1)–(4.4). $\square$

Now, we want to obtain a priori bounds on the derivative of the solution to (P_λ) for $\lambda \in [0, 1]$. We first establish an a priori bound in L^2-norm.

Proposition 4.3. *Assume* (H1)–(H3) *and* (H7) *are satisfied. Then there exists $K > 0$ such that $\|x'\|_{L^2} < K$ for all x solution to (P_λ) for $\lambda \in [0, 1]$.*

Proof. Let x be a solution to (P_λ) then there exits $b_x \in L^2(I, \mathbb{R}^n)$ such that $b_x(t) \in B(x(t))$ and $x''(t) - x(t) = b_x(t) + f_\lambda(t, x(t), x'(t))$ a.e. $t \in I$. From Proposition 4.2, $\|x(t) - v(t)\| \leqslant r(t)$ for all $t \in I$. Let z be given in (H7) and fix $b_z \in L^2(I, \mathbb{R}^n)$ such that $b_z(t) \in Bz(t)$ a.e. $t \in I$. By (H7) and since B is a maximal operator,

$$\begin{aligned}
&\bigl\langle x(t) - z(t), x''(t) - z''(t)\bigr\rangle \\
&\quad = \bigl\langle x(t) - z(t), b_x(t) - b_z(t) + \lambda f\bigl(t, x(t), x'(t)\bigr) - z''(t) + b_z(t)\bigr\rangle \\
&\qquad + (1-\lambda)\left\langle x(t) - z(t), v''(t) - b(t) + \left(1 + \frac{r''(t)}{r(t)}\right)\bigl(x(t) - v(t)\bigr)\right\rangle \\
&\quad \geqslant -k\bigl\|x'(t) - z'(t)\bigr\|^\mu - h(t) \\
&\qquad - \bigl\|x(t) - z(t)\bigr\|\bigl(r(t) + \bigl|r''(t)\bigr| + \bigl|v''(t)\bigr| + \bigl|b(t)\bigr| + \bigl|b_z(t)\bigr| + \bigl|z''(t)\bigr|\bigr) \\
&\quad = -k\bigl\|x'(t) - z'(t)\bigr\|^\mu - h_0(t),
\end{aligned}$$

where $h_0 \in L^1(I, \mathbb{R}^n)$. So,

$$\begin{aligned}
&\int_I \bigl\|x'(t) - z'(t)\bigr\|^2 - k\bigl\|x'(t) - z'(t)\bigr\|^\mu \, dt \\
&\quad \leqslant \|h_0\|_{L^1} + \bigl\langle x(1) - z(1), x'(1) - z'(1)\bigr\rangle - \bigl\langle x(0) - z(0), x'(0) - z'(0)\bigr\rangle.
\end{aligned}$$

Observe that if BC denotes (1.2), $z \in \mathrm{BC}$ and

$$\bigl\langle x(1) - z(1), x'(1) - z'(1)\bigr\rangle - \bigl\langle x(0) - z(0), x'(0) - z'(0)\bigr\rangle = 0.$$

On the other hand, in the case where BC denotes (1.3), if $\beta_0 = 0$, $z \in \mathrm{BC}$ and $x(0) = z(0) = A_0^{-1}\theta_0$. If $\beta_0 = 1$,

$$\bigl|\bigl\langle x(0) - z(0), x'(0) - z'(0)\bigr\rangle\bigr| = \bigl|\bigl\langle x(0) - z(0), A_0 x(0) - \theta_0 - z'(0)\bigr\rangle\bigr| \leqslant c_0$$

for some constant c_0 not depending on x. Similarly at $t = 1$, and hence there is a constant c_1 not depending on x such that

$$\bigl|\bigl\langle x(1) - z(1), x'(1) - z'(1)\bigr\rangle - \bigl\langle x(0) - z(0), x'(0) - z'(0)\bigr\rangle\bigr| \leqslant c_1.$$

This concludes the proof. $\square$

The remaining part of this section concerns the existence of a priori bounds in the norm of the uniform convergence of the derivative of solutions to (P_λ).

Proposition 4.4. *Let* BC *denote* (1.3) *with* $\beta_0 + \beta_1 > 0$. *Under* (H1)–(H4) *and* (H8), *there exists* $K > 0$ *such that* $\|x'\|_0 < K$ *for all x solution to* (P_λ) *for* $\lambda \in [0, 1]$.

Proof. The assumption (H4) implies that there exists $C \geqslant 0$ such that $\|B(x(t))\| \leqslant C$ for $t \in I$ for all $x \in T(v, r)$. Also, for all $x \in T(v, r)$,

$$\min\bigl\{\bigl\|x'(0)\bigr\|, \bigl\|x'(1)\bigr\|\bigr\} \leqslant c := \min\left\{\frac{1}{\beta_0}\max_{y \in E_0}\bigl\{\|A_0 y\|\bigr\}, \frac{1}{\beta_1}\max_{y \in E_1}\bigl\{\|A_1 y\|\bigr\}\right\},$$

where $E_i = \{y \in \mathbb{R}^n\colon \|y - v(i)\| \leqslant r(i)\}$, $i = 0, 1$.

So, for x a solution to (P_λ) and $b_x \in L^2(I,\mathbb{R}^n)$ such that $b_x(t) \in B(x(t))$ and $x''(t) - x(t) = b_x(t) + f_\lambda(t,x(t),x'(t))$ a.e. $t \in I$, we have that $x \in T(v,r)$ by Proposition 4.2. Therefore, a.e. $t \in \{t \in I\colon \|x'(t)\| > 0\}$,

$$\left|\frac{\langle x'(t), x''(t)\rangle}{\|x'(t)\|}\right| \leqslant \|b_x(t)\| + \|f(t,x(t),x'(t))\| + r(t) + \|v''(t)\| + \|b(t)\| + |r''(t)| \leqslant \gamma(t)\phi(\|x(t)\|) + \gamma_0(t),$$

with

$$\gamma_0(t) = 2C + r(t) + \|v''(t)\| + \|r''(t)\|. \tag{4.5}$$

It follows that for all $t_0, t_1 \in I$ such that $\|x'(t_0)\| = c$ and $\|x'(t)\| > c$ for all t between t_0 and t_1,

$$\int_c^{\|x'(t_1)\|} \frac{ds}{\phi(s)} = \left|\int_{t_0}^{t_1} \frac{\|x'(t)\|'}{\phi(\|x'(t)\|)}\,dt\right| \leqslant \|\gamma + \gamma_0\|_{L^1}.$$

We conclude in choosing K such that

$$\int_c^K \frac{ds}{\phi(s)} > \|\gamma + \gamma_0\|_{L^1}. \qquad \square$$

In order to obtain a priori bounds on the derivative of the solutions under the Wintner–Nagumo growth condition (H9), we recall the following results of [6].

Lemma 4.5. [6, Lemma 3.4] *Let $c, \kappa \geqslant 0$, $l \in L^1(I)$, and $\psi : [0,\infty[\to]0,\infty[$ a Borel measurable function such that*

$$\int_c^\infty \frac{s\,ds}{\psi(s)} > \|l\|_{L^1} + \kappa.$$

Then there exists $K > 0$ such that $\|x'\|_0 < K$ for every $x \in W^{2,1}(I,\mathbb{R}^n)$ satisfying

(i) $\min_{t\in I} \|x'(t)\| \leqslant c$;
(ii) $\|x'\|_{L^1[t_0,t_1]} \leqslant \kappa$ *if* $\|x'(t)\| \geqslant c$ *on* $[t_0,t_1]$;
(iii) $|\langle x'(t), x''(t)\rangle| \leqslant \psi(\|x'(t)\|)(l(t) + \|x'(t)\|)$ *a.e. on* $\{t \in I\colon \|x'(t)\| \geqslant c\}$.

To apply the previous result in order to obtain a priori bounds of the derivative x' with respect to norm of the uniform convergence, we need to obtain a priori bounds of x' in the L^1-norm. The two following results give sufficient conditions to ensure that. The first one relies on a condition introduced by Hartman [12], while the second one generalizes and simplifies [6, Lemma 3.3] (see also [8]).

Lemma 4.6. *Let $c \geqslant 0$ and $k \in L^1(I)$. Then there exists $\rho : [0,\infty[\to]0,\infty[$ an increasing function such that for every $x \in W^{2,1}(I,\mathbb{R}^n)$ satisfying*

$$\|x''(t)\| \leqslant c\big(\langle x(t), x''(t)\rangle + \|x'(t)\|^2\big) + k(t) \quad \text{a.e. } t \in I,$$

we have $\|x'\|_{L^1} \leqslant \rho(\|x\|_0)$.

Lemma 4.7. *Let $k > 0$, $\kappa \geqslant 0$, $c > 0$, $m \in L^1(I)$. Then there exists $\rho : [0, \infty[\to [0, \infty[$ an increasing function such that we have for any interval $[t_0, t_1]$ on which $\|x'(t)\| \geqslant c$,*

$$\|x'\|_{L^1[t_0,t_1]} \leqslant \rho(\|x\|_0),$$

and

$$\min_{t\in I}\|x'(t)\| \leqslant \max\{c, \rho(\|x\|_0)\},$$

for every $x \in W^{2,1}(I, \mathbb{R}^n)$ satisfying almost everywhere on $\{t \in I: \|x'(t)\| \geqslant c\}$,

$$(k + \kappa\|x(t)\|)\varsigma(t, x) + \frac{\kappa\langle x(t), x'(t)\rangle^2}{\|x(t)\|\,\|x'(t)\|} \geqslant \|x'(t)\| - m(t),$$

where

$$\varsigma(t, x) = \frac{\langle x(t), x''(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), x''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}.$$

We are now ready to obtain a priori bounds for $\|x'\|_0$ with x solution to (P_λ) under the Wintner–Nagumo growth condition (H9).

Proposition 4.8. *Assume* (H1)–(H4), (H9), *and* (H10) *or* (H11). *Then there exists $K > 0$ such that every solution x to (P_λ) satisfies $\|x'\|_0 < K$.*

Proof. By (H4), there exists $C \geqslant 0$ such that $\|B(x(t))\| \leqslant C$ for $t \in I$ and for all $x \in T(v, r)$. Let x be a solution of (P_λ) and $b_x \in L^2(I, \mathbb{R}^n)$ such that $b_x(t) \in B(x(t))$ and $x''(t) - x(t) = b_x(t) + f_\lambda(t, x(t), x'(t))$ a.e. $t \in I$. From Proposition 4.2, we know that $x \in T(v, r)$. We have by (H9) that

$$\begin{aligned}
|\langle x'(t), x''(t)\rangle| &= |\langle x'(t), f_\lambda(t, x(t), x'(t)) + x(t) + b_x(t)\rangle| \\
&\leqslant \lambda|\langle x'(t), f(t, x(t), x'(t))\rangle| \\
&\quad + \|x'(t)\|(r(t) + \|v''(t)\| + |r''(t)| + \|b_x(t)\| + \|b(t)\|) \\
&\leqslant \phi(\|x'(t)\|)(\gamma(t) + \|x'(t)\|) + \gamma_0(t)\|x'(t)\|,
\end{aligned}$$

where γ_0 is defined in (4.5). So,

$$|\langle x'(t), x''(t)\rangle| \leqslant (\phi(\|x'(t)\|) + \|x'(t)\|)(\gamma(t) + \gamma_0(t) + \|x'(t)\|). \tag{4.6}$$

Now, to verify assumptions (i) and (ii) of Lemma 4.5, we consider two cases.

Case 1: (H10) is satisfied. We have

$$\begin{aligned}
\|x''(t)\| &= \|f_\lambda(t, x(t), x'(t)) + x(t) + b_x(t)\| \\
&\leqslant \lambda\|f(t, x(t), x'(t))\| + r(t) + \|v''(t)\| + |r''(t)| + \|b(t)\| + \|b_x(t)\| \\
&\leqslant a\lambda(\langle x(t), f(t, x(t), x'(t))\rangle + \|x'(t)\|^2) + h(t) + \gamma_0(t) \\
&\leqslant a(\langle x(t), f_\lambda(t, x(t), x'(t)) + x(t) + b_x(t)\rangle + \|x'(t)\|^2) + h(t) + \gamma_0(t) \\
&\quad - a\left\langle x(t), b_x(t) + (1-\lambda)\left(v''(t) + b(t) + \left(\frac{r''(t)}{r(t)} + 1\right)(x(t) - v(t))\right)\right\rangle \\
&\leqslant a(\langle x(t), x''(t)\rangle + \|x'(t)\|^2) + h_1(t),
\end{aligned}$$

with

$$h_1(t) = h(t) + \gamma_0(t)\big(1 + a\big(r(t) + \|v(t)\|\big)\big).$$

This inequality with (4.6), and Lemmas 4.5 and 4.6 lead to the conclusion.

Case 2: (H11) is satisfied. Let ς be the function introduced in Lemma 4.7. Observe that

$$\begin{aligned}
\varsigma(t,x) &= \frac{\langle x(t), x''(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), x''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3} \\
&= \lambda\left(\frac{\langle x(t), f(t,x(t),x'(t))\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), f(t,x(t),x'(t))\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\right) \\
&\quad + (1-\lambda)\Bigg[\|x'(t)\| + \frac{\langle x(t), v''(t) - b(t)\rangle}{\|x'(t)\|} - \frac{\langle x'(t), v''(t) - b(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3} \\
&\quad + \left(1 + \frac{r''(t)}{r(t)}\right)\left(\frac{\langle x(t), x(t) - v(t)\rangle}{\|x'(t)\|} - \frac{\langle x'(t), x(t) - v(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\right)\Bigg] \\
&\quad + \frac{\langle x(t), b_x(t)\rangle}{\|x'(t)\|} - \frac{\langle x'(t), b_x(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3} \\
&\geqslant \lambda\left(\frac{\langle x(t), f(t,x(t),x'(t))\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), f(t,x(t),x'(t))\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\right) \\
&\quad + (1-\lambda)\|x'(t)\| - 2\frac{\|x(t)\|\gamma_0(t)}{\|x'(t)\|}.
\end{aligned}$$

It follows from (H11) that on $\{t \in [0,1]\colon \|x'(t)\| \geqslant R\}$,

$$\begin{aligned}
&\big(\delta + d\|x(t)\|\big)\varsigma(t,x) + \frac{d\langle x(t), x'(t)\rangle^2}{\|x(t)\|\|x'(t)\|} \\
&\quad \geqslant \lambda\big(\|x'(t)\| - h(t)\big) + \delta(1-\lambda)\|x'(t)\| - h_2(t),
\end{aligned} \tag{4.7}$$

with

$$h_2(t) = \frac{2}{R}\big(\delta + d\big(r(t) + \|v(t)\|\big)\big)\big(r(t) + \|v(t)\|\big)\gamma_0(t).$$

The conclusion follows from (4.6), (4.7) and Lemmas 4.5 and 4.7. □

5. Operators

We associate to f_λ an operator defined by

$$F(\lambda, x)(t) = -f_\lambda\big(t, x(t), x'(t)\big).$$

The following result establishes some properties of F.

Proposition 5.1. *Assume* (H1)–(H3).

(a) *The operator $F: I \times C^1(I,\mathbb{R}^n) \to L^2(I,\mathbb{R}^n)$ is continuous and integrably bounded on bounded sets of $C^1(I,\mathbb{R}^n)$; that is for every bounded set V in $C^1(I,\mathbb{R}^n)$, there exists $k \in L^2(I,\mathbb{R})$ such that $\|F(\lambda,y)(t)\| \leqslant k(t)$ a.e. $t \in I$ for all $y \in V$ and all $\lambda \in [0,1]$.*

(b) *In addition, if* (H6) *is satisfied then* $F : I \times W^{1,2}(I, \mathbb{R}^n) \to L^2(I, \mathbb{R}^n)$ *is continuous and integrably bounded on bounded sets of* $W^{1,2}(I, \mathbb{R}^n)$.

Proof. (a) The reader is referred to [6, Proposition 3.5].

(b) It follows from (H6) that F is integrably bounded on bounded sets of $W^{1,2}(I, \mathbb{R}^n)$.

Assume that there exists a sequence $\{x_n\}$ converging to x in $W^{1,2}(I, \mathbb{R}^n)$, and a sequence $\{\lambda_n\}$ converging to λ such that $F(\lambda_n, x_n) \nrightarrow F(\lambda, x)$ in $L^2(I, \mathbb{R}^n)$. So, there exists $\delta > 0$ and sequences $\{x_{n_k}\}$ and $\{\lambda_{n_k}\}$ such that

$$\left\| F(\lambda_{n_k}, x_{n_k}) - F(\lambda, x) \right\|_{L^2} > \delta.$$

Since $\{x_{n_k}\}$ converge to x in $W^{1,2}(I, \mathbb{R}^n)$, there exists a subsequence still denoted $\{x_{n_k}\}$ such that

$$x_{n_k}(t) \to x(t) \quad \text{and} \quad x'_{n_k}(t) \to x'(t) \quad \text{a.e. } t \in I.$$

Observe that

$$\big\langle x(t) - v(t), x'(t) - v'(t) \big\rangle = r(t) r'(t) \quad \text{a.e. on } \big\{ t\colon \ \| x(t) - v(t) \| = r(t) > 0 \big\},$$

and

$$x(t) = v(t), \quad x'(t) = v'(t), \quad r'(t) = 0, \quad r''(t) = 0$$
$$\text{a.e. on } \big\{ t\colon \ \| x(t) - v(t) \| = r(t) = 0 \big\}.$$

So,

$$\widehat{x'_{n_k}(t)} \to \widehat{x'(t)} \quad \text{a.e. } t \in I,$$

and hence,

$$f_{\lambda_{n_k}}\big(t, x_{n_k}(t), x'_{n_k}(t)\big) \to f_\lambda\big(t, x(t), x'(t)\big) \quad \text{a.e. } t \in I,$$

which is a contradiction. $\square$

Let us define $L : W_B^{2,2}(I, \mathbb{R}^n) \subset L^2(I, \mathbb{R}^n) \to L^2(I, \mathbb{R}^n)$ by $Lx = -x''$ and denote $M : \mathrm{dom}(M) \subset L^2(I, \mathbb{R}^n) \to L^2(I, \mathbb{R}^n)$ by

$$M = L + \widehat{B}.$$

Obviously, $\mathrm{dom}(M) = W_B^{2,2}(I, \mathbb{R}^n) \cap \mathrm{dom}(\widehat{B})$.

Proposition 5.2. *Under* (H2), (H4)*, M is a multi-valued maximal monotone operator.*

Proof. Let us show that M is monotone. First of all observe that $\mathrm{dom}(M) \neq \emptyset$. It is sufficient to show that L is monotone. Take $x, y \in W_B^{2,2}(I, \mathbb{R}^n)$. We have

$$\begin{aligned}
\langle -x'' + y'', x - y \rangle_{L^2} &= \int_I \big\langle y''(t) - x''(t), x(t) - y(t) \big\rangle \, dt \\
&= \big\langle x(0) - y(0), x'(0) - y'(0) \big\rangle - \big\langle x(1) - y(1), x'(1) - y'(1) \big\rangle \\
&\quad + \int_I \big\| x'(t) - y'(t) \big\|^2 \, dt \\
&\geqslant \big\langle x(0) - y(0), x'(0) - y'(0) \big\rangle - \big\langle x(1) - y(1), x'(1) - y'(1) \big\rangle.
\end{aligned}$$

Therefore, if BC denotes (1.2),

$$\langle -x'' + y'', x - y\rangle_{L^2} \geqslant 0.$$

On the other hand, if BC denotes (1.3), if $\beta_0 = 0$ then $\alpha_0 > 0$ and hence A_0 is invertible and $x(0) = y(0) = A_0^{-1}\theta_0$. So,

$$\big\langle x(0) - y(0), x'(0) - y'(0)\big\rangle = 0.$$

If $\beta_0 = 1$,

$$\begin{aligned}\big\langle x(0) - y(0), x'(0) - y'(0)\big\rangle &= \frac{1}{\beta_0}\big\langle x(0) - y(0), A_0\big(x(0) - y(0)\big)\big\rangle \\ &\geqslant \frac{\alpha_0}{\beta_0}\big\|x(0) - y(0)\big\|^2 \geqslant 0.\end{aligned}$$

Similarly,

$$\big\langle x(1) - y(1), x'(1) - y'(1)\big\rangle \leqslant 0.$$

So,

$$\langle -x'' + y'', x - y\rangle_{L^2} \geqslant 0,$$

and hence L and M are monotone.

Now, we have to show that M is maximal. By Lemma 2.1, we have to show that $\mathrm{id} + M$ is surjective. It is well known that $\mathrm{id} + L$ is invertible and hence surjective. By Lemma 2.1, L is maximal monotone. Since for $\lambda > 0$, B_λ is single-valued, monotone and Lipschitzian, $L + B_\lambda$ is maximal monotone, and hence $\mathrm{id} + L + B_\lambda$ is surjective by Lemmas 2.1 and 2.3. So, for $h \in L^2(I, \mathbb{R}^n)$, there exists $x_\lambda \in W_B^{2,2}(I, \mathbb{R}^n)$ such that

$$(\mathrm{id} + L + B_\lambda)x_\lambda = h. \tag{5.1}$$

Let w be the function given in (H2), and denote

$$h_{\lambda,w} := (\mathrm{id} + L + B_\lambda)w.$$

Using the fact that $x_\lambda \in \mathrm{BC}$, and B_λ is monotone, we have that

$$\begin{aligned}&\int_I \big\langle h(t) - h_{\lambda,w}(t), x_\lambda(t) - w(t)\big\rangle\,dt \\ &\quad= \int_I \big\langle x_\lambda(t) - w(t) - \big(x_\lambda''(t) - w''(t)\big) + B_\lambda\big(x_\lambda(t)\big) - B_\lambda\big(w(t)\big), x_\lambda(t) - w(t)\big\rangle\,dt \\ &\quad= \|x_\lambda - w\|_{1,2}^2 + \big\langle x_\lambda(0) - w(0), x_\lambda'(0) - w'(0)\big\rangle - \big\langle x_\lambda(1) - w(1), x_\lambda'(1) - w'(1)\big\rangle \\ &\qquad+ \int_I \big\langle B_\lambda\big(x_\lambda(t)\big) - B_\lambda\big(w(t)\big), x_\lambda(t) - w(t)\big\rangle\,dt \\ &\quad\geqslant \|x_\lambda - w\|_{1,2}^2.\end{aligned}$$

So, $\{x_\lambda\}$ is bounded in $W^{1,2}(I, \mathbb{R}^n)$ as $\lambda \to 0^+$, and hence in $C(I, \mathbb{R}^n)$ by a constant c. Since B is bounded on bounded sets,

$$\left\|B_\lambda\big(x_\lambda(t)\big)\right\| \leqslant \inf\big\{\|y\|\colon\ y \in B(z) \text{ with } \|z\| \leqslant c\big\}.$$

So $\{B_\lambda(x_\lambda)\}$ is bounded in $L^2(I, \mathbb{R}^n)$. If follows from Lemma 2.2 that M is maximal. $\square$

When M is maximal monotone, $\mathrm{id} + M$ is surjective and invertible, so we denote for $x \in L^2(I, \mathbb{R}^n)$,

$$S(x) := (\mathrm{id} + M)^{-1}(x) \in W_B^{2,2}\big(I, \mathbb{R}^n\big).$$

Proposition 5.3. *Under* (H2) *and* (H4), *the operator* $S\colon L^2(I,\mathbb{R}^n) \to W^{2,2}(I,\mathbb{R}^n)$, *where* $W^{2,2}(I,\mathbb{R}^n)$ *is endowed with the topology of* $C^1(I,\mathbb{R}^n)$ *is continuous and completely continuous.*

Proof. Since $L^2(I,\mathbb{R}^n)$ is a Hilbert space, it is sufficient to show that if $x_n \rightharpoonup x$ weakly in $L^2(I,\mathbb{R}^n)$, $y_n = S(x_n) \to y = S(x)$ in $C^1(I,\mathbb{R}^n)$. There exists $u_n \in B(y_n)$ such that $x_n = y_n - y_n'' + u_n$. Let w, b_w be given in (H2) and denote $h_w := w - w'' + b_w$. So, using the boundary condition and the fact that B is monotone, we obtain

$$\begin{aligned}\langle y_n - w, x_n - h_w\rangle_{L^2} &= \|y_n - w\|_{1,2}^2 + \big\langle y_n(0) - w(0), y_n'(0) - w'(0)\big\rangle \\ &\quad - \big\langle y_n(1) - w(1), y_n'(1) - w'(1)\big\rangle + \langle y_n - w, u_n - b_w\rangle_{L^2} \\ &\geqslant \|y_n - w\|_{1,2}^2.\end{aligned}$$

It follows that $\{y_n\}$ is bounded in $W^{1,2}(I,\mathbb{R}^n)$. The compactness of the inclusion $W^{1,2}(I,\mathbb{R}^n) \hookrightarrow L^2(I,\mathbb{R}^n)$ implies that up to a subsequence still denoted $\{y_n\}$, $y_n \to y$ weakly in $W^{1,2}(I,\mathbb{R}^n)$ and strongly in $L^2(I,\mathbb{R}^n)$. We know that since $\mathrm{id} + M$ is maximal monotone $\mathrm{Gr}(\mathrm{id} + M)$ is closed in $(L^2(I,\mathbb{R}^n), \mathcal{T}_s) \times (L^2(I,\mathbb{R}^n), \mathcal{T}_w)$. So, $(y, x) \in \mathrm{Gr}(\mathrm{id} + M)$; i.e., $y = S(x)$.

Now, we want to show that $y_n \to y$ in $C^1(I,\mathbb{R}^n)$. We deduce that $\{u_n\}$ is bounded in $L^2(I,\mathbb{R}^n)$ from (H4). Therefore, $\{y_n\}$ is bounded in $W^{2,2}(I,\mathbb{R}^n)$. So, there are subsequence still denoted $\{y_n\}$ and $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in $L^2(I,\mathbb{R}^n)$, and $y_n \to y$ weakly in $W^{2,2}(I,\mathbb{R}^n)$ and strongly in $C^1(I,\mathbb{R}^n)$. Since B is maximal monotone, we deduce that $(y, u) \in \mathrm{Gr}(B)$ which is closed in $(L^2(I,\mathbb{R}^n), \mathcal{T}_s) \times (L^2(I,\mathbb{R}^n), \mathcal{T}_w)$. It follows that $y_n = S(x_n) \to y = S(x)$ strongly in $C^1(I,\mathbb{R}^n)$. $\square$

Remark 5.4. This result is true if we consider the problem (3.2) with $\widehat{B}$ satisfying (H2′) and (H4′).

If we do not assume that B is bounded on bounded sets, it can be shown that M is maximal monotone under an extra assumption. The reader is referred to [13, Propositions 2 and 3] for the proof (see also [17]).

Proposition 5.5. *Under* (H2), (H5), *M is a multi-valued maximal monotone operator. Moreover, the operator* $S\colon L^2(I,\mathbb{R}^n) \to W^{2,2}(I,\mathbb{R}^n)$, *where* $W^{2,2}(I,\mathbb{R}^n)$ *is endowed with the topology of* $W^{1,2}(I,\mathbb{R}^n)$ *is continuous and completely continuous.*

6. Proofs of existence results

In order to prove our main theorem, we first establish the following general result.

Theorem 6.1. *Assume* (H1)–(H3). *Assume also that one of the following conditions is satisfied*:

(i) (H4) *is satisfied and there exists a constant* $K > 0$ *such that every solution* x *of* (P_λ) *with* $\lambda \in [0,1]$, *satisfies* $\|x'\|_0 < K$;
(ii) (H6) *holds and there exists a constant* $K > 0$ *such that every solution* x *of* (P_λ) *with* $\lambda \in [0,1]$, *satisfies* $\|x'\|_{L^2} < K$; *moreover,* (H4) *or* (H5) *is satisfied.*

Then the problem (1.1) *has a solution* $x \in T(v,r)$.

Proof. Denote $X = C^1(I,\mathbb{R}^n)$ (respectively $W^{1,2}(I,\mathbb{R}^n)$). Let us define

$$H, \overline{H} : [0,1] \times X \to X$$

by $H(\lambda,x) = S \circ F(\lambda,x)$, and $\overline{H}(\lambda,x) = \lambda H(0,x)$, respectively. Propositions 5.1 and 5.3 (respectively 5.5) imply that H, and hence $\overline{H}$, are continuous and completely continuous. Observe that $\overline{H}$ is bounded, so we can find an open bounded set $W \subset X$ such that $\overline{H}([0,1] \times X) \subset W$. Without lost of generality, we can assume that

$$\bigl\{x \in X:\ x \in T(v,r),\ \|x\|_X < K\bigr\} \subset W.$$

From Proposition 4.2 and by assumption, $H(\lambda,\cdot)$ has no fixed point on ∂W. Therefore, degree theory (see [10]) implies

$$1 = \deg(\mathrm{id}, W, 0) = \deg\bigl(\mathrm{id} - \overline{H}(1,\cdot), W, 0\bigr) = \deg\bigl(\mathrm{id} - H(1,\cdot), W, 0\bigr).$$

Hence, there exists $x \in T(v,M)$ a solution of (P_1), and hence of (1.1). □

With this general result, we can prove our existence theorems.

Proof of Theorem 3.6. The proof is a direct consequence of Proposition 4.3 and Theorem 6.1. □

Proof of Theorem 3.7. The proof is a direct consequence of Proposition 4.4 and Theorem 6.1. □

Proof of Theorem 3.8. The proof is a direct consequence of Proposition 4.8 and Theorem 6.1. □

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