NoDEA
Nonlinear differ. equ. appl. 13 (2006) 369–384
1021–9722/06/030369–16
DOI 10.1007/s00030-006-4012-1

Nonlinear Differential Equations
and Applications NoDEA

Systems of coupled Poisson equations with critical growth in unbounded domains

Fabrice COLIN
Département de mathématiques et informatique
Université Lourentienne, Sudbury, (Ontario)
P3E, 2C6, Canada
e-mail: fcolin@cs.laurentian.ca

and

Marlène FRIGON
Département de mathématiques et de statistique
Université de Montréal
CP 6128 succ. Centre-Ville
Montréal, (Québec)
H3C 3J7, Canada
e-mail: frigon@dms.umontreal.ca

Abstract. We establish the existence of a nontrivial solution to systems of coupled Poisson equations with critical growth in unbounded domains. The proofs rely on a generalized linking theorem due to Krysewski and Szulkin, and on a concentration-compactness argument since the Palais-Smale condition fails at all critical levels.

2000 *Mathematics Subject Classification*: 35J50, 35J55.
Key words: Coupled Poisson equations, critical exponant, unbounded domains, variational methods, generalized linking theorem.

This work was partially supported by CRSNG Canada and FCAR Québec.

1 Introduction

In this paper, we study the existence of nontrivial solutions to the following systems of coupled Poisson equations with critical growth in unbounded domains

$$\begin{array}{l} -\Delta u = |v|^{2^*-2}v \\ -\Delta v = |u|^{2^*-2}u \\ u, v \in D_0^{1,2}(\Omega_*); \end{array} \qquad (*)$$

and

$$\begin{array}{l} -\Delta u = \gamma v + |v|^{2^*-2}v \\ -\Delta v = \lambda u + |u|^{2^*-2}u \\ u, v \in H_0^1(\Omega_{**}), \end{array} \qquad (**)$$

where $\Omega_* = \mathbb{R}^N \setminus E$ with $E = \bigcup_{a\in\mathbb{Z}^N} a + \omega_*$ for a domain containing the origin $\omega_* \subset \overline{\omega_*} \subset B(0, 1/2)$ the open ball centered at the origin of radius 1/2; and $\Omega_{**} = \mathbb{R}^l \times \omega_{**}$ is a cylindrical domain with ω_{**} a bounded domain of $\mathbb{R}^{N-l}$ for $1 \le l \le N-1$, and $0 < \gamma, \lambda < \lambda_1(\Omega_{**})$, with $\lambda_1(\Omega_{**})$ the best constant in the Poincaré Inequality:

$$\lambda_1(\Omega_{**}) := \inf \left\{ \int_{\Omega_{**}} |\nabla u|^2 \, dx \ : \ u \in H_0^1(\Omega_{**}), \int_{\Omega_{**}} u^2 \, dx = 1 \right\}.$$

More precisely, using variational methods, we shall establish our following main results:

Theorem 1.1 *Problem* $(*)$ *has a non trivial solution.*

Theorem 1.2 *If* $\gamma, \lambda \in]0, \lambda_1(\Omega_{**})[$, *then Problem* $(**)$ *has a non trivial solution.*

This type of problems on bounded domains were studied in the subcritical growth case by Husholf and van der Vorst [8] using the Indefinite Functional Theorem due to Benci and Rabinowitz [1]; and by Felmer and Wang [7] who obtained multiplicity results using Galerkin type methods. The critical growth case was studied by Husholf, Mitidieri and van der Vorst [9] where they used a dual formulation due to Clarke and Ekeland [3]. To our knowledge, there are no results in the literature establishing the existence of solutions to these problems in unbounded domains.

Observe that these problems have a variational structure. Indeed, they constitute the Euler-Lagrange equations for the functionals:

$$\varphi(u, v) := \int_{\Omega_*} \nabla u \cdot \nabla v - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} \, dx$$

$$\varphi_1(u, v) := \int_{\Omega_{**}} \nabla u \cdot \nabla v - \frac{\gamma u^2}{2} - \frac{\lambda v^2}{2} - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} \, dx,$$

respectively. Moreover, the first part of these functionals is a strongly indefinite operator. In particular, the functionals have the form

$$\frac{\|Q(u,v)\|^2}{2} - \frac{\|P(u,v)\|^2}{2} - \psi(u,v),$$

where (u,v) belongs to a Hilbert space $X = Y \oplus Z$ with Y and Z infinite dimensional subspaces, and P and Q being respectively the orthogonal projections on Y and Z. Therefore, the proofs of our main results can not rely on classical min-max results. Indeed, the classical min-max results use the fact that the functional satisfies appropriate lower and upper bounds on suitable sets S, M, with $S \subset Y$ linking M, and where Y is finite dimensional. The linking between S and M is established using the Brouwer degree theory.

An other difficulty arises from the fact that the functionals φ and φ_1 are invariant under $\mathbb{Z}^N$-translations and $\mathbb{R}^l$-translations respectively. Therefore, the Palais-Smale condition fails at all critical levels. Indeed, $\varphi(u,v) = \varphi(\hat{u},\hat{v})$ and $\varphi_1(u,v) = \varphi_1(\tilde{u},\tilde{v})$ where $(\hat{u},\hat{v})(x) = (u,v)(x+a)$ for some $a \in \mathbb{Z}^N$, and $(\tilde{u},\tilde{v})$ $(y,z) = (u,v)(y+b,z)$ for $(y,z) \in \mathbb{R}^l \times \omega_{**}$, and some $b \in \mathbb{R}^l$.

Here, we shall use concentration-compactness techniques and we shall apply a generalized linking theorem due to Krysewski and Szulkin [10] for suitable functional defined on a Hilbert space $X = Y \oplus Z$ where Y and Z could be infinite dimensional.

The paper is organized as follows. In the next section, we recall some results and present some technical lemmas on Sobolev spaces that will be used in the following. The proofs of Theorems 1.1 and 1.2 are presented in section 4, while properties of the functional φ are studied in section 3.

2 Preliminaries and main technical lemma

In the sequel $N \geq 3$ and $2^* := 2N/(N-2)$. We consider the Hilbert space

$$D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) \ : \ \nabla u \in L^2(\mathbb{R}^N)\},$$

endowed with the inner product

$$\int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla v(x)\, dx,$$

and the associated norm noted $\|u\|$. The Sobolev Imbedding Theorem asserts that the imbedding

$$D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$$

is continuous.

For a domain $\Omega \subset \mathbb{R}^N$, we denote by $D_0^{1,2}(\Omega)$ the closure of $D(\Omega)$ in $D^{1,2}(\mathbb{R}^N)$, where $D(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω. Obviously $D^{1,2}(\mathbb{R}^N) = D_0^{1,2}(\mathbb{R}^N)$, and when the Poincaré Inequality is satisfied, $D_0^{1,2}(\Omega) = H_0^1(\Omega)$.

Let us recall that the imbedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, but we have the following well known result:

Lemma 2.1 *If $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, then $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$.*

The reader can consult [14] for a proof of this result, and Wang and Willem [13] for a generalization.

The following lemma is a corollary in cylindrical domains of a result due to Ramos, Wang and Willem [12] in $\mathbb{R}^N$. It gives sufficient conditions to ensure the convergence to 0 in $L^{2^*}(\mathbb{R}^N)$ of a sequence in $H^1_0(\Omega_{**})$. This type of results was firstly established by P. L. Lions [11] for an exponent $p < 2^*$. See also Colin [4] or [5] for a similar result in weighted spaces on a cylindrical domain.

Lemma 2.2 (Ramos, Wang and Willem [12]) *Let $r > 0$. If (u_n) is bounded in $H^1_0(\Omega_{**})$ and if*

$$\sup_{x\in\Omega_{**}} \int_{B(x,r)} |u_n|^{2^*}\, dx \to 0 \quad \text{as } n \to \infty$$

then $u_n \to 0$ in $L^{2^*}(\Omega_{**})$.

Here, we establish an analogous result in $D^{1,2}_0(\Omega_*)$ that we shall use in the sequel.

Lemma 2.3 *Let $r \geq \sqrt{N}$, and $(u_n) \subset D^{1,2}_0(\Omega_*)$ be a bounded sequence. If*

$$\sup_{a\in\mathbb{Z}^N} \int_{B(a,r)\cap\Omega_*} |u_n|^{2^*} \to 0, \quad \text{when } n \to \infty,$$

then $u_n \to 0$ in $L^{2^*}(\Omega_*)$.

Proof. Let $u \in D(\Omega_*)$, and $a \in \mathbb{Z}^N$. By the invariance by $\mathbb{Z}^N$ translations, we may assume that $a = 0$. Denote $U = B(0,r)\backslash\omega_*$, and define $H : \overline{U} \to \mathbb{R}^N$ by

$$H(x) = \left(-1 + \frac{2r}{|x|}\right) x.$$

Let us denote $W = H(U)$, and $V = W \cup (\overline{B(0,r)}\backslash\omega_*)$. For every $\phi \in D(V)$, we obtain by the Divergence Theorem

$$\begin{aligned}
&\int_W u(H^{-1}(x))\frac{\partial\phi}{\partial x_i}(x)\,dx\\
&\quad = -\int_W \phi(x)\frac{\partial}{\partial x_i}\big(u(H^{-1}(x))\big)\,dx + \int_{\partial W} u(H^{-1}(x))\phi(x)\nu\,dS\\
&\quad = -\int_W \phi(x)\frac{\partial}{\partial x_i}\big(u(H^{-1}(x))\big)\,dx - \int_{\partial B(0,r)} u(H^{-1}(x))\phi(x)\nu\,dS\\
&\qquad + \int_{H(\partial\omega_*)} u(H^{-1}(x))\phi(x)\nu\,dS\\
&\quad = -\int_W \phi(x)\frac{\partial}{\partial x_i}\big(u(H^{-1}(x))\big)\,dx - \int_{\partial B(0,r)} u(x)\phi(x)\nu\,dS;
\end{aligned}$$

and similarly

$$\int_U u(x)\frac{\partial\phi}{\partial x_i}(x)\,dx = -\int_U \phi(x)\frac{\partial u}{\partial x_i}(x)\,dx + \int_{\partial B(0,r)} u(x)\phi(x)\nu\,dS.$$

Therefore, if we define $u^*: V \to \mathbb{R}$ by

$$u^*(x) = \begin{cases} u(x), & \text{if } x \in U,\\ u(H^{-1}(x)), & \text{otherwise,}\end{cases}$$

combining the previous equalities gives

$$\begin{aligned}
&\int_V u^*(x)\frac{\partial\phi}{\partial x_i}(x)\,dx\\
&\qquad = -\int_V \phi(x)\left(\frac{\partial}{\partial x_i}u(x)\chi_U(x) + \frac{\partial}{\partial x_i}\big(u(H^{-1}(x))\big)\chi_W(x)\right)\,dx. \qquad (2.1)
\end{aligned}$$

Also, by the change of variable formula

$$\begin{aligned}
\int_W \left|\nabla_x u(H^{-1}(x))\right|^2\,dx &= \int_U |\nabla_x u(y)|^2\,|J_H(y)|\,dy\\
&= \int_U |\nabla_y u(y)\cdot J_{H^{-1}}(y)|^2\,|J_H(y)|\,dy \qquad (2.2)\\
&\le c\int_U |\nabla_y u(y)|^2\,dy,
\end{aligned}$$

with c independant of u. Thus, from (2.1) and (2.2), we deduce that $u^* \in H_0^1(V)$, and

$$\|\nabla u^*\|_{L^2(V)} \le c\|\nabla u\|_{L^2(U)} = c\|\nabla u\|_{L^2(U\cap\Omega_*)}.$$

This inequality, the fact that $u = u^*$ on U, the Sobolev Imbedding Theorem, and the Poincaré Inequality applied to the function u^* lead to

$$\int_{U\cap\Omega_*} |u|^{2^*}\,dx \le C\left(\int_{U\cap\Omega_*} |\nabla u|^2\,dx\right)^{2^*/2}.$$

Using the $\mathbb{Z}^N$ invariance and a density argument, we obtain for every $a \in \mathbb{Z}^N$ and for every $u \in D_0^{1,2}(\Omega_*)$,

$$\int_{B(a,r)\cap\Omega_*} |u|^{2^*}\,dx \le C\left(\int_{B(a,r)\cap\Omega_*} |\nabla u|^2\,dx\right)^{2^*/2},$$

and hence, for any $\lambda > 0$,

$$\int_{B(a,r)\cap\Omega_*} |u|^{2^*}\,dx \le C^\lambda\left(\int_{B(a,r)\cap\Omega_*} |\nabla u|^2\,dx\right)^{(2^*/2)\lambda}\left(\int_{B(a,r)\cap\Omega_*} |u|^{2^*}\,dx\right)^{1-\lambda}.$$

Choosing $\lambda = 2/2^*$, we may write

$$\int_{\Omega_*} |u|^{2^*}\,dx \le C_0\left(\int_{\Omega_*} |\nabla u|^2\,dx\right)\sup_{a\in\mathbb{Z}^N}\left(\int_{B(a,r)\cap\Omega_*} |u|^{2^*}\,dx\right)^{(2^*-2)/2^*},$$

since $\Omega_* \subset \cup_{a\in\mathbb{Z}^N} B(a,r)$. Hence, by the assumptions of the lemma, we conclude that $u_n \to 0$ in $L^{2^*}(\Omega_*)$. □

For sake of completeness, we state a corollary of a result of Kryszewski and Szulkin [10] that will be used in what follows (see also [14]).

Let $X = Y \oplus Z$ be a Hilbert space with Y a separable subspace of X which could be infinite dimensional and $Z := Y^\perp$. Let $P : X \to Y, Q : X \to Z$ be the orthogonal projections. Now, let $\rho > r > 0$ and let $z \in Z$ be such that $\|z\| = 1$. Define

$$M := \{u = y + \lambda z \ :\ \|u\| \le \rho,\ \lambda \ge 0,\ y \in Y\}, \tag{2.3}$$
$$M_0 := \{u = y + \lambda z \ :\ y \in Y,\ \|u\| = \rho \text{ and } \lambda \ge 0 \text{ or } \|u\| \le \rho \text{ and } \lambda = 0\}, \tag{2.4}$$
$$M_1 := \{u \in Z \ :\ \|u\| = \rho\}. \tag{2.5}$$

Theorem 2.4 (Kryszewski-Szulkin, 1998). *Let $\psi \in C^1(X,\mathbb{R})$ be weakly sequentially lower semicontinuous, bounded below and such that ψ' is weakly sequentially continuous. If*

$$\varphi(u) := \frac{\|Qu\|^2}{2} - \frac{\|Pu\|^2}{2} - \psi(u)$$

satisfies

$$b := \inf_{M_1}\varphi > 0 = \sup_{M_0}\varphi, \quad d := \sup_M \varphi < \infty, \tag{2.6}$$

then there exists $c \in [b, d]$ and a sequence $(u_n) \subset X$ such that

$$\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0.$$

3 Properties of the functional φ

In this section, we establish general results related to the functional φ mentionned in the introduction. Let Ω be a domain in $\mathbb{R}^N$. Denote $X := D_0^{1,2}(\Omega) \times D_0^{1,2}(\Omega)$ the Hilbert space endowed with the inner product:

$$\langle (u, v), (u_1, v_1) \rangle := \int_\Omega \nabla u \cdot \nabla u_1 + \nabla v \cdot \nabla v_1 \, dx, \tag{3.1}$$

If we set

$$Y := \{(-v, v) \in X\} \quad \text{and} \quad Z := \{(u, u) \in X\}, \tag{3.2}$$

it is easy to check that $X = Y \oplus Z$ since

$$(u, v) = \frac{1}{2}(u + v, u + v) + \frac{1}{2}(-v + u, v - u).$$

Let us denote by P (resp. Q) the projection of X onto Y (resp. Z).

Define the functional $\varphi : X \to \mathbb{R}$ by

$$\begin{aligned} \varphi(u, v) &:= \int_\Omega \nabla u \cdot \nabla v - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} \, dx \\ &= \frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v), \end{aligned}$$

where

$$\psi(u, v) := \int_\Omega \frac{|u|^{2^*}}{2^*} + \frac{|v|^{2^*}}{2^*} \, dx.$$

Lemma 3.1 *The function ψ is C^1 and weakly sequentially lower semi-continuous. Moreover, for every $(u, v), (w, z) \in X$,*

$$\langle \varphi'(u, v), (w, z) \rangle = \int_\Omega \nabla u \cdot \nabla z + \nabla v \cdot \nabla w - |u|^{2^*-2} uw - |v|^{2^*-2} vz \, dx.$$

Proof. It is clear that ψ is weakly sequentially lower semi-continuous since the imbedding $X \hookrightarrow L^{2^*}(\Omega) \times L^{2^*}(\Omega)$ is linear continuous and the norm on $L^{2^*}(\Omega)$ is weakly sequentially continuous.

Let $(u, v), (w, z) \in X$. For $x \in \Omega$ and $|t| \in]0, 1[$, there exists $\lambda \in]0, 1[$ such that

$$\begin{aligned} \frac{\big||u(x) + tw(x)|^{2^*} - |u(x)|^{2^*}\big|}{2^*|t|} &= \big||u(x) + \lambda tw(x)|^{2^*-2}(u(x) + \lambda tw(x))w(x)\big| \\ &= |u(x) + \lambda tw(x)|^{2^*-1}|w(x)| \\ &\le \big(|u(x)| + |w(x)|\big)^{2^*-1}|w(x)|. \end{aligned}$$

The term on the right hand side is in L^1 by the Hölder inequality. The Lebesgue Dominated Convergence Theorem implies that

$$\lim_{t\to 0}\frac{1}{2^*|t|}\int_\Omega |u(x)+tw(x)|^{2^*}-|u(x)|^{2^*}\,dx=\int_\Omega |u(x)|^{2^*-2}u(x)w(x)\,dx.$$

Similarly,

$$\lim_{t\to 0}\frac{1}{2^*|t|}\int_\Omega |v(x)+tz(x)|^{2^*}-|v(x)|^{2^*}\,dx=\int_\Omega |v(x)|^{2^*-2}v(x)z(x)\,dx.$$

Now, assume that $(u_n,v_n)\to(u,v)$ in X. From the continuous imbedding $D_0^{1,2}(\Omega)\hookrightarrow L^{2^*}(\Omega)$, we deduce that

$$\left(|u_n|^{2^*-2}u_n,|v_n|^{2^*-2}v_n\right)\to\left(|u|^{2^*-2}u,|v|^{2^*-2}v\right)\quad\text{in } L^{2^*/(2^*-1)}\times L^{2^*/(2^*-1)},$$

and,

$$\begin{aligned}
&\bigg|\int_\Omega |u_n(x)|^{2^*-2}u_n(x)w(x)-|u(x)|^{2^*-2}u(x)w(x)\\
&\qquad+|v_n(x)|^{2^*-2}v_n(x)z(x)-|v(x)|^{2^*-2}v(x)z(x)\,dx\bigg|\\
&\le \|w\|_{L^{2^*}}\left(\int_\Omega \left||u_n(x)|^{2^*-2}u_n(x)-|u(x)|^{2^*-2}u(x)\right|^{2^*/2^*-1}dx\right)^{(2^*-1)/2^*}\\
&\qquad+\|z\|_{L^{2^*}}\left(\int_\Omega \left||v_n(x)|^{2^*-2}v_n(x)-|v(x)|^{2^*-2}v(x)\right|^{2^*/2^*-1}dx\right)^{(2^*-1)/2^*}\\
&\le k\Bigg(\left(\int_\Omega \left||u_n(x)|^{2^*-2}u_n(x)-|u(x)|^{2^*-2}u(x)\right|^{2^*/2^*-1}dx\right)^{(2^*-1)/2^*}\\
&\qquad+\left(\int_\Omega \left||v_n(x)|^{2^*-2}v_n(x)-|v(x)|^{2^*-2}v(x)\right|^{2^*/2^*-1}dx\right)^{(2^*-1)/2^*}\Bigg),
\end{aligned}$$

for every $(w,z)\in X$ such that $\|(w,z)\|\le 1$. This shows that the Gâteau derivative of ψ is continuous and hence ψ is C^1.

On the other hand,

$$\lim_{t\to 0}\frac{1}{|t|}\int_\Omega \nabla(u+tw)\cdot\nabla(v+tz)-\nabla u\cdot\nabla v\,dx=\langle(u,v),(z,w)\rangle.$$

This Gâteau derivative is obviously continuous; so φ is C^1 and

$$\langle\varphi'(u,v),(w,z)\rangle=\int_\Omega \nabla u\cdot\nabla z+\nabla v\cdot\nabla w-|u|^{2^*-2}uw-|v|^{2^*-2}vz\,dx.$$

□

Lemma 3.2 *The map ψ' is weakly sequentially continuous.*

Proof. Suppose that $(u_n, v_n) \rightharpoonup (u, v)$ in X. Thus, $\{u_n\}$ and $\{v_n\}$ are bounded in $L^{2^*}(\Omega)$ and hence $\{|u_n|^{2^*-2}u_n\}$ and $\{|v_n|^{2^*-2}v_n\}$ are bounded in $\left(L^{2^*}(\Omega)\right)'$. Lemma 2.1 implies that $u_n \to u$ and $v_n \to v$ in $L^2_{loc}(\Omega)$. Therefore, for every $w, z \in D(\Omega)$,

$$\int_\Omega |u_n|^{2^*-2}u_n w + |v_n|^{2^*-2}v_n z\,dx \to \int_\Omega |u|^{2^*-2}uw + |v|^{2^*-2}vz\,dx;$$

i.e. $\langle \psi'(u_n, v_n), (w, z)\rangle \to \langle \psi'(u, v), (w, z)\rangle$. Moreover, $\{\psi'(u_n, v_n)\}$ is bounded in X, so $\psi'(u_n, v_n) \rightharpoonup \psi'(u, v)$. □

Now, we want to show that φ satisfies the inequalities (2.6) of Kryszewski-Szulkin's result (Theorem 2.4). Let us fix $(z, z) \in Z$ such that $\|(z, z)\| = 1$.

Lemma 3.3 *There exists $r > 0$ such that*

$$b := \inf_{\substack{(u,u)\in Z\\ \|(u,u)\|=r}} \varphi(u, u) > 0 = \min_{\substack{(u,u)\in Z\\ \|(u,u)\|\le r}} \varphi(u, u). \tag{3.3}$$

Moreover, there exists $\rho > r$ such that

$$\max_{M_0} \varphi = 0 \quad \textit{and} \quad d := \sup_M \varphi < \infty, \tag{3.4}$$

where M and M_0 are given respectively by (2.3) *and* (2.4).

Proof. The Sobolev Imbedding Theorem regarding $D_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ implies directly (3.3) since for $(u, u) \in Z$,

$$\varphi(u, u) \ge \frac{\|(u, u)\|^2}{2} - C\|(u, u)\|^{2^*}.$$

Observe that on Y, we have

$$\varphi(-v, v) = \frac{-\|(-v, v)\|^2}{2} - \frac{2}{2^*}\int_\Omega |v|^{2^*}\,dx \le 0.$$

Denote by W the closure of $Y \oplus \mathbb{R}(z, z)$ in $L^{2^*}(\Omega) \times L^{2^*}(\Omega)$. Since there exists a continuous projection of W onto $\mathbb{R}(z, z)$ and all the norms are equivalent on the latter space, we have

$$\begin{aligned}\varphi((-v, v) + \lambda(z, z)) &= -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2}\|(z, z)\|^2\\ &\quad -\frac{1}{2^*}(|-v + \lambda z|^{2^*}_{L^{2^*}} + |v + \lambda z|^{2^*}_{L^{2^*}})\\ &\le -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2} - C\lambda^{2^*}.\end{aligned}$$

It follows that for $w \in Y \oplus \mathbb{R}(z,z)$

$$\varphi(w) \to -\infty \quad \text{whenever } \|w\| \to \infty$$

and so, for some $\rho > r$, $\max_{M_0} \varphi = 0$.

Finally, the Cauchy-Schwarz Inequality and the Sobolev Inequality imply that φ maps bounded sets into bounded sets, hence $\sup_M \varphi < \infty$. □

Lemma 3.4 *There exists $c \in [b,d]$ and a bounded sequence $\{(u_n, v_n)\}$ in X such that*

$$\varphi(u_n, v_n) \to c > 0, \quad \varphi'(u_n, v_n) \to 0. \tag{3.5}$$

Proof. It follows from Theorem 2.4 and Lemmas 3.1–3.3 that there exist $c \in [b,d]$ and a sequence $\{(u_n, v_n)\}$ in X satisfying (3.5).

Observe that

$$\begin{aligned}\varphi(u_n, v_n) - \frac{1}{2}\langle \varphi'(u_n, v_n), (u_n, v_n)\rangle &= \int_\Omega \nabla u_n \cdot \nabla v_n - \frac{|u_n|^{2^*}}{2^*} - \frac{|v_n|^{2^*}}{2^*} \\ &\quad - \frac{1}{2}\left(2\nabla u_n \cdot \nabla v_n - |u_n|^{2^*} - |v_n|^{2^*}\right) \\ &= \mu\left(\|u_n\|_{L^{2^*}}^{2^*} + \|v_n\|_{L^{2^*}}^{2^*}\right),\end{aligned}$$

with $\mu = 1/2 - 1/2^*$. So, for $\varepsilon > 0$, and $n \in \mathbb{N}$ large enough,

$$c + \varepsilon + \varepsilon\|(u_n, v_n)\| \geq \mu\left(\|u_n\|_{L^{2^*}}^{2^*} + \|v_n\|_{L^{2^*}}^{2^*}\right) \geq c - \varepsilon - \varepsilon\|(u_n, v_n)\|. \tag{3.6}$$

On the other hand,

$$\begin{aligned}\|Q(u_n, v_n)\|^2 &- \varepsilon\|Q(u_n, v_n)\| \\ &\leq \left|\|Q(u_n, v_n)\|^2 - \frac{1}{2}\langle \varphi'(u_n, v_n), (u_n + v_n, u_n + v_n)\rangle\right| \\ &= \int_\Omega (|u_n|^{2^*-2}u_n + |v_n|^{2^*-2}v_n)\left(\frac{u_n + v_n}{2}\right) \\ &\leq \|Q(u_n, v_n)\|_{L^{2^*}\times L^{2^*}}(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1}) \\ &\leq k\|Q(u_n, v_n)\|(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1}).\end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned}\|P(u_n, v_n)\|^2 &- \varepsilon\|P(u_n, v_n)\| \\ &\leq \left|-\|P(u_n, v_n)\|^2 - \frac{1}{2}\langle \varphi'(u_n, v_n), (u_n - v_n, v_n - u_n)\rangle\right| \\ &\leq k\|P(u_n, v_n)\|(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1}).\end{aligned} \tag{3.8}$$

Therefore,

$$\|Q(u_n, v_n)\| - \varepsilon \ \leq \ k(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1}), \tag{3.9}$$

$$\|P(u_n, v_n)\| - \varepsilon \ \leq \ k(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1}). \tag{3.10}$$

Adding (3.9) and (3.10), and combining the result with (3.6) leads to

$$\|(u_n, v_n)\| - 2\varepsilon \leq c_1 + c_2\|(u_n, v_n)\|^{(2^*-1)/2^*};$$

so, $\{(u_n, v_n)\}$ is bounded in X. □

4 Proofs of the main theorems

In this section, we present the proofs of our main Theorems. We start in proving Theorem 1.1; that is we establish the existence of a nontrivial solution to the following system of two coupled Poisson equations:

$$\begin{aligned} -\Delta u &= |v|^{2^*-2}v \\ -\Delta v &= |u|^{2^*-2}u \\ u,\, v &\in D_0^{1,2}(\Omega_*). \end{aligned} \tag{$*$}$$

Notice that we look for weak solutions to the problem $(*)$; that is, the problem $(*)$ allows a variationnal formulation and its solutions are critical points of φ for $\Omega = \Omega_*$.

Proof of Theorem 1.1 Take $\Omega = \Omega_*$. By lemma 3.4, there exists a bounded sequence $(u_n, v_n) \subset X$ satisfying (3.5)

Now, let us assume that:

$$\delta_1 \ := \ \overline{\lim_{n\to\infty}} \sup_{a\in\mathbb{Z}^N} \int_{B(a,\sqrt{N})} |u_n|^{2^*} = 0$$

$$\delta_2 \ := \ \overline{\lim_{n\to\infty}} \sup_{a\in\mathbb{Z}^N} \int_{B(a,\sqrt{N})} |v_n|^{2^*} = 0$$

then Lemma 2.3 implies that $u_n, v_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$. This fact and (3.6) lead to a contradiction since $c > 0$.

Therefore, we must have $\delta := \max\{\delta_1, \delta_2\} > 0$. Taking a subsequence if necessary, we may assume the existence of $a_n \in \mathbb{Z}^N$ such that

$$\int_{B\left(a_n,\sqrt{N}\right)} |u_n|^{2^*} + |v_n|^{2^*}\, dx > \frac{\delta}{2}.$$

The sequence $(\hat{u}_n, \hat{v}_n)$ defined by $\hat{u}_n(x) := u_n(x + a_n)$ and $\hat{v}_n(x) := v_n(x + a_n)$ is such that

$$\int_{B\left(0,\sqrt{N}\right)} |\hat{u}_n|^{2^*} + |\hat{v}_n|^{2^*}\, dx > \frac{\delta}{2}$$

and satisfies (3.5) by $\mathbb{Z}^N$ invariance. Taking again a subsequence, if needed, we may assume that

$$(\hat{u}_n, \hat{v}_n) \rightharpoonup (u, v) \quad \text{in } X.$$

Since $\hat{u}_n \to u$, $\hat{v}_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$, then $(u, v) \neq 0$. Finally, the weakly sequentially continuity of φ' gives

$$\|\varphi'(u, v)\| \leq \liminf_{n\to\infty} \|\varphi'(\hat{u}_n, \hat{v}_n)\| = 0.$$

Consequently (u, v) is a non trivial solution of $(*)$. □

Now, we shall study the following problem in $\Omega_{**} := \mathbb{R}^l \times \omega_{**}$ a cylindrical domain with ω_{**} a bounded domain of $\mathbb{R}^{N-l}$ with for $1 \leq l \leq N-1$:

$$\begin{aligned} -\Delta u &= \gamma v + |v|^{2^*-2} v \\ -\Delta v &= \lambda u + |u|^{2^*-2} u \\ u, v &\in H^1_0(\Omega_{**}), \end{aligned} \tag{$**$}$$

where $0 < \gamma, \lambda < \lambda_1(\Omega_{**})$, with $\lambda_1(\Omega_{**})$ the best constant in the Poincaré Inequality. It is shown in [6] that $\lambda_1(\Omega_{**}) > 0$.

The Poincaré Inequality implies that the usual norm on $X := H^1_0(\Omega_{**}) \times H^1_0(\Omega_{**})$ is equivalent to the norm induced by the scalar product defined in (3.1). If we set Y and Z as in (3.2), and we denote by P (resp. Q) the projection of X onto Y (resp. Z), the problem $(**)$ allows a variationnal formulation and its (weak) solutions are critical points of the functional $\varphi_1 : X \to \mathbb{R}$ defined by

$$\begin{aligned} \varphi_1(u, v) &:= \int_{\Omega_{**}} \nabla u \cdot \nabla v - \frac{\gamma u^2}{2} - \frac{\lambda v^2}{2} - \frac{|u|^{2^*}}{2^*} - \frac{|v|^{2^*}}{2^*} \, dx \\ &= \frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v) - \hat{\psi}(u, v) \\ &= \varphi(u, v) - \hat{\psi}(u, v), \end{aligned}$$

where

$$\hat{\psi}(u, v) := \int_{\Omega_{**}} \gamma \frac{u^2}{2} + \lambda \frac{v^2}{2} \, dx.$$

and where φ and ψ are defined at the beginning of the previous section. It is easy to see that $\hat{\psi}$ is C^1, weakly sequentially lower semi-continuous, and $\hat{\psi}'$ is weakly sequentially continuous. Also,

$$\langle \hat{\psi}'(u, v), (w, z) \rangle = \int_{\Omega_{**}} \gamma u w + \lambda v z \, dx.$$

The following result establishes the existence of a bounded Palais-Smale sequence of φ_1. However, writing $x := (y, z)$, where $y \in \mathbb{R}^l$ and $z \in \omega_{**}$, we observe that for each $(u, v) \in X$ and each $a \in \mathbb{R}^l$

$$\varphi_1(u, v) = \varphi_1(\hat{u}, \hat{v})$$

where $\hat{u} := u(y+a,z), \hat{v} := v(y+a,z)$. So φ_1 is invariant under $\mathbb{R}^l$-translations and consequently, the Palais-Smale condition fails at all critical levels.

Lemma 4.5 *There exists $c > 0$ and a bounded sequence $\{(u_n, v_n)\}$ in X such that*

$$\varphi_1(u_n, v_n) \to c > 0, \quad \varphi_1'(u_n, v_n) \to 0. \tag{4.1}$$

Proof. Set $\beta := (\gamma + \lambda)/2$. From the Poincaré Inequality and the Sobolev Imbedding Theorem, it follows that there is a constant $C \geq 0$ such that for every $(u, u) \in Z$

$$\varphi_1(u,u) \geq \left(\frac{\lambda_1(\Omega_{**}) - \beta}{2\lambda_1(\Omega_{**})}\right) \|(u,u)\|^2 - C\|(u,u)\|^{2^*}.$$

Therefore, there exists $r > 0$ such that

$$\inf_{\substack{(u,u)\in Z \\ \|(u,u)\|=r}} \varphi_1(u,u) > 0 = \min_{\substack{(u,u)\in Z \\ \|(u,u)\|\leq r}} \varphi_1(u,u). \tag{4.2}$$

The fact that for $(u, v) \in X$

$$\varphi_1(u,v) \leq \varphi(u,v)$$

allows us to use the arguments contained in the proof of Lemma 3.3 in order to show that there exists $\rho > r$ such that

$$\max_{M_0} \varphi_1 = 0 \quad \text{and} \quad \sup_M \varphi_1 < \infty, \tag{4.3}$$

where M and M_0 are given respectively by (2.3) and (2.4).

Theorem 2.4 ensures the existence of a sequence $\{(u_n, v_n)\}$ in X satisfying (4.1) for some $c > 0$. It remains to show that the sequence is bounded. Let us observe that

$$\varphi_1(u_n,v_n) - \frac{1}{2}\big\langle \varphi_1'(u_n,v_n), (u_n,v_n)\big\rangle = \varphi(u_n,v_n) - \frac{1}{2}\big\langle \varphi'(u_n,v_n), (u_n,v_n)\big\rangle,$$

which implies (3.6).

Without lost of generality, we may assume that $\lambda \geq \gamma$. A straightforward computation leads to

$$\begin{aligned}
\frac{1}{2}\big\langle \hat{\psi}'(u_n,v_n), (u_n+v_n, u_n+v_n)\big\rangle &\leq \frac{\lambda}{2}\|u_n+v_n\|_{L^2}^2 \\
&\leq \frac{\lambda}{2\lambda_1(\Omega_{**})}\|\nabla(u_n+v_n)\|_{L^2}^2 \\
&= \frac{\lambda}{4\lambda_1(\Omega_{**})}\|(u_n+v_n, u_n+v_n)\|^2 \\
&= \frac{\lambda}{\lambda_1(\Omega_{**})}\|Q(u_n,v_n)\|^2.
\end{aligned}$$

Similarly, we have

$$\frac{1}{2}\left\langle \hat{\psi}'(u_n, v_n), (u_n - v_n, v_n - u_n) \right\rangle \leq \frac{\lambda}{\lambda_1(\Omega_{**})} \|P(u_n, v_n)\|^2.$$

In combining respectively the last two equations with (3.7) and (3.8), the inequalities (3.9) and (3.10) become respectively:

$$\begin{aligned} \left(\frac{\lambda_1(\Omega_{**}) - \lambda}{\lambda_1(\Omega_{**})}\right) \|Q(u_n, v_n)\| - \varepsilon &\leq k \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right), \\ \left(\frac{\lambda_1(\Omega_{**}) - \lambda}{\lambda_1(\Omega_{**})}\right) \|P(u_n, v_n)\| - \varepsilon &\leq k \left(\|u_n\|_{L^{2^*}}^{2^*-1} + \|v_n\|_{L^{2^*}}^{2^*-1} \right), \end{aligned}$$

and the conclusion follows as in the proof of Lemma 3.4. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2 Let $(u_n, v_n) \subset X$ be a bounded sequence given in Lemma 4.5, and define

$$\begin{aligned} \delta_1 &:= \overline{\lim_{n\to\infty}} \sup_{x \in \Omega_{**}} \int_{B(x,1)} |u_n|^{2^*}, \\ \delta_2 &:= \overline{\lim_{n\to\infty}} \sup_{x \in \Omega_{**}} \int_{B(x,1)} |v_n|^{2^*}. \end{aligned}$$

In arguing as in the proof of Theorem 1.1 with Lemma 2.2, we show that $\delta := \max\{\delta_1, \delta_2\} > 0$, and hence we deduce the existence of a sequence $x_n = (y_n, z_n) \in \Omega_{**}$ such that

$$\int_{B(x_n,1)} |u_n|^{2^*} + |v_n|^{2^*} \, dx > \frac{\delta}{2}.$$

The sequence $(\hat{u}_n, \hat{v}_n)$ defined by $\hat{u}_n(x) := u_n(y + y_n, z)$ and $\hat{v}_n(x) := v_n(y + y_n, z)$ is such that

$$\int_{B((0,z_n),1)} |\hat{u}_n|^{2^*} + |\hat{v}_n|^{2^*} \, dx > \frac{\delta}{2},$$

and satisfies (4.1) by $\mathbb{R}^l$ invariance. By letting $R := \max_{z \in \omega_{**}} |z|$, we obtain

$$\int_{B(0,R+1)} |\hat{u}_n|^{2^*} + |\hat{v}_n|^{2^*} \, dx > \frac{\delta}{2}.$$

Finally, taking a subsequence, if needed, we may assume that

$$(\hat{u}_n, \hat{v}_n) \rightharpoonup (u, v) \quad \text{in } X.$$

As in the proof of Theorem 1.1, we conclude that (u, v) is a non trivial solution of $(**)$. □

Acknowledgement

The first author wishes to thank Marlène Frigon and the Department of Mathematics and Statistic of the Université de Montréal for the hospitality during his postdoctoral studies.

References

[1] V. BENCI and P. H. RABINOWITZ, Critical point theorems for indefinite functionals, *Invent. Math.* **52** (1979), 241–273.

[2] H. BREZIS, "Analyse fonctionnelle, théorie et applications", Masson, Paris, 1983.

[3] F. CLARKE and I. EKELAND, Hamiltonian trajectories having prescribed minimal period, *Comm. Pure Appl. Math.* **33** (1980), 103–116.

[4] F. COLIN, Inégalités de Hardy sur des domaines non bornés, Thèse de doctorat (Ph.D.), Université de Sherbrooke, Sherbrooke, 2002.

[5] F. COLIN, Lemmes de décomposition appliqués à des inégalités de type Hardy-Sobolev, **23** (2005), 181–206. Potential Analysis.

[6] M. J. ESTEBAN, Nonlinear elliptic problems in strip-like domains: symmetry of positive vortex rings, *Nonlinear Anal.* **7** (1983), 365–379.

[7] P. FELMER and Z. Q. WANG, Multiplicity for symmetric indefinite functionals: application to Hamiltonian and elliptic systems, *Topol. Methods Nonlinear Anal.* **12** (1998), 207–226.

[8] J. HULSHOF and R. VAN DER VORST, Differential systems with strongly indefinite variational structure, *J. Funct. Anal.* **114** (1993), 32–58.

[9] J. HULSHOF, E. MITIDIERI and R. VAN DER VORST, Strongly indefinite systems with critical Sobolev exponents, *Trans. Amer. Math. Soc.* **350** (1998), 2349–2365.

[10] W. KRYSZEWSKI and A. SZULKIN, Generalized linking theorem with an application to a semilinear Schrödinger equation, *Adv. Differential Equations* **3** (1998), 441–472.

[11] P. -L. LIONS, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, *Ann. Inst. Henri-Poincaré* **1** (1984), 223–283.

[12] M. RAMOS, Z. -Q. WANG, and M. WILLEM, Positive solutions for elliptic equations with critical growth in unbounded domains, in Calculus of variations and differential equations, Technion 1988, Chapman and Hall, Boca Raton, 1999, 192-199.

[13] Z. -Q. WANG and M. WILLEM, Singular minimization problems, *J. Differential Equations* **161** (2000), 307–320.

[14] M. WILLEM, Minimax Theorems, Birkhauser, Boston, 1996.

Received 27 January 2004; accepted 27 July 2004