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On some generalizations of Ekeland's principle and inward contractions in gauge spaces

M. Frigon

To Richard S. Palais

Abstract. We give generalizations in complete gauge spaces of the following results: Bishop-Phelps' theorem, Ekeland's variational principle, Caristi's fixed point theorem, the drop theorem and the flower petal theorem. We show that our generalizations are equivalent. We apply those results to obtain fixed point theorems for multivalued contractions defined on a closed subset of a complete gauge space and satisfying a generalized inwardness condition.

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1. Introduction and preliminaries

In 1963, Bishop and Phelps [3] established the following principle. We state the formulation which appeared in [12] (see also [18] and [28]).

Theorem 1.1 (Bishop–Phelps' theorem). *Let M be a complete metric space, $\phi : M \to \mathbb{R}$ lower semicontinuous and bounded from below and $c > 0$. Then for any $x_0 \in M$, there exists $x^* \in M$ such that*

(i) $\phi(x^*) + cd(x_0, x^*) \leq \phi(x_0)$;
(ii) $\phi(x^*) < \phi(x) + cd(x, x^*)$ *for every* $x \neq x^*$.

The well-known Ekeland variational principle [13] obtained in 1972 is one of the most frequently applied results of nonlinear analysis (see also [14]).

Theorem 1.2 (Ekeland's variational principle). *Let M be a complete metric space, and $\phi : M \to \mathbb{R}\cup\{\infty\}$ proper, lower semicontinuous and bounded from* *below. For each $c > 0$, $\delta > 0$ and $x_0 \in M$ such that $\phi(x_0) \leq \inf \phi(M) + c\delta$, there exists $x^* \in M$ such that*

(i) $\phi(x^*) \leq \phi(x_0)$;
(ii) $d(x_0, x^*) \leq \delta$;
(iii) $\phi(x^*) < \phi(x) + cd(x, x^*)$ *for all* $x \neq x^*$.

The same year, Daneš [8] established the so-called drop theorem. Here is the generalized version published in 1985 in [9].

Theorem 1.3 (Drop theorem). *Let E be a Banach space, A a closed subset of E and B a closed bounded convex subset of E with $d(A, B) > 0$. Then, for every $x_0 \in A$, there exists $x^* \in A \cap K(x_0, B)$ such that $\{x^*\} = A \cap K(x^*, B)$, where*

$$K(x, B) = co\big(B \cup \{x\}\big) = \{u = (1 - \theta)x + \theta y : \theta \in [0, 1], y \in B\}.$$

In 1976, Caristi [5] established the existence of a fixed point for maps f for which the distance between x and $f(x)$ is suitably controlled. Here is the generalized formulation presented by Caristi and Kirk [6].

Theorem 1.4 (Caristi's theorem). *Let M be a complete metric space, $f : M \to M$ a map, and $\phi : M \to \mathbb{R}$ lower semicontinuous and bounded from below such that*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)) \quad \forall x \in M.$$

Then f has a fixed point.

Ten years later, the flower petal theorem was obtained by Penot [27].

Theorem 1.5 (Flower petal theorem). *Let A be a complete subset of a metric space M. Let $x_0 \in A$ and $b \in M \backslash \{x_0\}$. Then, for every $\delta > 0$, there exists $x^* \in P_\delta(x_0, b)$ such that $P_\delta(x^*, b) \cap A = \{x^*\}$, where*

$$P_\delta(x, b) = \{u \in M : d(u, b) + \delta d(x, u) \leq d(x, b)\}.$$

It appeared that the five previous theorems are equivalent; see [4, 9, 27].

Many generalizations of those results were obtained. One approach was to consider more general spaces such as locally convex spaces [7, 19], or suitable uniform spaces or gauges spaces [2, 15, 20, 22, 24]. Another approach was to generalize Ekeland's variational principle to maps $\phi : X \to Y \cup \{\infty\}$ defined on a metric space or, more generally, on a suitable uniform space, with values in an ordered vectorial topological space Y [17, 20, 26, 30].

The results presented in this paper lie between those two approaches. Indeed, we consider a gauge space $(X, \{d_n\})$ with $\{\phi_n : X \to \mathbb{R} \cup \{\infty\}\}$, a family of lower semicontinuous maps bounded from below, and we look for $x^* \in X$ such that

$$\phi_n(x^*) + c_n d_n(x^*, x_0) \leq \phi_n(x_0) \quad \forall n.$$

This new approach leads us to generalize the previous five theorems. Then, we show that our generalizations are equivalent results.

Our results permit us to obtain fixed point theorems for multivalued contractions defined on a closed subset of X and satisfying generalized inwardness conditions. Fixed point theorems for multivalued contractions in Banach spaces satisfying different inwardness conditions were obtained by Downing and Kirk [10] and by Lim [21]. Some generalizations of their results to metric spaces were obtained by Song [29], Maciejewski [23], and recently in [16]. Here, our generalized inwardness conditions, combined with our approach presented above, permit us to extend their results to gauge spaces. However, some of our fixed point theorems are new even in metric spaces. Let us mention that fixed point results were also obtained for inward multivalued contractions in the sense of the measure of noncompactness in Fréchet spaces by Agarwal and O'Regan [1].

Finally, in the last section, we apply our generalization of Ekeland's principle to obtain a Palais–Smale-star (PS*) type sequence of an extended real-valued map defined on a gauge space X.

We start with some preliminary definitions.

In what follows, X is a gauge space endowed with a complete gauge structure $\{d_n : n \in \mathbb{N}\}$ satisfying the following condition:

$$d_1(x, y) \leq d_2(x, y) \leq \cdots \quad \text{for every } x, y \in X; \tag{1.1}$$

see [11] for definitions. Let us mention that conditions on the family of gauges are imposed for the sake of simplicity. We could have considered a sequentially complete gauge space endowed with a family of gauges parametrized by a directed set, not necessarily countable, and with a sequentially lower closed quasi order on X; see [20] for more details.

For A, $B \subset X$ and $x \in X$, we denote

$$\begin{aligned} d_n(x, B) &= \inf_{y \in B} d_n(x, y), \qquad d_n(A, B) = \inf_{x \in A} d_n(x, B), \\ \rho_n(x, B) &= \sup_{y \in B} d_n(x, y), \qquad \rho_n(A, B) = \sup_{x \in A} \rho_n(x, B), \\ D_n(A, B) &= \max\Big\{ \inf_{x \in A} \rho_n(x, B), \inf_{y \in B} \rho_n(y, A) \Big\}. \end{aligned}$$

Definition 1.6. Let $A \subset X$. A multivalued map $F : A \to X$ with nonempty closed values is a *contraction* if for every $n \in \mathbb{N}$, there exists $k_n \in [0, 1[$ such that

$$D_n\big(F(x), F(y)\big) \leq k_n d(x, y) \quad \text{for every } x, y \in A.$$

2. Variational principles and generalization of Caristi's theorem

In this section, we generalize Bishop–Phelps' theorem, Ekeland's variational principle and Caristi's fixed point theorem to the complete gauge space X. We recall that X is endowed with a complete gauge structure $\{d_n : n \in \mathbb{N}\}$ satisfying condition (1.1).

The following result is a generalization of Bishop–Phelps' theorem.

Theorem 2.1. *For every $n \in \mathbb{N}$, let $c_n > 0$ and $\phi_n : X \to \mathbb{R}$ lower semicontinuous and bounded from below. Then for every $x_0 \in X$, there exists $x^* \in X$ such that*

(i) $\phi_n(x^*) + c_n d_n(x_0, x^*) \leq \phi_n(x_0)$ *for every* $n \in \mathbb{N}$*;*
(ii) *for every* $x \neq x^*$*, there exists* $n \in \mathbb{N}$ *such that* $\phi_n(x^*) < \phi_n(x) + c_n d_n(x, x^*)$*.*

Proof. For $x \in X$, denote

$$S(x) = \bigcap_{n \in \mathbb{N}} \{y \in X : \phi_n(y) + c_n d_n(y, x) \leq \phi_n(x)\}.$$

This set is closed and nonempty since $x \in S(x)$ and the maps ϕ_n are lower semicontinuous. Choose inductively $x_n \in S(x_{n-1})$ such that $\phi_n(x_n) \leq c_n/n + \inf \phi_n(S(x_{n-1}))$. It is easy to check that

$$S(x_0) \supset S(x_1) \supset \cdots .$$

Observe that for $x \in S(x_n)$,

$$\phi_n(x) + c_n d_n(x, x_n) \leq \phi_n(x_n) \leq \frac{c_n}{n} + \inf \phi_n(S(x_{n-1})) \leq \frac{c_n}{n} + \phi_n(x).$$

Thus,

$$d_n(x, x_n) \leq \frac{1}{n} \quad \forall x \in S(x_n). \tag{2.1}$$

So, for $k < n < p$,

$$d_k(x_p, x_n) \leq d_n(x_p, x_n) \leq \frac{1}{n}.$$

Therefore, $\{x_n\}$ is a Cauchy sequence, and hence converges to some

$$x^* \in \bigcap_{n \geq 0} S(x_n).$$

Assume there is $x \neq x^*$ such that $\phi_n(x) + c_n d_n(x, x^*) \leq \phi_n(x^*)$ for every $n \in \mathbb{N}$. Then for every $k \in \mathbb{N}$,

$$\begin{aligned} \phi_n(x) + c_n d_n(x, x_k) &\leq \phi_n(x) + c_n d_n(x, x^*) + c_n d_n(x^*, x_k) \\ &\leq \phi_n(x^*) + c_n d_n(x^*, x_k) \\ &\leq \phi_n(x_k); \end{aligned}$$

i.e., $x \in S(x_k)$ for every $k \in \mathbb{N}$. By (2.1),

$$d_k(x, x^*) \leq d_n(x, x^*) \leq d_n(x, x_n) + d_n(x_n, x^*) \leq \frac{2}{n} \quad \forall n \geq k.$$

This leads to a contradiction since $x \neq x^*$. □

Theorem 2.1 permits to obtain the following generalization of Ekeland's variational principle.

Theorem 2.2. *For every $n \in \mathbb{N}$, let $\phi_n : X \to \,]-\infty, \infty]$ be proper, lower semicontinuous and bounded from below. For every $x_0 \in X$ and every sequences of positive numbers $\{c_n\}$ and $\{\delta_n\}$ such that $\phi_n(x_0) \leq \inf \phi_n(X) + c_n \delta_n$, there exists $x^* \in X$ such that*

(i) $\phi_n(x^*) \leq \phi_n(x_0)$ *for every* $n \in \mathbb{N}$*;*
(ii) $d_n(x_0, x^*) \leq \delta_n$ *for every* $n \in \mathbb{N}$*;*
(iii) *for every* $x \neq x^*$*, there exists* $n \in \mathbb{N}$ *such that* $\phi_n(x^*) < \phi_n(x) + c_n d_n(x, x^*)$*.*

Proof. Observe that $\phi_n(x_0) < \infty$ for every $n \in \mathbb{N}$. Consider the gauge space

$$Y = \bigcap_{n \in \mathbb{N}} \{y \in X : \phi_n(y) \leq \phi_n(x_0)\}.$$

The previous theorem applied to the restrictions of ϕ_n to this space insures the existence of $x^* \in Y$ such that (iii) is satisfied for every $x \in Y$ and

$$\phi_n(x^*) + c_n d_n(x^*, x_0) \leq \phi_n(x_0) \leq c_n \delta_n + \inf \phi_n(X) \quad \forall n \in \mathbb{N}.$$

So (i) and (ii) are satisfied.

If $x \in X \backslash Y$, there exists $n \in \mathbb{N}$ such that

$$\phi_n(x^*) \leq \phi_n(x_0) < \phi_n(x) \leq \phi_n(x) + c_n d_n(x, x^*);$$

so (iii) is satisfied for every $x \in X$. □

Now, using the generalization of Bishop–Phelps' theorem (Theorem 2.1), we obtain the following generalization of Caristi's fixed point theorem.

Theorem 2.3. *Let* $f : X \to X$*. For every* $n \in \mathbb{N}$*, let* $\phi_n : X \to \mathbb{R}$ *be lower semicontinuous and bounded from below such that*

$$d_n(x, f(x)) \leq \phi_n(x) - \phi_n(f(x)) \quad \forall x \in X.$$

Then f *has a fixed point.*

Proof. Fix $x_0 \in X$, and let x^* be as given by Theorem 2.1. If $x^* \neq f(x^*)$, there exists $n \in \mathbb{N}$ such that

$$\phi_n(x^*) < d_n(x^*, f(x^*)) + \phi_n(f(x^*)).$$

This contradicts our assumption. □

Proposition 2.4. *Theorems* 2.1*,* 2.2 *and* 2.3 *are equivalent.*

Proof. We already know that Theorem 2.1 implies Theorems 2.2 and 2.3.

We first show that Theorem 2.2 implies Theorem 2.1. Take

$$Y = \{x \in X : \phi_n(x) + c_n d_n(x, x_0) \leq \phi_n(x_0) \ \forall n \in \mathbb{N}\}.$$

Set $\delta_n \geq (\phi_n(x_0) - \inf \phi_n(X))/c_n$. Let x^* be given by Theorem 2.2 applied on the gauge space Y. It satisfies conditions (i) and (ii) of Theorem 2.1 for every $x \in Y \backslash \{x^*\}$. If $x \notin Y$, there exists $n \in \mathbb{N}$ such that $\phi_n(x) + c_n d_n(x, x_0) > \phi_n(x_0)$. So,

$$\begin{aligned}\phi_n(x^*) &\leq \phi_n(x_0) - c_n d_n(x^*, x_0)\\ &< \phi_n(x) + c_n(d_n(x, x_0) - d_n(x^*, x_0))\\ &\leq \phi_n(x) + c_n d_n(x, x^*).\end{aligned}$$

It remains to show that Theorem 2.3 implies Theorem 2.1. Assume the conclusion is false. Let Y be as above. For $x \in Y$, there exists a point denoted

$f(x) \neq x$ such that $\phi_n(f(x)) + c_n d_n(x, f(x)) \leq \phi_n(x)$ for all $n \in \mathbb{N}$. Arguing as above, one can check that $f(x) \in Y$. By Theorem 2.3, f has a fixed point. Contradiction. $\square$

A multivalued version of Theorem 2.3 generalizes the multivalued version of Caristi's fixed point theorem due to Mizoguchi and Takahashi [25].

Theorem 2.5. *Let $F : X \to X$ be a multivalued map with nonempty values. For every $n \in \mathbb{N}$, let $\phi_n : X \to \mathbb{R}$ be lower semicontinuous and bounded from below. Assume that for every $x \in X$, there exists $y \in F(x)$ such that*

$$d_n(x, y) \leq \phi_n(x) - \phi_n(y) \quad \textit{for every } n \in \mathbb{N}.$$

Then F has a fixed point.

3. Generalizations of the drop and the flower petal theorems

In order to generalize the drop theorem and the flower petal theorem, we extend the notions of drop and petal.

Let $x \in X$, let $B \subset X$ be nonempty, closed, bounded, and let $\alpha = (\alpha_1, \alpha_2, \ldots) \in [0, 1]^{\mathbb{N}}$. For $\gamma = (\gamma_1, \gamma_2, \ldots) \in]0, 1]^{\mathbb{N}}$, we define the *generalized petal* by

$$\begin{aligned}\mathcal{P}_{\alpha,\gamma}(x, B) = \Big\{u \in X :\ & \text{for every } n \in \mathbb{N}, \\ & \alpha_n d_n(u, B) + (1 - \alpha_n)\rho_n(u, B) + \gamma_n d_n(x, u) \\ & \qquad \leq \alpha_n d_n(x, B) + (1 - \alpha_n)\rho_n(x, B)\Big\}.\end{aligned}$$

For $\sigma = (\omega, \mu, \nu) \in]0, \infty[^{\mathbb{N}} \times [0, \infty[^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, we define the *generalized drop* by

$$\begin{aligned}\mathcal{D}_{\alpha,\sigma}(x, B) = \Big\{u \in X :\ & \forall n \in \mathbb{N}, \exists \theta_n \geq 0 \text{ such that} \\ & d_n(x, u) \leq \theta_n \omega_n \rho_n(x, B) \text{ and} \\ & \quad \alpha_n d_n(u, B) + (1 - \alpha_n)\rho_n(u, B) \\ & \qquad \leq (\alpha_n - \theta_n \mu_n) d_n(x, B) + (1 - \alpha_n + \theta_n \nu_n)\rho_n(x, B)\Big\}.\end{aligned}$$

Example 3.1. Figures 3.1 and 3.2 show different generalized petals and generalized drops in the case where $X = \mathbb{R}^2$, $x = (-1, -1)$ and $B = \{(x_1, x_2) \in \mathbb{R} \times [0, \infty[: x_1^2 + x_2^2 \leq 1\}$.

In the following two results, for A a subset of X, we look for x^*, the unique element of $A \cap \mathcal{P}_{\alpha,\gamma}(x^*, B)$ and $A \cap \mathcal{D}_{\alpha,\sigma}(x^*, B)$, respectively.

Theorem 3.2. *Let A be a nonempty closed subset of X and B a nonempty, closed, bounded subset of X. Then for every $\alpha \in [0, 1]^{\mathbb{N}}$ and every $\gamma \in]0, 1]^{\mathbb{N}}$, there exists $x^* \in A$ such that $\{x^*\} = A \cap \mathcal{P}_{\alpha,\gamma}(x^*, B)$.*

Proof. For $n \in \mathbb{N}$, define $\phi_n : A \to \mathbb{R}$ by

$$\phi_n(x) = \alpha_n d_n(x, B) + (1 - \alpha_n)\rho_n(x, B).$$

The map ϕ_n is lower semicontinuous. Choose $x_0 \in A$. Let $x^* \in A$ be the point

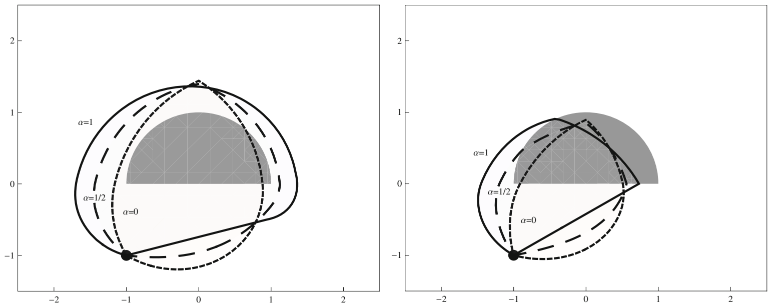

FIGURE 3.1. The generalized petal $\mathcal{P}_{\alpha,\gamma}(x, B)$ for $\gamma = 0.25$ and $\gamma = 0.5$.

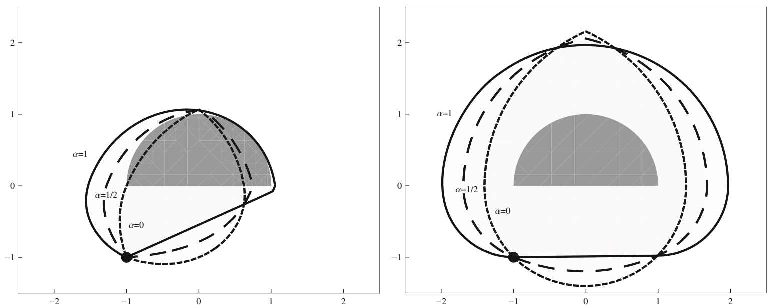

FIGURE 3.2. The generalized drop $\mathcal{D}_{\alpha,\sigma}(x, B)$ for $\sigma = (0.5, 0.5, 0)$ and $\sigma = (1, 0.75, 0.3)$.

given by Theorem 2.1 with $c_n = \gamma_n$. If there exists $x \in A \cap \mathcal{P}_{\alpha,\gamma}(x^*, B)\backslash\{x^*\}$, then there exists $n \in \mathbb{N}$ such that

$$\begin{aligned}\phi_n(x^*) &< \phi_n(x) + \gamma_n d_n(x, x^*)\\ &= \alpha_n d_n(x, B) + (1 - \alpha_n)\rho_n(x, B) + \gamma_n d_n(x, x^*)\\ &\leq \alpha_n d_n(x^*, B) + (1 - \alpha_n)\rho_n(x^*, B).\end{aligned}$$

Contradiction. □

Theorem 3.3. *Let A be a nonempty closed subset of X and B a nonempty, closed, bounded subset of X. Let $\sigma = (\omega, \mu, \nu) \in\ ]0, \infty[^{\mathbb{N}} \times [0, \infty[^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Assume that for every $n \in \mathbb{N}$,*

$$\nu_n \in\ ] - \infty, \mu_n d_n(A, B)/\rho_n(A, B)[.$$

Then for every $\alpha \in\ ]0, 1]^{\mathbb{N}}$, there exists $x^* \in A$ such that

$$\{x^*\} = A \cap \mathcal{D}_{\alpha,\sigma}(x^*, B).$$

Proof. For $n \in \mathbb{N}$, if $\nu_n < 0$, choose $\gamma_n \in\ ]0, \min\{1, -\nu_n/\omega_n\}]$. Otherwise, $\mu_n > 0$, $d_n(A, B) > 0$, $\rho_n(A, B) < \infty$, and we can choose $\gamma_n \in\ ]0, 1]$ such that

$$(\nu_n + \gamma_n\omega_n)\rho_n(A, B) \leq \mu_n d_n(A, B).$$

The conclusion follows from Theorem 3.2 if we show that

$$\mathcal{D}_{\alpha,\sigma}(x,B) \subset \mathcal{P}_{\alpha,\gamma}(x,B) \quad \text{for every } x \in A.$$

Observe that if $u \in \mathcal{D}_{\alpha,\sigma}(x,B)$, for every $n \in \mathbb{N}$, there exists $\theta_n \geq 0$ such that

$$\begin{aligned}
\alpha_n d_n(u,B) &+ (1-\alpha_n)\rho_n(u,B) + \gamma_n d_n(x,u) \\
&\leq \alpha_n d_n(x,B) + (1-\alpha_n)\rho_n(x,B) \\
&\quad + \theta_n\big((\nu_n + \gamma_n\omega_n)\rho_n(x,B) - \mu_n d_n(x,B)\big) \\
&\leq \alpha_n d_n(x,B) + (1-\alpha_n)\rho_n(x,B).
\end{aligned}$$

□

Remark 3.4. In Theorems 3.2 and 3.3, it is not necessary to assume that B is bounded if $\alpha = (1,1,\ldots)$.

Proposition 3.5. *Theorems* 3.2 *and* 3.3 *are equivalent.*

Proof. Since the proof of Theorem 3.3 relies on Theorem 3.2, to conclude, we need to show that Theorem 3.3 implies Theorem 3.2.

Observe that if $u \in \mathcal{P}_{\alpha,\gamma}(x,B)$, for every $n \in \mathbb{N}$, there exists $\theta_n \in [0,1]$ such that

$$d_n(x,u) = \frac{\theta_n}{\gamma_n}\big(\alpha_n d_n(x,B) + (1-\alpha_n)\rho_n(x,B)\big) \leq \frac{\theta_n}{\gamma_n}\rho_n(x,B).$$

So,

$$\begin{aligned}
\alpha_n d_n(u,B) &+ (1-\alpha_n)\rho_n(u,B) \\
&\leq \alpha_n d_n(x,B) + \big(1-\alpha_n\big)\rho_n(x,B) - \gamma_n d_n(x,u) \\
&= \big(\alpha_n - \theta_n\alpha_n\big)d_n(x,B) + \big(1-\alpha_n - \theta_n(1-\alpha_n)\big)\rho_n(x,B).
\end{aligned}$$

Hence,

$$\mathcal{P}_{\alpha,\gamma}(x,B) \subset \mathcal{D}_{\alpha,\sigma}\big(x,B\big) \quad \text{with} \quad \sigma = \left(\frac{1}{\gamma}, \alpha, \alpha - 1\right).$$

The conclusion follows from Theorem 3.3. □

We show the equivalence between those results and Theorems 2.1, 2.2 and 2.3.

Proposition 3.6. *Theorems* 2.1*,* 2.2*,* 2.3*,* 3.2 *and* 3.3 *are equivalent.*

Proof. We already know that Theorems 2.1, 2.2 and 2.3 are equivalent, and that Theorems 3.2 and 3.3 are equivalent. Since the proof of Theorem 3.2 relies on Theorem 2.1, to conclude, we need to show that Theorem 3.2 implies Theorem 2.1.

Under the assumptions of Theorem 2.1, fix $r_n = 2/c_n$. Consider the space

$$\widehat{X} = \{(x, t_1, t_2, \ldots) \in X \times \mathbb{R}^{\mathbb{N}} : \phi_n(x) \leq t_n \ \forall n \in \mathbb{N}\},$$

endowed with a complete gauge structure $\{\hat{d}_n\}$ defined by

$$\hat{d}_n\big((x,t_1,t_2,\ldots),(y,s_1,s_2,\ldots)\big)=d_n(x,y)+\sum_{i=1}^{n} r_i|t_i-s_i|.$$

Obviously, (1.1) is satisfied.

Set

$$\begin{aligned}A&=\{(x,t_1,t_2,\ldots)\in\widehat{X}: t_n+c_nd_n(x,x_0)\le\phi_n(x_0)\ \forall n\in\mathbb{N}\},\\ B&=\{(x_0,\inf\phi_1(X),\inf\phi_2(X),\ldots)\}.\end{aligned}$$

Those sets are closed and nonempty, and B is bounded. Fix

$$\begin{aligned}\alpha&=(\alpha_1,\alpha_2,\ldots)=(1,1,\ldots),\\ \gamma&=(\gamma_1,\gamma_2,\ldots)=\left(\frac{1}{3},\frac{1}{3},\ldots\right).\end{aligned}$$

Theorem 3.2 implies that there exists $(x^*,t_1^*,t_2^*,\ldots)$ such that

$$\{(x^*,t_1^*,t_2^*,\ldots)\}=A\cap\mathcal{P}_{\alpha,\gamma}\big((x^*,t_1^*,t_2^*,\ldots),B\big). \tag{3.1}$$

We claim that $(x^*,t_1^*,t_2^*,\ldots)=(x^*,\phi_1(x^*),\phi_2(x^*),\ldots)$. Indeed, for every $n\in\mathbb{N}$,

$$\begin{aligned}&\frac{1}{3}\hat{d}_n\big((x^*,t_1^*,t_2^*,\ldots),(x^*,\phi_1(x^*),\phi_2(x^*),\ldots)\big)\\ &\qquad+\hat{d}_n\big((x^*,\phi_1(x^*),\phi_2(x^*),\ldots),B\big)\\ &\quad=\frac{1}{3}\sum_{i=1}^{n} r_i|t_i^*-\phi_i(x^*)|+d_n(x^*,x_0)\\ &\qquad+\sum_{i=1}^{n} r_i|\phi_i(x^*)-\inf\phi_i(X)|\\ &\quad=\hat{d}_n\big((x^*,t_1^*,t_2^*,\ldots),B\big)-\frac{2}{3}\sum_{i=1}^{n} r_i|t_i^*-\phi_i(x^*)|\\ &\quad\le\hat{d}_n\big((x^*,t_1^*,t_2^*,\ldots),B\big).\end{aligned}$$

So,

$$(x^*,\phi_1(x^*),\phi_2(x^*),\ldots)\in A\cap\mathcal{P}_{\alpha,\gamma}\big((x^*,t_1^*,t_2^*,\ldots),B\big).$$

This, combined with (3.1), implies that

$$t_n^*=\phi_n(x^*)\quad\forall n\in\mathbb{N}. \tag{3.2}$$

If $(x, \phi_1(x), \phi_2(x), \dots) \in A$ and $\phi_n(x) + c_n d_n(x, x^*) \leq \phi_n(x^*)$ for all $n \in \mathbb{N}$, observe that

$$\begin{aligned}
\frac{1}{3}\hat{d}_n\big((x,&\phi_1(x), \phi_2(x), \dots), (x^*, \phi_1(x^*), \phi_2(x^*), \dots)\big) \\
&\quad + \hat{d}_n\big((x, \phi_1(x), \phi_2(x), \dots), B\big) \\
&= \frac{1}{3}d_n(x, x^*) + \frac{1}{3}\sum_{i=1}^{n} r_i(\phi_i(x^*) - \phi_i(x)) + d_n(x, x_0) \\
&\quad + \sum_{i=1}^{n} r_i(\phi_i(x) - \inf \phi_i(X)) \\
&= \hat{d}_n\big((x^*, \phi_1(x^*), \phi_2(x^*), \dots), B\big) + \frac{1}{3}d_n(x, x^*) \\
&\quad + d_n(x, x_0) - d_n(x^*, x_0) - \frac{2}{3}\sum_{i=1}^{n} r_i(\phi_i(x^*) - \phi_i(x)) \\
&\leq \hat{d}_n\big((x^*, \phi_1(x^*), \phi_2(x^*), \dots), B\big) + \frac{4}{3}d_n(x, x^*) \\
&\quad - \frac{2}{3}\sum_{i=1}^{n} r_i(\phi_i(x^*) - \phi_i(x)) \\
&\leq \hat{d}_n\big((x^*, \phi_1(x^*), \phi_2(x^*), \dots), B\big) + \frac{4}{3c_n}\big(\phi_n(x^*) - \phi_n(x)\big) \\
&\quad - \frac{2}{3}\sum_{i=1}^{n} r_i(\phi_i(x^*) - \phi_i(x)) \\
&\leq \hat{d}_n\big((x^*, \phi_1(x^*), \phi_2(x^*), \dots), B\big) \\
&\quad + \left(\frac{4}{3c_n} - \frac{2r_n}{3}\right)\big(\phi_n(x^*) - \phi_n(x)\big) - \frac{2}{3}\sum_{i=1}^{n-1} r_i(\phi_i(x^*) - \phi_i(x)) \\
&\leq \hat{d}_n\big((x^*, \phi_1(x^*), \phi_2(x^*), \dots), B\big).
\end{aligned}$$

So,

$$(x, \phi_1(x), \phi_2(x), \dots) \in A \cap \mathcal{P}_{\alpha,\gamma}\big((x^*, \phi_1(x^*), \phi_2(x^*), \dots), B\big).$$

This fact, combined with (3.1) and (3.2), implies that $x = x^*$.

On the other hand, statement (ii) of Theorem 2.1 is satisfied if $x \in X$ is such that $(x, \phi_1(x), \phi_2(x), \dots) \notin A$. Indeed, in this case, there exists $n \in \mathbb{N}$ such that $\phi_n(x) + c_n d_n(x, x_0) > \phi_n(x_0)$. Since $(x^*, \phi_1(x^*), \phi_2(x^*), \dots) \in A$,

$$\begin{aligned}
\phi(x) + c_n d_n(x, x^*) &> \phi_n(x^*) + c_n\big(d_n(x^*, x_0) + d_n(x, x^*) - d_n(x, x_0)\big) \\
&\geq \phi_n(x^*). \qquad \square
\end{aligned}$$

Now, we present some corollaries of Theorems 3.2 and 3.3. Observe that in those results, there is no convexity assumption on B since there is no vectorial structure on the space. However, if X is a Fréchet space and B is convex, we also obtain, as a corollary, a generalization in Fréchet spaces of the drop theorem [8, 9].

Corollary 3.7. *Let E be a Fréchet space endowed with a family of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ such that $\|x\|_1 \le \|x\|_2 \le \cdots$ for every $x \in E$. Let A, B be nonempty closed subsets of E such that $d_n(A,B) > 0$ for every $n \in \mathbb{N}$. In addition, assume that B is convex and bounded. Then, there exists $x^* \in A$ such that $\{x^*\} = A \cap \mathcal{K}(x^*, B)$, where*

$$\begin{aligned}\mathcal{K}(x^*, B) = \{x^*\} \cup \{u \in X : &\forall n \in \mathbb{N}, \exists \lambda_n \ge 1 \text{ such that} \\ &d_n(x^* + \lambda_n(u - x^*), B) = 0\}.\end{aligned}$$

Proof. Let $x \in A$ and $u \in \mathcal{K}(x,B)\backslash\{x\}$. For every $n \in \mathbb{N}$, there exists $\lambda_n \ge 1$ such that $d_n(x + \lambda_n(u-x), B) = 0$, So, for every $\varepsilon > 0$, there exists $y \in B$ such that $\|x + \lambda_n(u-x) - y\|_n < \lambda_n \varepsilon$. Observe that

$$\|x-u\|_n \le \|x + \frac{1}{\lambda_n}(y-x) - u\|_n + \frac{1}{\lambda_n}\|x-y\|_n < \varepsilon + \frac{1}{\lambda_n}\rho_n(x,B).$$

Also, the convexity of B implies that for every $z \in B$,

$$\begin{aligned}d_n(u,B) &\le \left\| \frac{1}{\lambda_n} y + \left(1 - \frac{1}{\lambda_n}\right) z - u \right\|_n \\ &\le \left\| x + \frac{1}{\lambda_n}(y-x) - u \right\|_n + \left(1 - \frac{1}{\lambda_n}\right)\|x-z\|_n \\ &< \varepsilon + \left(1 - \frac{1}{\lambda_n}\right)\|x-z\|_n.\end{aligned}$$

Since ε and z are arbitrary,

$$\|x-u\|_n \le \frac{1}{\lambda_n}\rho_n(x,B) \qquad \text{and} \qquad d_n(u,B) \le \left(1 - \frac{1}{\lambda_n}\right) d_n(x,B).$$

Therefore,

$$\mathcal{K}(x,B) \subset \mathcal{D}_{\alpha,\sigma}(x,B) \quad \forall x \in A$$

with $\alpha = (1, 1, \ldots)$ and $\sigma = \big((1,1,\ldots),(1,1,\ldots),(0,0,\ldots)\big)$.

The conclusion follows from Theorem 3.3. □

In the previous result, more precision on the location of x^* can be obtained.

Corollary 3.8. *Let E be a Fréchet space endowed with a family of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ such that $\|x\|_1 \le \|x\|_2 \le \cdots$ for every $x \in E$. Let B be a nonempty closed, convex, bounded subset of E and A a nonempty closed subset of E such that $d_n(A,B) > 0$ for every $n \in \mathbb{N}$. Then, for every $x_0 \in A$, there exists $x^* \in \mathcal{K}(x_0, B)$ such that $\{x^*\} = A \cap \mathcal{K}(x^*, B)$.*

Proof. Denote $\tilde{A} = A \cap \mathcal{K}(x_0, B)$. The conclusion follows from the previous corollary and the observation that for every $x \in \tilde{A}$,

$$A \cap \mathcal{K}(x,B) = \tilde{A} \cap \mathcal{K}(x,B).$$ □

As a corollary of Theorem 3.2, we obtain a generalization in gauge spaces of the flower petal theorem due to Penot [27].

Corollary 3.9. *Let A be a nonempty closed subset of X, and $b \in X\backslash A$. Then for every $x_0 \in A$ and every $\gamma \in]0,1]^{\mathbb{N}}$, there exists $x^* \in A \cap \mathcal{P}_{1,\gamma}(x_0, \{b\})$ such that $A \cap \mathcal{P}_{1,\gamma}(x^*, \{b\}) = \{x^*\}$.*

Proof. Fix $\tilde{A} = A \cap \mathcal{P}_{1,\gamma}(x_0, \{b\})$. Theorem 3.2 implies that there exists $x^* \in \tilde{A}$ such that $\{x^*\} = \tilde{A} \cap \mathcal{P}_{1,\gamma}(x^*, \{b\})$. The conclusion follows from the fact that

$$x^* \in A \cap \mathcal{P}_{1,\gamma}(x^*, \{b\}) \subset \tilde{A} \cap \mathcal{P}_{1,\gamma}(x^*, \{b\}). \qquad \square$$

In complete metric spaces, Theorems 3.2 and 3.3 are, respectively, generalizations of the flower petal theorem and the drop theorem.

Corollary 3.10. *Let M be a complete metric space and A, B nonempty closed subsets of M with B bounded. Then for every $\alpha \in [0,1]$ and every $\gamma \in]0,1]$, there exists $x^* \in A$ such that*

$$\{x^*\} = A \cap \big\{u \in M : \alpha d(u,B) + (1-\alpha)\rho(u,B) + \gamma d(u,x^*) \\ \le \alpha d(x^*,B) + (1-\alpha)\rho(x^*,B)\big\}.$$

Corollary 3.11. *Let M be a complete metric space, A, B nonempty closed subsets of M with B bounded, and let*

$$\mu \in [0,\infty[\qquad \text{and} \qquad \nu \in]-\infty, \mu d(A,B)/\rho(A,B)[.$$

Then for every $\alpha \in [0,1]$ and $\omega \in]0,\infty[$, there exists $x^* \in A$ such that

$$\{x^*\} = A \cap \Big\{u \in M : \exists\, \theta \in [0,1] \text{ such that} \\ d(u,x^*) \le \theta\omega\rho_n(x^*,B) \text{ and} \\ \alpha d(u,B) + (1-\alpha)\rho(u,B) \\ \le (\alpha - \theta\mu) d(x^*,B) + (1-\alpha+\theta\nu)\rho(x^*,B)\Big\}.$$

As a direct consequence of Theorems 3.2 and 3.3, we obtain the following fixed point results which are new even in the case where the space is a complete metric space.

Theorem 3.12. *Let A be a nonempty, closed subset of X, and $F : A \to X$ a multivalued map. Assume there exists B a closed, bounded subset of X, $\alpha \in [0,1]^{\mathbb{N}}$, and $\gamma \in]0,1]^{\mathbb{N}}$ such that for every $x \in A$, $F(x) \cap A \cap \mathcal{P}_{\alpha,\gamma}(x,B) \neq \emptyset$. Then F has a fixed point.*

Theorem 3.13. *Let A be a nonempty, closed subset of X, B a closed, bounded subset of X, and $\sigma = (\omega,\mu,\nu) \in]0,\infty[^{\mathbb{N}} \times [0,\infty[^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$,*

$$\nu_n \in]-\infty, \mu_n d_n(A,B)/\rho_n(A,B)[.$$

Assume that $F : A \to X$ is a multivalued map such that for every $x \in A$,

$$F(x) \cap A \cap \mathcal{D}_{\alpha,\sigma}(x, B) \neq \emptyset.$$

Then F has a fixed point.

In the following section, instead of considering the generalized petals $\mathcal{P}_{\alpha,\gamma}(x, B)$ (resp., the generalized drops $\mathcal{D}_{\alpha,\sigma}(x, B)$) for some fixed set B, we will obtain fixed point results considering the generalized petals $\mathcal{P}_{\alpha,\gamma}(x, F(x))$ (resp., the generalized drops $\mathcal{D}_{\alpha,\sigma}(x, F(x))$).

4. Generalized inwardness condition

In this section, we present fixed point results for multivalued contractions defined on a closed subset of the gauge space X and satisfying generalized inwardness conditions. More precisely, we ask that the generalized petal (resp., generalized drop) of x and $F(x)$ intersects the domain of the multivalued contraction F.

Theorem 4.1. *Let A be a closed subset of X, and $F : A \to X$ a multivalued contraction with nonempty, closed, bounded values, and constants of contraction $\{k_n\}$. Assume that there exist $\alpha \in [0, 1]^{\mathbb{N}}$ and $\gamma \in]0, 1]^{\mathbb{N}}$ such that $k_n < \gamma_n$ and for every $x \in A$, one has*

$$x \in F(x) \qquad \text{or} \qquad \mathcal{P}_{\alpha,\gamma}(x, F(x)) \cap A\backslash\{x\} \neq \emptyset. \tag{4.1}$$

Then F has a fixed point.

Proof. Choose $c_n < \gamma_n - k_n$ and define $\phi_n : A \to \mathbb{R}$ by

$$\phi_n(x) = \alpha_n d_n(x, F(x)) + (1 - \alpha_n)\rho_n(x, F(x)).$$

Since F is a contraction, ϕ_n is lower semicontinuous. Fix $x_0 \in A$ and let $x^* \in A$ be as given by Theorem 2.1. By (4.1), if $x^* \notin F(x^*)$, there exists $x \in \mathcal{P}_{\alpha,\gamma}(x^*, F(x^*)) \cap A\backslash\{x^*\}$. So, for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\phi_n(x) + c_n d_n(x, x^*) \\
&\qquad \le \alpha_n\big(d_n(x, F(x^*)) + D_n(F(x), F(x^*))\big) \\
&\qquad\quad + (1 - \alpha_n)\big(\rho_n(x, F(x^*)) + D_n(F(x), F(x^*))\big) + c_n d_n(x, x^*) \\
&\qquad \le \alpha_n d_n(x, F(x^*)) + (1 - \alpha_n)\rho_n(x, F(x^*)) + (k_n + c_n)d_n(x, x^*) \\
&\qquad \le \phi_n(x^*).
\end{aligned}$$

This contradicts (ii) of Theorem 2.1. □

The previous result is original even in the particular case where X is a complete metric space. It generalizes the following result obtained in [16].

Corollary 4.2. *Let A be a closed subset of M, a complete metric space, and let $F : A \to M$ be a multivalued contraction with constant $k \in [0, 1[$. Assume that there exits $\theta \in]k, 1]$ such that for every $x \in A$, $x \in F(x)$ or*

$$F(x) \in \big\{Y \subset X \text{ nonempty and closed} : \exists\, u \in K\backslash\{x\} \text{ such that } \theta d(x, u) + d(u, Y) \leq d(x, Y)\big\}.$$

Then F has a fixed point.

Another type of inward contraction occurs when the generalized drop of x and $F(x)$ intersects the domain of F.

Theorem 4.3. *Let A be a closed subset of X, and $F : A \to X$ a multivalued contraction with nonempty, closed, bounded values, and constants of contraction $\{k_n\}$. Assume that there exist $\alpha \in [0, 1]^{\mathbb{N}}$ and $\sigma = (\omega, \mu, \nu) \in]0, \infty[^{\mathbb{N}} \times [0, \infty[^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ such that for every $x \in A$, one has*

$$x \in F(x) \qquad \text{or} \qquad \mathcal{D}_{\alpha,\sigma}(x, F(x)) \cap A\backslash\{x\} \neq \emptyset. \tag{4.2}$$

Moreover, for every $n \in \mathbb{N}$,

$$\text{there exists } a_n > \omega_n k_n + \nu_n \text{ such that for every } x \in A, \quad a_n \rho_n(x, F(x)) \leq \mu_n d_n(x, F(x)). \tag{4.3}$$

Then F has a fixed point.

Proof. For $n \in \mathbb{N}$, choose $\gamma_n \in]k_n, 1]$ such that

$$\omega_n \gamma_n + \nu_n < a_n.$$

Arguing as in the proof of Theorem 3.3, one can show that

$$\mathcal{D}_{\alpha,\sigma}(x, F(x)) \subset \mathcal{P}_{\alpha,\gamma}(x, F(x)) \quad \forall x \in A.$$

The conclusion follows from the previous theorem. □

Remark 4.4. If F is a single-valued contraction, condition (4.3) can be written as $\omega_n k_n < \mu_n$ since one can take $\nu_n = 0$ and $a_n = \mu_n$.

In fact, inspired by the results of Song [29] on directional contractions, one sees that the contraction assumption imposed on F can be weakened in Theorem 4.1. The proof of the following result is analogous to the proof of Theorem 4.1.

Theorem 4.5. *Let A be a closed subset of X, $\alpha \in [0, 1]^{\mathbb{N}}$, and $F : A \to X$ a multivalued map with nonempty, closed, bounded values such that, for every $n \in \mathbb{N}$, the map $x \mapsto \alpha_n d_n(x, F(x)) + (1 - \alpha_n)\rho_n(x, F(x))$ is lower* *semicontinuous. Assume that there exist $k, \gamma \in [0,1]^{\mathbb{N}}$ such that $0 \le k_n < \gamma_n \le 1$, and for every $x \in A$, one has $x \in F(x)$ or*

$$\begin{aligned}\emptyset \neq \mathcal{P}_{\alpha,\gamma}(x, F(x)) \cap \Big\{u \in A\backslash\{x\} :\ & \textit{for every } n \in \mathbb{N} \\ & \alpha_n \sup_{y \in F(x)} d_n(y, F(u)) \\ & + (1-\alpha_n) \sup_{z \in F(u)} d_n(z, F(x)) \le k_n d_n(x,u)\Big\}.\end{aligned} \tag{4.4}$$

Then F has a fixed point.

In Theorem 4.1, we have considered a multivalued contraction satisfying a generalized inwardness condition involving the generalized petal $\mathcal{P}_{\alpha,\gamma}(x, F(x))$. We can also consider another type of inwardness condition involving the family of generalized petals $\mathcal{P}_{\alpha,\gamma}(x, \{y\})$ for $y \in F(x)$. Notice that since $\{y\}$ is a singleton, $\mathcal{P}_{\alpha,\gamma}(x, \{y\}) = \mathcal{P}_{\hat{\alpha},\gamma}(x, \{y\})$ for every $\alpha, \hat{\alpha} \in [0,1]^{\mathbb{N}}$. So, by $\mathcal{P}_{1,\gamma}(x, \{y\})$, we mean $\mathcal{P}_{\alpha,\gamma}(x, \{y\})$ with $\alpha = (1, 1, \dots)$.

Theorem 4.6. *Let A be a closed subset of X, and $F : A \to X$ a multivalued map with nonempty values and closed graph. Assume that there are a nondecreasing sequence $\{k_n\}$ in $[0,1[$ and $\gamma \in\ ]0,1]^{\mathbb{N}}$ such that $k_n < \gamma_n$, and for every $x \in A$ and every $y \in F(x)\backslash\{x\}$,*

$$\begin{aligned}\emptyset \neq \mathcal{P}_{1,\gamma}(x, \{y\}) \cap \Big\{u \in A\backslash\{x\} :\ & \textit{there exists } v \in F(u) \textit{ such that} \\ & d_n(y,v) \le k_n d_n(x,u)\ \forall n \in \mathbb{N}\Big\}.\end{aligned} \tag{4.5}$$

Then F has a fixed point.

Proof. For every $n \in \mathbb{N}$, define on $\operatorname{graph} F$, the gauge

$$\hat{d}_n\big((x,y),(\hat{x},\hat{y})\big) = k_n d_n(x,\hat{x}) + d_n(y,\hat{y}).$$

Since F has closed graph and $\{k_n\}$ is nondecreasing, $(\operatorname{graph} F, \{\hat{d}_n\})$ is a complete gauge space satisfying condition (1.1). Let $(x_0, y_0) \in \operatorname{graph} F$. For every $n \in \mathbb{N}$, define $\phi_n : \operatorname{graph} F \to \mathbb{R}$ by $\phi_n(x,y) = d_n(x,y)$ and choose $c_n > 0$ such that $(1+2c_n)k_n \le \gamma_n$. Theorem 2.1 guarantees the existence of $(x^*, y^*) \in \operatorname{graph} F$ such that for every $(x,y) \in \operatorname{graph} F\backslash\{(x^*,y^*)\}$,

$$\exists\, n \in \mathbb{N} \text{ such that } d_n(x^*,y^*) < d_n(x,y) + c_n \hat{d}_n\big((x,y),(x^*,y^*)\big).$$

If $x^* \neq y^*$, by assumption, there exists $(x,y) \in \operatorname{graph} F$ such that $x \neq x^*$ and

$$d_n(y,y^*) \le k_n d_n(x,x^*) \le \gamma_n d_n(x,x^*) + d_n(x,y^*) \le d_n(x^*,y^*).$$

So,

$$\begin{aligned} d_n(x,y) + c_n\hat{d}_n\big((x,y),(x^*,y^*)\big) &\le d_n(x,y^*) + c_nk_nd_n(x,x^*) + (1+c_n)d_n(y,y^*) \\ &\le d_n(x,y^*) + k_n(1+2c_n)d_n(x,x^*) \\ &\le d_n(x,y^*) + \gamma_nd_n(x,x^*) \\ &\le d_n(x^*,y^*). \end{aligned}$$

Contradiction. Therefore, F has a fixed point. □

We obtain the following corollary in Fréchet spaces for multivalued contractions with a classical inwardness condition.

Corollary 4.7. *Let E be a Fréchet space endowed with a family of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ such that $\|x\|_1 \le \|x\|_2 \le \cdots$ for every $x \in E$. Let A be a closed subset of E, and $F : A \to E$ a multivalued contraction with constants of contraction $\{k_n\}$, a nondecreasing sequence in $[0,1[$. Assume that for every $x \in A$,*

$$F(x) \subset \{x + \lambda(u-x) : u \in A, \lambda \ge 1\}.$$

Then F has a fixed point.

Proof. The conclusion follows directly from Theorem 4.6 since $y = x + \lambda(u - x) \in F(x)$ for some $u \in A$ and $\lambda \ge 1$ implies that $u \in \mathcal{P}_{1,1}(x,\{y\})$. □

5. Admissibly differentiable maps and (PS)* sequence

In this section, we present an application of Theorem 2.2. More precisely, we introduce the notion of admissibly d-differentiable maps for which we obtain a kind of Palais–Smale-star (PS*) sequence.

We consider E a Fréchet space endowed with a family of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ such that $\|x\|_1 \le \|x\|_2 \le \cdots$ for every $x \in E$.

Definition 5.1. A map $f : E \to \mathbb{R} \cup \{\infty\}$ is *admissibly d-differentiable at* $x \in \operatorname{dom} f$ if

(i) $\operatorname{dir}(f,x) = \{y \in E : \sup_n \|y\|_n \in\,]0,\infty[$, and x is an accumulation point of the sets $\operatorname{dom} f \cap (x + \mathbb{R}^+y)$ and $\operatorname{dom} f \cap (x + \mathbb{R}^-y)\} \ne \emptyset$;

(ii) for every $y \in \operatorname{dir}(f,x)$, there exists $L(y) \in \mathbb{R}$ such that

$$\lim_{\substack{t\to 0\\ x+ty\in\operatorname{dom} f}} \frac{f(x+ty)-f(x)}{t} = L(y);$$

(iii) $\|L\| = \sup\left\{\frac{|L(y)|}{\sup_n \|y\|_n} : y \in \operatorname{dir}(f,x)\right\} < \infty.$

We call L the *admissible d-derivative of f at x* and we denote it $Df(x)$.

For $\phi : E \to \mathbb{R} \cup \{\pm\infty\}$ and nonempty, closed subsets $X_1 \subset X_2 \subset \cdots \subset E$, we define $\phi_n : E \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\phi_n(x) = \begin{cases} \phi(x) & \text{if } x \in X_n, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 5.2. *Let $X_1 \subset X_2 \subset \cdots$ be a sequence of nonempty, closed subsets of E, $\phi : E \to \mathbb{R} \cup \{\pm\infty\}$ a proper map such that ϕ_n is lower semicontinuous, bounded from below and admissibly d-differentiable. If $\{x_n\}$ is a minimizing sequence of ϕ such that $x_n \in X_n$, there exists a sequence $\{x_n^*\}$ such that $x_n^* \in X_n$, $\phi(x_n^*) \leq \phi(x_n)$ for every $n \in \mathbb{N}$, and $x_n - x_n^* \to 0$, $\|D\phi_n(x_n^*)\| \to 0$.*

Proof. Denote $\gamma_n = (\phi_n(x_n) - \inf \phi_n(E))^{1/2}$. By Theorem 2.2 applied with $x_0 = x_1$, $c_n = \delta_n = \gamma_1$ for all $n \in \mathbb{N}$, there exists x_1^* such that $\phi_n(x_1^*) \leq \phi_n(x_1)$, $\|x_1 - x_1^*\|_n \leq \gamma_1$ for every $n \in \mathbb{N}$, and for every $x \neq x_1^*$, there exists k such that

$$\phi_k(x_1^*) < \phi_k(x) + \gamma_1 \|x - x_1^*\|_k.$$

Observe that $x_1^* \in X_1$, and for every $x \in \operatorname{dom} \phi_1$, $\phi_k(x) = \phi_1(x)$ for every $k \in \mathbb{N}$. So, for every $y \in \operatorname{dir}(\phi_1, x_1^*)$,

$$\begin{aligned} -\gamma_1 \sup_k \|y\|_k &\leq \lim_{\substack{t \to 0^+ \\ x_1^* + ty \in \operatorname{dom} \phi_1}} \frac{\phi_1(x_1^* + ty) - \phi_1(x_1^*)}{t} \\ &= D\phi_1(x_1^*)(y) \\ &= \lim_{\substack{t \to 0^- \\ x_1^* + ty \in \operatorname{dom} \phi_1}} \frac{\phi_1(x_1^* + ty) - \phi_1(x_1^*)}{t} \\ &\leq \gamma_1 \sup_k \|y\|_k. \end{aligned}$$

Thus, $\|D\phi_1(x_1^*)\| \leq \gamma_1$.

Repeat the argument inductively for $k = 2, 3, \ldots$, with $x_0 = x_k$ and $c_n = \delta_n = \gamma_k$, to obtain $x_k^* \in X_k$ such that $\phi(x_k^*) \leq \phi(x_k)$, $\|x_k^* - x_k\|_n \leq \gamma_k$ for every $n \in \mathbb{N}$, and $\|D\phi_k(x_k^*)\| \leq \gamma_k$. The conclusion follows from the fact that $\gamma_k \to 0$. □

Example 5.3. Let $E = L^\infty_{\mathrm{loc}}[0, \infty[$ endowed with the family of seminorms $\|x\|_n = \|x\chi_{[0,n]}\|_\infty$. Let $g \in L^1_{\mathrm{loc}}[0, \infty[$ and let $f : [0, \infty[\times \mathbb{R} \to \mathbb{R}$ be L^1_{loc}-Carathéodory; that is, $s \mapsto f(t, s)$ is continuous for almost every t; $t \mapsto f(t, s)$ is measurable for every s; and for every $r > 0$ and $n \in \mathbb{R}$, there exists $k_{n,r} \in L^1[0, n]$ such that $|f(t, s)| \leq k_{n,r}(t)$ for a.e. $t \in [0, n]$ and every $s \in [-r, r]$.

Let $\phi : E \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$\phi(x) = \int_0^\infty \left(x^2(t) g(t) - \int_0^{x(t)} f(t, s)\, ds \right) dt.$$

Let $X_n = \{x \in E : x(t) = 0 \text{ a.e. } t \in]n, \infty[\}$. For every $n \in \mathbb{N}$, ϕ_n is lower semicontinuous and it is admissibly d-differentiable. Indeed, for $x \in \operatorname{dom} \phi_n =$

X_n, $\operatorname{dir}(\phi_n, x) = X_n \backslash \{x\}$ and, for $y \in \operatorname{dir}(\phi_n, x)$,

$$\begin{aligned} D\phi_n(x)(y) &= \int_0^n \big(2x(t)g(t) - f(t, x(t))\big) y(t)\, dt \\ &= \int_0^\infty \big(2x(t)g(t) - f(t, x(t))\big) y(t)\, dt, \end{aligned}$$

since, for some $\theta(h)$ between 0 and h,

$$\begin{aligned} \lim_{h\to 0} \frac{\phi_n(x+hy) - \phi_n(x)}{h} &= \lim_{h\to 0} \int_0^\infty \frac{1}{h}\Big(((x+hy)^2(t) - x^2(t))g(t) \\ &\quad - \int_{x(t)}^{x(t)+hy(t)} f(t,s)\, ds\Big)\, dt \\ &= \lim_{h\to 0} \int_0^n \big(2x(t)y(t)g(t) + hy^2(t)g(t) \\ &\quad - f(t, x(t) + \theta(h)y(t))y(t)\big)\, dt \\ &= \int_0^n \big(2x(t)y(t)g(t) - f(t, x(t))y(t)\big)\, dt. \end{aligned}$$

Moreover,

$$\begin{aligned} \|D\phi_n(x)\| &= \sup \left\{ \frac{\left| \int_0^\infty 2x(t)y(t)g(t) - f(t,x(t))y(t)\, dt \right|}{\|y\|_n} : y \in \operatorname{dir}(\phi_n, x) \right\} \\ &\le 2\|x\|_n \|g\|_{L^1[0,n]} + \|k_{n,\|x\|_n}\|_{L^1[0,n]} < \infty. \end{aligned}$$

By Theorem 5.2, if $x_n \in X_n$ is such that $\{x_n\}$ is a minimizing sequence of ϕ, there exists $x_n^* \in X_n$ such that $\phi(x_n^*) \le \phi(x_n)$, $x_n - x_n^* \to 0$, and $\|D\phi_n(x_n^*)\| \to 0$.

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M. Frigon
Département de mathématiques et de statistique,
Université de Montréal
CP 6128, succ. Centre-Ville
Montréal, Québec H3C 3J7
Canada
e-mail: `frigon@dms.umontreal.ca`