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EXISTENCE THEOREMS FOR SYSTEMS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we establish the existence of solutions to systems of third order differential equations. A notion of solution-tube to these problems is introduced. This notion extends to systems the notion of upper and lower solutions of third order differential equations. Our proofs rely on the Leray-Schauder degree and the theory of multivalued mappings.

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1. INTRODUCTION

In this paper we establish existence results for the following system of third order differential equations:

$$\begin{aligned} &x'''(t) = f(t, x(t), x'(t), x''(t)), \quad \text{a.e. } t \in [0,1], \\ &x(0) = x_0, \ x' \in (BC). \end{aligned} \tag{1.1}$$

Here $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ is a Carathéodory function, $x_0 \in \mathbb{R}^n$ and (BC) denotes one of the following boundary conditions:

$$\begin{aligned} A_0 x(0) - \rho_0 x'(0) = r_0, \\ A_1 x(1) + \rho_1 x'(1) = r_1; \end{aligned} \tag{1.2}$$

$$\begin{aligned} x(0) = x(1), \\ x'(0) = x'(1); \end{aligned} \tag{1.3}$$

where for $i = 0, 1,$ A_i is a $n \times n$ matrix such that there exists $\alpha_i \geq 0$ satisfying $\langle x, A_i x\rangle \geq \alpha_i \|x\|$ for all $x \in \mathbb{R}^n$; $\rho_i = 0, 1$; $\alpha_i + \rho_i > 0$.

The literature counts only few papers on existence theorems for third order nonlinear systems of differential equations. Let us mention the result of Miao [17] for a

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general N^{th} order system ($N \geq 2$) and relying on generalized upper and lower solutions. Other existence results for N^{th} order systems may be found in [15] and [16].

In the particular case of a boundary value problem for a single third order differential equation, more results were obtained. We refer to [3, 4, 5, 19, 20, 21, 22, 23] and the references therein. One of our results generalizes to systems and in the scalar case too, a theorem due to Grossinho and Minhós [12] in the particular case of the boundary condition (1.2). Let us precise that in [12], it is not assumed that $A_0 \geq 0$ and $A_1 \geq 0$ but $x_0 = 0$.

In this paper, we introduce a notion of solution-tube. This notion is inspired by the definition of solution-tube introduced in [7] for systems of second order differential equations. Our notion of solution-tube for third order systems of differential equations is related to the definitions of upper and lower solutions to the third order differential equation. We recall this definition used in [12] for (1.1), (1.2).

Definition 1.1. For $n = 1$ and $x_0 = 0$, a function $\alpha \in C^3([0,1])$ is called a *lower solution of* (1.1), (1.2) if

(i) $\alpha'''(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t))$ for every $t \in [0,1]$;
(ii) $\alpha(0) = 0$;
(iii) $A_0\alpha'(0) - \rho_0\alpha''(0) \leq r_0$ and $A_1\alpha'(0) + \rho_1\alpha''(0) \leq r_1$.

A function $\beta \in C^3([0,1])$ is called an *upper solution of* (1.1), (1.2) if it satisfies (i)–(iii) with the reversed inequalities.

Let us mention that other attempts to generalize the method of upper and lower solutions to systems of third order differential equations were made, see for example [17]. However, our notion of solution-tube is much simpler than the other approaches.

This paper is organized as follows. We start with some notations, definitions and results which are used throughout this paper. The third section presents a general existence theorem for the system (1.1) on which our further results will be based. In section 4, other existence results are obtained under some growth conditions. In the last section, an existence theorem is established under a Nagumo-Wintner type growth condition. To this aim, we work with differential inclusions and hence, we use the theory of multivalued mappings.

2. PRELIMINARIES

In this section, we state notations, definitions, and results which are used throughout this paper. We denote $\langle , \rangle$ the scalar product, and $\| \cdot \|$ the Euclidian norm in $\mathbb{R}^n$. The Banach space of k-times continuously differentiable functions x is denoted by $C^k([0,1],\mathbb{R}^n)$ with the norm

$$\|x\|_k = \max\{\|x\|_0, \|x'\|_0, \dots, \|x^{(k)}\|_0\}, \quad \text{where } \|x\|_0 = \max\{\|x(t)\| : t \in [0,1]\}.$$

For $k \geq 1$, the Sobolev space of functions in $C^{k-1}([0,1],\mathbb{R}^n)$ with the derivative being absolutely continuous is denoted by $W^{k,1}([0,1],\mathbb{R}^n)$. We define $C_{x_0}([0,1],\mathbb{R}^n) = \{x \in C([0,1],\mathbb{R}^n) : x(0) = x_0\}$, $C^k_B([0,1],\mathbb{R}^n)$ (resp. $W^{k,1}_B([0,1],\mathbb{R}^N)$) the set of functions $x \in C^k([0,1],\mathbb{R}^n)$ (resp. $W^{k,1}([0,1],\mathbb{R}^N)$) satisfying the condition $x \in (BC)$, and $C^{k+1}_{x_0,B}([0,1],\mathbb{R}^N)$ (resp. $W^{k+1,1}_{x_0,B}([0,1],\mathbb{R}^n)$) denotes the set of functions $x \in C^{k+1}([0,1],\mathbb{R}^n)$ (resp. $W^{k+1,1}([0,1],\mathbb{R}^N)$) satisfying the boundary conditions $x(0) = x_0$ and $x^{(k)} \in (BC)$. Let $L^1([0,1],\mathbb{R}^n)$ denote the space of integrable functions, with the usual norm $\|\cdot\|_{L^1}$.

Let X, Y be topological spaces and $G : X \to Y$ a multivalued map. G is *upper semi-continuous* (u.s.c.) if $\{x \in X : G(x) \cap C \neq \emptyset\}$ is closed for every closed set $C \subset Y$ and it is *compact* if $G(X) = \bigcup_{x\in X} G(x)$ is relatively compact. For X metrizable, G is *completely continuous* if for all bounded subsets $A \subset X$, $\overline{G(A)}$ is compact. Let Ω be a measurable space, we say that a multivalued map $G : \Omega \to X$ is *measurable* if $\{t \in \Omega : G(t) \cap C \neq \emptyset\}$ is measurable for every closed set $C \subset X$.

Let X be a subset of $\mathbb{R}^m$. We say that a single-valued map $f : [0,1] \times X \to \mathbb{R}^n$ (resp. $G : [0,1] \times X \to \mathbb{R}^n$ a multivalued map with nonempty, closed, convex values) is a *Carathéodory function* if: (i) for every $x \in X$, the function $t \mapsto f(t,x)$ (resp. $t \mapsto G(t,x)$) is measurable; (ii) the function $x \mapsto f(t,x)$ (resp. $x \mapsto G(t,x)$) is continuous (resp. upper semi-continuous) for almost every $t \in [0,1]$; (iii) for every $R > 0$, there exists a function $h_R \in L^1([0,1],[0,\infty))$ such that $\|f(t,x)\| \leq h_R(t)$ (resp. $\|G(t,x)\| \leq h_R(t)$, i.e. $\|v\| \leq h_R(t)$ for all $v \in G(t,x)$) for almost every $t \in [0,1]$ and for all x such that $\|x\| \leq R$.

A function $\mathcal{F} : C^k([0,1],\mathbb{R}^n) \times [0,1] \to L^1([0,1],\mathbb{R}^n)$ is *integrably bounded on bounded* if for every bounded subset $B \subset C^k([0,1],\mathbb{R}^n)$, there exists an integrable function $h_B \in L^1([0,1],[0,\infty))$ such that $\|\mathcal{F}(x,\lambda)(t)\| \leq h_B(t)$ for almost every $t \in [0,1]$ and for all $(x,\lambda) \in B \times [0,1]$. We associate to $\mathcal{F}$ an operator

$$N_{\mathcal{F}} : C^k([0,1],\mathbb{R}^n) \times [0,1] \to C_0([0,1],\mathbb{R}^n)$$

defined by

$$N_{\mathcal{F}}(x)(t) = \int_0^t \mathcal{F}(x,\lambda)(s)ds.$$

We now state the following result. To prove it, we could argue as in the proof of Theorem 1.3 in [11].

Theorem 2.1. *Let $\mathcal{F} : C^k([0,1],\mathbb{R}^n) \times [0,1] \to L^1([0,1],\mathbb{R}^n)$ be continuous and integrably bounded on bounded, then the associate operator $N_{\mathcal{F}}$ is continuous and completely continuous.*

We recall a change of variable rule for integrals and the Banach Lemma. Their proofs may be found respectively in [6] and [18].

Lemma 2.2. *Let $f \in W^{1,1}([a,b])$ such that $f([a,b]) \subset [c,d]$ and let $g : [c,d] \to \mathbb{R}$ be a Borel measurable function such that $g \in L^1([c,d])$ and $g(f)f' \in L^1([a,b])$. Then,*

$$\int_{f(a)}^{f(b)} g(s)ds = \int_a^b g(f(t))f'(t)dt.$$

Lemma 2.3. *Let E be a Banach space and $u : [0,1] \to E$ be an absolutely continuous function, then the measure of the set $\{t \in [0,1] : \ u(t) = 0 \text{ and } u'(t) \neq 0\}$ is zero.*

For sake of completeness, we state the following result which will be used later in this paper.

Theorem 2.4 (Kuratowski-Ryll-Nardzewski)**.** *Let X be a separable Banach space and $F : [0,1] \to X$ be a measurable multivalued map, then there exists a measurable function $f : [0,1] \to X$ such that $f(t) \in F(t)$ for almost every $t \in [0,1]$.*

3. GENERAL EXISTENCE THEOREM

We now introduce the notion of solution-tube. This definition will play a fundamental role in our general existence result and hence, in our other results.

Definition 3.1. Let $(v, M) \in W^{3,1}([0,1], \mathbb{R}^n) \times W^{3,1}([0,1], [0,\infty[)$. We say that (v, M) is a *solution-tube of* (1.1) if

(i) $M'(t) \geq 0$ for all $t \in [0,1]$;
(ii) $\langle y - v'(t), f(t,x,y,z) - v'''(t)\rangle + \|z - v''(t)\|^2 \geq M'(t)M'''(t) + (M''(t))^2$ for almost every $t \in [0,1]$ and for all $(x,y,z) \in \mathbb{R}^{3n}$ such that $\|x - v(t)\| \leq M(t)$, $\|y - v'(t)\| = M'(t)$, $\langle y - v'(t), z - v''(t)\rangle = M'(t)M''(t)$;
(iii) $v'''(t) = f(t, x, v'(t), v''(t))$ for almost every $t \in [0,1]$ such that $M'(t) = 0$ and for all $x \in \mathbb{R}^n$ such that $\|x - v(t)\| \leq M(t)$;
(iv) If (BC) denotes (1.2), $\|r_0 - (A_0 v'(0) - \rho_0 v''(0))\| \leq \alpha_0 M'(0) - \rho_0 M''(0)$, $\|r_1 - (A_1 v'(1) + \rho_1 v''(1))\| \leq \alpha_1 M'(1) + \rho_1 M''(1)$;
if (BC) denotes (1.3), then $v'(0) = v'(1)$, $M'(0) = M'(1)$ and $\|v''(1) - v''(0)\| \leq M''(1) - M''(0)$;
(v) $\|x_0 - v(0)\| \leq M(0)$.

We denote

$$T(v,M) = \{x \in C^1([0,1], \mathbb{R}^n) : \|x'(t) - v'(t)\| \leq M'(t),\ \|x(t) - v(t)\| \leq M(t)\ \forall t \in I\}.$$

Remark 3.2. In the scalar case ($n = 1$) and with $x_0 = 0$, consider the following assumptions:

(A) There exist $\alpha \leq \beta$ respectively lower and upper solutions of (1.1), (1.2) (see Definition 1.1).

(B) There exists (v, M) a solution-tube of (1.1), (1.2).

(C) There exist $\alpha \leq \beta$ respectively lower and upper solutions of (1.1), (1.2) such that
  (i) $\alpha'(t) \leq \beta'(t)$ for all $t \in [0,1]$;
  (ii) $f(t, \beta(t), y, z) \leq f(t, x, y, z) \leq f(t, \alpha(t), y, z)$ for all $t \in [0,1]$ and $(x, y, z) \in \mathbb{R}^{3n}$ such that $\alpha(t) \leq x \leq \beta(t)$.

It is easy to check that

if (B) *is satisfied with v and M of class C^3, $v(0) = 0$, and $M(0) = 0$, then* (A) *is satisfied.*

Indeed $\alpha = v - M$ and $\beta = v + M$ are respectively lower and upper solutions of (1.1), (1.2). The opposite implication is false.

It can also be verified that (B) is more general than (C) which is the assumption used in [12]; i.e,

if (C) *is satisfied then* (B) *is satisfied.*

Indeed, take $v = (\alpha+\beta)/2$ and $M = (\beta-\alpha)/2$. Observe that the opposite implication is false. In particular (B) does not imply (C)(ii) and $\alpha(0) = \beta(0) = 0$.

For the moment, we only assume the following hypothesis:

(H1) There exists a solution-tube (v, M) of (1.1).

We consider the following family of problems.

$$\begin{aligned} &x'''(t) - \epsilon x'(t) = f_\lambda^\epsilon(t, x(t), x'(t), x''(t)), \quad \text{a.e. } t \in [0,1], \\ &x(0) = x_0,\ x' \in (BC); \end{aligned} \tag{3.1}$$

where $\epsilon, \lambda \in [0,1]$ and $f_\lambda^\epsilon : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ is defined by

$$f_\lambda^\epsilon(t,x,y,z) = \begin{cases} \lambda\Big(\frac{M'(t)}{\|y-v'(t)\|} f_1(t,x,\hat{y},\tilde{z}) - \epsilon\hat{y}\Big) - \epsilon(1-\lambda)v'(t) & \\ \quad +\Big(1 - \frac{\lambda M'(t)}{\|y-v'(t)\|}\Big)\Big(v'''(t) + \frac{M'''(t)(y-v'(t))}{\|y-v'(t)\|}\Big) & \text{if } \|y - v'(t)\| > M'(t), \\ \lambda(f_1(t,x,y,z) - \epsilon y) - \epsilon(1-\lambda)v'(t) & \\ \quad +(1-\lambda)\Big(v'''(t) + \frac{M'''(t)}{M'(t)}(y - v'(t))\Big) & \text{otherwise;} \end{cases}$$

where (v, M) is the solution-tube of (1.1) given in (H1),

$$f_1(t,x,y,z) = \begin{cases} f(t,\overline{x},y,z) & \text{if } \|x - v(t)\| > M(t), \\ f(t,x,y,z) & \text{otherwise;} \end{cases} \tag{3.2}$$

$$\overline{x} = \frac{M(t)}{\|x - v(t)\|}\big(x - v(t)\big) + v(t), \tag{3.3}$$

$$\hat{y} = \frac{M'(t)}{\|y - v'(t)\|}\big(y - v'(t)\big) + v'(t), \tag{3.4}$$

$$\tilde{z} = z + \Big(M''(t) - \frac{\langle y - v'(t), z - v''(t)\rangle}{\|y - v'(t)\|}\Big)\Big(\frac{y - v'(t)}{\|y - v'(t)\|}\Big); \tag{3.5}$$

and where we mean

$$\frac{M'''(t)\big(y - v'(t)\big)}{M'(t)} = 0 \quad \text{on } \{t \in [0,1] : \|x'(t) - v'(t)\| = M'(t) = 0\}.$$

We will need the following lemma to find an a priori bound for solutions of (3.1).

Lemma 3.3. *Let (v, M) be a solution-tube of (1.1). If a function $x \in W^{3,1}_{x_0,B}([0,1], \mathbb{R}^n)$ satisfies*

$$\frac{\langle x'(t) - v'(t), x'''(t) - v'''(t)\rangle + \|x''(t) - v''(t)\|^2}{\|x'(t) - v'(t)\|} - \frac{\langle x'(t) - v'(t), x''(t) - v''(t)\rangle^2}{\|x'(t) - v'(t)\|^3} - \epsilon\|x'(t) - v'(t)\| \geq M'''(t) - \epsilon M'(t)$$

almost everywhere on $\{t \in [0,1] : \|x'(t) - v'(t)\| > M'(t)\}$, then $x \in T(v, M)$.

Proof. By assumption, $x' \in W^{2,1}_B([0,1], \mathbb{R}^n)$ and thus, from Lemma 3.2 of [8] applied to x', we get $\|x'(t) - v'(t)\| \leq M'(t)$, for all $t \in [0,1]$. On $\{t \in [0,1] : \|x(t) - v(t)\| > 0\}$, $\|x(t) - v(t)\|' \leq \|x'(t) - v'(t)\| \leq M'(t)$, and hence, $t \mapsto \|x(t) - v(t)\| - M(t)$ is nonincreasing. Since $\|x_0 - v(0)\| \leq M(0)$, we get $\|x(t) - v(t)\| \leq M(t)$ for all $t \in [0,1]$. □

We now associate to f^ϵ_λ an operator $\mathcal{F}^\epsilon : C^2([0,1], \mathbb{R}^n) \times [0,1] \to L^1([0,1], \mathbb{R}^n)$ defined by

$$\mathcal{F}^\epsilon(x, \lambda)(t) = f^\epsilon_\lambda(t, x(t), x'(t), x''(t)).$$

This operator has nice properties.

Proposition 3.4. *Let $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function satisfying* (H1). *Then the operator $\mathcal{F}^\epsilon$ previously defined is continuous and integrably bounded on bounded.*

Proof. We first show that $\mathcal{F}^\epsilon$ is integrably bounded on bounded. Let B be a bounded set of $C^2([0,1], \mathbb{R}^n)$. If $x \in B$, there exists a constant $K > 0$ such that $\|x^{(i)}(t)\| \leq K$, for every $t \in [0,1]$ and for $i = 0, 1, 2$. Observe that

$$\begin{aligned}\|\mathcal{F}^\epsilon(x,\lambda)(t)\| &= \|f^\epsilon_\lambda(t, x(t), x'(t), x''(t))\| \\ &\leq \max\{\|f(t,u,y,z)\| : (u,y,z) \in E\} \\ &\quad + |M'(t)| + \|v'(t)\| + \|v'''(t)\| + \|M'''(t)\|\end{aligned}$$

for all $\lambda \in [0,1]$ and almost every $t \in [0,1]$, where

$$E = \{(u,y,z) \in \mathbb{R}^{3n} : \|u\| \leq \|v\|_0 + \|M\|_0,$$

$$\|y\| \leq \|v'\|_0 + \|M'\|_0, \quad \|z\| \leq 2\|x''\|_0 + \|v''\|_0 + \|M''\|_0 \}.$$

Since f is Carathéodory, $v \in W^{3,1}([0,1], \mathbb{R}^n)$ and $M \in W^{3,1}([0,1], \mathbb{R})$, it is clear that $\mathcal{F}^\epsilon$ is integrably bounded on bounded.

To show the continuity of the operator, we first show that if $(x_n, \lambda_n) \to (x, \lambda)$ in $C^2([0,1], \mathbb{R}^n) \times [0,1]$, then

$$f^\epsilon_{\lambda_n}(t, x_n(t), x'_n(t), x''_n(t)) \to f^\epsilon_\lambda(t, x(t), x'(t), x''(t)) \text{ a.e. } t \in [0,1]. \tag{3.6}$$

Since f is Carathéodory, it is clear by definition of f^ϵ_λ that (3.6) is true almost everywhere on $\{t \in [0,1] : \|x'(t) - v'(t)\| \neq M'(t)\}$. Moreover, it could be shown, using Lemma 2.3 and the same ideas as in the proof of Proposition 3.5 of [8], that $\widetilde{x}''_n(t) \to x''(t)$ almost everywhere on $\{t \in [0,1] : \|x'(t) - v'(t)\| = M'(t) > 0\}$. Then, (3.6) is true almost everywhere on $\{t \in [0,1] : \|x'(t) - v'(t)\| = M'(t) > 0\}$.

On the set $A = \{t \in [0,1] : \|x'(t) - v'(t)\| = M'(t) = 0\}$, $x'(t) = v'(t)$, and by Lemma 2.3, we must have $x''(t) = v''(t)$, $M''(t) = 0$ and $M'''(t) = 0$ for almost every $t \in A$. So, by Definition 3.1 (iii),

$$\begin{aligned}
f^\epsilon_\lambda(t, x(t), x'(t), x''(t)) &= \lambda(f_1(t, x(t), x'(t), x''(t)) - \epsilon x'(t)) \\
&\quad + (1-\lambda)(v'''(t) - \epsilon v'(t)) \\
&= \lambda f_1(t, x(t), v'(t), v''(t)) + (1-\lambda)v'''(t) - \epsilon v'(t) \\
&= \lambda v'''(t) + (1-\lambda)v'''(t) - \epsilon v'(t) \\
&= v'''(t) - \epsilon v'(t)
\end{aligned}$$

almost everywhere on A. This is now clear that (3.6) is true almost everywhere on $[0,1]$. The conclusion follows from the Lebesgue Dominated Convergence Theorem and the fact that $\mathcal{F}^\epsilon$ is integrably bounded on bounded. □

Fix $\epsilon \in [0,1]$ such that the operator $L_\epsilon : C^1_B([0,1], \mathbb{R}^n) \to C_0([0,1], \mathbb{R}^n)$ defined by

$$L_\epsilon(x)(t) = x'(t) - x'(0) - \epsilon \int_0^t x(s)ds$$

is invertible. We are now ready to obtain our general existence result.

Theorem 3.5. *Let $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function satisfying* (H1). Assume that there exists $K > 0$ such that every solution x of (3.1) satisfies

$$\|x''(t)\| < K \text{ for every } t \in [0,1].$$

Then, the problem (1.1) has a solution such that $x \in T(v, M) \cap W^{3,1}([0,1], \mathbb{R}^n)$.

Proof. We first show that if $x \in W^{3,1}_{x_0,B}([0,1],\mathbb{R}^n)$ is a solution of (3.1), then $\|x'(t) - v'(t)\| \le M'(t)$ for all $t \in [0,1]$. On the set $\{t \in [0,1] : \|x'(t) - v'(t)\| > M'(t)\}$, we have

$$\|\widehat{x'}(t) - v'(t)\| = M'(t), \tag{3.7}$$

$$\langle \widehat{x'}(t) - v'(t), \widetilde{x''}(t) - v''(t)\rangle = M'(t)M''(t), \tag{3.8}$$

and

$$\begin{aligned}\|\widetilde{x''}(t) - v''(t)\|^2 &= \|x''(t) - v''(t)\|^2 \\ &\quad + (M''(t))^2 - \frac{\langle x'(t) - v'(t), x''(t) - v''(t)\rangle^2}{\|x'(t) - v'(t)\|^2}.\end{aligned} \tag{3.9}$$

Thus, using (H1) we obtain

$$\begin{aligned}
&\frac{\langle x'(t) - v'(t), x'''(t) - v'''(t)\rangle + \|x''(t) - v''(t)\|^2}{\|x'(t) - v'(t)\|} \\
&\quad - \frac{\langle x'(t) - v'(t), x''(t) - v''(t)\rangle^2}{\|x'(t) - v'(t)\|^3} - \epsilon\|x'(t) - v'(t)\| \\
&= \frac{\left\langle x'(t) - v'(t), \frac{\lambda M'(t)}{\|x'(t)-v'(t)\|}\big(f_1(t, x(t), \widehat{x'}(t), \widetilde{x''}(t)) - v'''(t) + \epsilon(x'(t) - v'(t))\big)\right\rangle}{\|x'(t) - v'(t)\|} \\
&\quad + \frac{\left\langle x'(t) - v'(t), \epsilon(x'(t) - v'(t)) + \left(1 - \frac{\lambda M'(t)}{\|x'(t)-v'(t)\|}\right)\frac{M'''(t)(x'(t)-v'(t))}{\|x'(t)-v'(t)\|}\right\rangle}{\|x'(t) - v'(t)\|} \\
&\quad + \frac{\|\widetilde{x''}(t) - v''(t)\|^2 - (M''(t))^2}{\|x'(t) - v'(t)\|} - \epsilon\|x'(t) - v'(t)\| \\
&= \frac{\lambda}{\|x'(t) - v'(t)\|}\Big(\langle \widehat{x'}(t) - v'(t), f_1(t, x(t), \widehat{x'}(t), \widetilde{x''}(t)) - v'''(t)\rangle \\
&\quad + \|\widetilde{x''}(t) - v''(t)\|^2 - (M''(t))^2\Big) + \frac{(1-\lambda)\big(\|\widetilde{x''}(t) - v''(t)\|^2 - (M''(t))^2\big)}{\|x'(t) - v'(t)\|} \\
&\quad + \left(1 - \frac{\lambda M'(t)}{\|x'(t) - v'(t)\|}\right)M'''(t) - \lambda\epsilon M'(t) \\
&\ge M'''(t) - \epsilon M'(t)
\end{aligned}$$

almost everywhere on $\{t \in [0,1] : \|x'(t) - v'(t)\| > M'(t)\}$. From Lemma 3.3, all solutions of (3.1) are in $T(v, M)$ and hence, in U by assumption, where

$$U = \{x \in C^2([0,1],\mathbb{R}^n) : \|x^{(i)}\|_0 < \|v^{(i)}\|_0 + \|M^{(i)}\|_0 + 1; i = 0,1; \|x''\|_0 < K\}.$$

Consider the linear operator $D : C^2_{x_0,B}([0,1],\mathbb{R}^n) \to C^1_B([0,1],\mathbb{R}^n)$ defined by

$$D(x) = x'.$$

It is easy to check that D is invertible.

A solution of (3.1) is a fixed point of the operator

$$H = D^{-1} \circ L_\epsilon^{-1} \circ N_{\mathcal{F}^\epsilon} : C^2([0,1],\mathbb{R}^n) \times [0,1] \to C^2_{x_0,B}([0,1],\mathbb{R}^n) \subset C^2([0,1],\mathbb{R}^n).$$

We deduce from Proposition 3.4, Theorem 2.1, and the continuity of the operators D and L_ϵ that H is completely continuous. This operator is fixed point free on ∂U.

Now, define

$$H_0 : C^2([0,1],\mathbb{R}^n) \times [0,1] \to C^2([0,1],\mathbb{R}^n)$$

by $H_0(x,\lambda) = \lambda H(x,0)$. Since $\mathcal{F}_\epsilon(\cdot,0)$ is integrably bounded, there exists an open bounded set $W \subset C^2([0,1],\mathbb{R}^n)$ such that $U \subset W$ and

$$H_0(C^2([0,1],\mathbb{R}^n) \times [0,1]) \subset W.$$

The homotopic and the excision properties of the Leray-Schauder degree imply that

$$\begin{aligned} 1 = d(I,W,0) &= d(I - H_0(\cdot,1),W,0) = d(I - H(\cdot,0),W,0) \\ &= d(I - H(\cdot,0),U,0) = d(I - H(\cdot,1),U,0). \end{aligned}$$

Therefore, (3.1) has a solution $x \in T(v,M)$ for $\lambda = 1$ which is also a solution of (1.1) by definition of f_1^ϵ. □

4. OTHER EXISTENCE RESULTS

In this section, we present existence results which will follow from our general existence Theorem (Theorem 3.5). In other words, we present assumptions on f which will imply the existence of an a priori bound on the second order derivative of solutions of (3.1). We will impose on f some of the following assumptions:

(H2) There exist a function $\gamma \in L^1([0,1],[0,\infty[)$ and a Borel measurable function $\phi : [0,\infty[\to [1,\infty[$ such that
  - (i) $\|f(t,x,y,z)\| \le \gamma(t)\phi(\|z\|)$ for almost every $t \in [0,1]$ and for all $(x,y,z) \in \mathbb{R}^{3n}$ such that $\|x - v(t)\| \le M(t)$ and $\|y - v'(t)\| \le M'(t)$;
  - (ii) for all $c \ge 0$, $\int_c^\infty \frac{ds}{\phi(s)} = \infty$.

(H3) There exist a function $\gamma \in L^1([0,1],[0,\infty[)$ and a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ such that
  - (i) $|\langle z, f(t,x,y,z)\rangle| \le \phi(\|z\|)(\gamma(t) + \|z\|)$ for almost every $t \in [0,1]$ and for all $(x,y,z) \in \mathbb{R}^{3n}$ such that $\|x - v(t)\| \le M(t)$ and $\|y - v'(t)\| \le M'(t)$;
  - (ii) for all $c \ge 0$, $\int_c^\infty \frac{s\,ds}{\phi(s)+s} = \infty$.

(H4) There exist constants $r, b > 0$, $c \ge 0$ and a function $h \in L^1([0,1],\mathbb{R})$ such that for all $t \in [0,1]$ and all $(x,y,z) \in \mathbb{R}^{3n}$ such that $\|x - v(t)\| \le M(t)$, $\|y - v'(t)\| \le M'(t)$ and $\|z\| \ge r$, then

$$(b + c\|y\|)\sigma(t,x,y,z) \ge \|z\| - h(t),$$

where

$$\sigma(t,x,y,z) = \frac{\langle y, f(t,x,y,z)\rangle + \|z\|^2}{\|z\|} - \frac{\langle z, f(t,x,y,z)\rangle\langle y,z\rangle}{\|z\|^3}.$$

(H5) There exist a constant $a \geq 0$ and a function $l \in L^1([0,1],\mathbb{R})$ such that

$$\|f(t,x,y,z)\| \leq a\big(\langle y, f(t,x,y,z)\rangle + \|z\|^2\big) + l(t)$$

for almost every $t \in [0,1]$ and all $(x,y,z) \in \mathbb{R}^n$ such that $\|x - v(t)\| \leq M(t)$ and $\|y - v'(t)\| \leq M'(t)$.

Assumption (H5) is inspired by a famous condition introduced in 1960 in Hartman's article [14] in which, to our knowledge, he obtained the first existence result for boundary value problems for systems of second order differential equations.

Assumption (H4) is inspired by a condition introduced by Frigon [8] in 1995. Notice that (H4) is trivially satisfied in the scalar case.

The growth condition (H2) combined with (H1) are sufficient to guarantee the existence of a solution of (1.1) for some particular boundary conditions as it is shown in the following result.

Theorem 4.1. *Let $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function satisfying* (H1) *and* (H2)*. If (BC) denotes* (1.2) *with* $\max\{\rho_0, \rho_1\} > 0$*, then the problem* (1.1) *has at least one solution $x \in T(v,M) \cap W^{3,1}([0,1],\mathbb{R}^n)$.*

Proof. Again, the existence of a solution will be guaranteed by Theorem 3.5 if we obtain an a priori bound on the second derivative of all solutions x of (3.1). From the proof of Theorem 3.5, we already know that $x \in T(v,M)$. Hence, since (BC) denotes (1.2) with $\max\{\rho_0, \rho_1\} > 0$, there exists a constant $k > 0$ such that $\min\{\|x''(0)\|, \|x''(1)\|\} \leq k$. Let $K > k$ be such that

$$\int_k^K \frac{ds}{\phi(s)} > L := \|\gamma\|_{L^1} + \epsilon\|M'\|_0 + \|v'''\|_{L^1} + \|M'''\|_{L^1}.$$

Suppose there exists $t_1 \in [0,1]$ such that $\|x''(t_1)\| \geq K$. Then, there exists $t_0 \neq t_1 \in [0,1]$ such that $\|x''(t_0)\| = k$ and $\|x''(t)\| > k$ for every t between t_0 and t_1. Without loss of generality, we may suppose that $t_0 < t_1$. Thus, by (H2), almost everywhere on $[t_0, t_1]$, we have

$$\begin{aligned}
\|x''(t)\|' = \frac{\langle x''(t), x'''(t)\rangle}{\|x''(t)\|} &\leq \|x'''(t)\| \\
&\leq \|f(t,x(t),x'(t),x''(t))\| + \epsilon\|x'(t) - v'(t)\| + \|v'''(t)\| + |M'''(t)| \\
&\leq \|\gamma(t)\|\phi(\|x''(t)\|) + \epsilon\|M'\|_0 + \|v'''\|_{L^1} + \|M'''\|_{L^1}.
\end{aligned}$$

So,

$$\int_{t_0}^{t_1} \frac{\|x''(t)\|'}{\phi(\|x''(t)\|)} dt \leq L.$$

On the other hand, by the change variable rule (Lemma 2.2), we get

$$\int_{t_0}^{t_1} \frac{\|x''(t)\|'}{\phi(\|x''(t)\|)}dt = \int_{\|x''(t_0)\|}^{\|x''(t_1)\|} \frac{ds}{\phi(s)} \geq \int_{k}^{K} \frac{ds}{\phi(s)} > L;$$

which is a contradiction. Then, for all solution x of (3.1), there exists a constant $K > 0$ such that $\|x''(t)\| < K$, for all $t \in [0,1]$. □

Now, if we want to replace (H2) by the more general growth condition (H3), extra assumptions are needed.

Theorem 4.2. *Let $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1), (H3), *and* (H4) *or* (H5). *Then, there exists $x \in T(v, M) \cap W^{3,1}([0,1], \mathbb{R}^n)$ a solution of* (1.1).

In order to prove this result, we will need the three following lemmas. Their proofs can be found in [8] and [10].

Lemma 4.3. *Let $r, k \geq 0$, $m \in L^1([0,1], \mathbb{R})$ and $\psi : [0, \infty[\to]0, \infty[$ be a Borel measurable function such that*

$$\int_r^{\infty} \frac{sds}{\psi(s)} > \|m\|_{L^1} + k.$$

Then there exists a constant $K > 0$ such that $\|x'\|_0 < K$ for all $x \in W^{2,1}([0,1], \mathbb{R}^n)$ satisfying the following conditions:

(i) $\min_{t\in[0,1]} \|x'(t)\| \leq r$;
(ii) $\|x'\|_{L^1([t_0,t_1])} \leq k$ *for every interval* $[t_0, t_1] \subset \{t \in [0,1] : \|x'(t)\| \geq r\}$;
(iii) $|\langle x'(t), x''(t)\rangle| \leq \psi(\|x'(t)\|)(m(t) + \|x'(t)\|)$ *almost everywhere on* $\{t \in [0,1] : \|x'(t)\| \geq r\}$.

Lemma 4.4. *Let $r, \beta > 0$, $\theta \geq 0$ and $m \in L^1([0,1], \mathbb{R})$. Then there exists a nondecreasing function $\omega : [0, \infty[\to [0, \infty[$ such that*

$$\|x'\|_{L^1([t_0,t_1])} \leq \omega(\|x\|_0),$$

and

$$\min_{t\in[0,1]} \|x'(t)\| \leq \max\{r, \omega(\|x\|_0)\}.$$

for every $x \in W^{2,1}([0,1], \mathbb{R}^n)$ and every interval $[t_0, t_1]$ such that almost everywhere on $\{t \in [t_0, t_1] : \|x'(t)\| \geq r\}$, the following inequality

$$(\beta + \theta\|x(t)\|)\sigma_0(t, x) + \frac{\theta\langle x(t), x'(t)\rangle^2}{\|x(t)\|\|x'(t)\|} \geq \|x'(t)\| - m(t)$$

is satisfied, where

$$\sigma_0(t, x) = \frac{\langle x(t), x''(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), x''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}.$$

Lemma 4.5. *Let $k \geq 0$ and $m \in L^1([0,1],\mathbb{R})$. Then there exists an increasing function $\omega : [0,\infty[\to [0,\infty[$ such that $\|x'\|_{L^1} \leq \omega(\|x\|_0)$ for all $x \in W^{2,1}([0,1],\mathbb{R}^n)$ satisfying*

$$\|x''(t)\| \leq k(\langle x(t), x''(t)\rangle + \|x'(t)\|^2) + m(t)$$

for almost every $t \in [0,1]$.

Proof of Theorem 4.2. The conclusion will follow from Theorem 3.5 if we prove the existence of a constant $K > 0$ such that $\|x''\|_0 < K$ for all solutions x of (3.1). Let x be a solution of (3.1). From the proof of Theorem 3.5, we already know that $x \in T(v,M)$. Using (H3), we obtain

$$\begin{aligned}
&|\langle x''(t), x'''(t)\rangle| \\
&\leq |\langle x''(t), f(t,x(t),x'(t),x''(t))\rangle| + \big(\epsilon\|x'(t)-v'(t)\| + \|v'''(t)\| + |M'''(t)|\big)\|x''(t)\| \\
&\leq (\gamma(t) + \|x''(t)\|)\phi(\|x''(t)\|) + \big(\epsilon|M'(t)| + \|v'''(t)\| + |M'''(t)|\big)\|x''(t)\| \\
&\leq \big(\phi(\|x''(t)\|) + \|x''(t)\|\big)\big(\gamma(t) + \|x''(t)\| + \epsilon|M'(t)| + \|v'''(t)\| + |M'''(t)|\big)
\end{aligned}$$

for almost every $t \in [0,1]$. Thus, condition (iii) of Lemma 4.3 is satisfied with $\psi(s) = \phi(s) + s$ and $m(t) = \gamma(t) + \epsilon|M'(t)| + \|v'''(t)\| + |M'''(t)|$. To conclude, it suffices to show that conditions (i) and (ii) of this lemma are satisfied.

If (H4) holds, observe that almost everywhere on $\{t \in [0,1] : \|x''(t)\| \geq r\}$, we have

$$\begin{aligned}
\sigma_0(t,x') &= \frac{\langle x'(t), x'''(t)\rangle + \|x''(t)\|^2}{\|x''(t)\|} - \frac{\langle x''(t), x'''(t)\rangle\langle x'(t), x''(t)\rangle}{\|x''(t)\|^3} \\
&= \lambda\sigma(t,x(t),x'(t),x''(t)) + (1-\lambda)\|x''(t)\| \\
&\quad + \frac{(1-\lambda)\big\langle x'(t), v'''(t) + (\epsilon + \frac{M'''(t)}{M'(t)})(x'(t)-v'(t))\big\rangle}{\|x''(t)\|} \\
&\quad - \frac{(1-\lambda)\big\langle x''(t), v'''(t) + (\epsilon + \frac{M'''(t)}{M'(t)})(x'(t)-v'(t))\big\rangle\big\langle x'(t), x''(t)\big\rangle}{\|x''(t)\|^3} \\
&\geq \lambda\sigma(t,x(t),x'(t),x''(t)) + (1-\lambda)\|x''(t)\| \\
&\quad - \frac{2(\|v'(t)\| + |M'(t)|)(\|v'''(t)\| + \epsilon|M'(t)| + |M'''(t)|)}{r}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&(b + c\|x'(t)\|)\sigma_0(t,x') + c\frac{\langle x'(t), x''(t)\rangle^2}{\|x'(t)\|\|x''(t)\|} \\
&\qquad\qquad \geq \lambda\|x''(t)\| + b(1-\lambda)\|x''(t)\| - h(t) - \delta_0(t),
\end{aligned}$$

where

$$\delta_0(t) = \frac{2}{r}(b + c\|v'(t)\| + c|M'(t)|)(\|v'(t)\| + |M'(t)|)(\|v'''(t)\| + \epsilon|M'(t)| + |M'''(t)|).$$

If we take $\nu = \min\{\lambda + b(1-\lambda) : \lambda \in [0,1]\}$, $\beta = b/\nu$ and $\theta = c/\nu$, we can apply Lemma 4.4 to $x' \in W^{2,1}([0,1], \mathbb{R}^n)$. Therefore, all conditions of Lemma 4.3 are satisfied.

On the other hand, if (H5) is satisfied, we have

$$\begin{aligned}
\|x'''(t)\| &\le \lambda \|f(t, x(t), x'(t), x''(t))\| + \epsilon \|x'(t) - v'(t)\| + \|v'''(t)\| + |M'''(t)| \\
&\le \lambda a(\langle x'(t), f(t, x(t), x'(t), x''(t))\rangle + \|x''(t)\|^2) + l(t) \\
&\quad + \epsilon |M'(t)| + \|v'''(t)\| + |M'''(t)| \\
&\le a(\langle x'(t), x'''(t)\rangle + \|x''(t)\|^2) + \epsilon |M'(t)| + \|v'''(t)\| + |M'''(t)| \\
&\quad - a(1-\lambda)\langle x'(t), v'''(t) + (\frac{M'''(t)}{M'(t)} + \epsilon)(x'(t) - v'(t))\rangle \\
&\le a(\langle x'(t), x'''(t)\rangle + \|x''(t)\|^2) + \epsilon |M'(t)| + \|v'''(t)\| + |M'''(t)| \\
&\quad + a(\|v'(t)\| + |M'(t)|)(\|v'''(t)\| + |M'''(t)| + \epsilon M'(t)).
\end{aligned}$$

Hence, if we apply Lemma 4.5 to $x' \in W^{2,1}([0,1], \mathbb{R}^n)$, all conditions of Lemma 4.3 are satisfied. Hence, there exists a constant $K > 0$ such that $\|x''\|_0 < K$ for all solutions x of (3.1). $\square$

From the previous result, we see that in Theorem 4.1, we can consider other boundary conditions if we impose an extra assumption.

Corollary 4.6. *Let $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1)*,* (H2) *and* (H4) *or* (H5)*. Then, there exists $x \in T(v, M) \cap W^{3,1}([0,1], \mathbb{R}^n)$ a solution of* (1.1)*.*

Example 4.7. Consider the system

$$\begin{aligned}
&x'''(t) = x''(t) + \|x''(t)\|(\|x'(t)\|^2 x(t) - \langle x(t), x'(t)\rangle x'(t)) - a \\
&x(0) = 0,\ A_0 x'(0) = 0,\ A_1 x'(1) + \rho_1 x''(1) = 0,
\end{aligned} \tag{4.1}$$

where $a \in \mathbb{R}^n$, $\|a\| = 1$ and where A_i and ρ_1 are defined as in the introduction for $i = \{0, 1\}$. Verify that with $v(t) \equiv 0$, $M(t) = t^2/2$, (v, M) is a solution-tube of (4.1). Also, assumptions (H3) and (H4) are satisfied with

$$\phi(s) = 3s + 1,\ \gamma(t) \equiv 0,\ b = 1,\ c = 0,\ r > 0 \text{ and } h(t) = t^5 + 2t/r.$$

From Theorem 4.2, the system (4.1) has at least one solution x such that $\|x(t)\| \le t^2/2$ and $\|x'(t)\| \le t$ for all $t \in [0,1]$.

Example 4.8. Consider the system

$$\begin{aligned}
&x'''(t) = x'(t)(\|x''(t)\|^2 + 1) + h(t) \\
&x(0) = 0,\ x'(0) = x'(1),\ x''(0) = x''(1),
\end{aligned} \tag{4.2}$$

where $h \in L^\infty([0,1],\mathbb{R}^n)$ with $\|h\|_{L^\infty} \le 1$. Verify that with $v(t) \equiv 0$, $M(t) = t$, (v, M) is a solution-tube of (4.2). Also, assumptions (H3) and (H5) are satisfied with

$$\phi(s) = s^2 + 2,\ \ \gamma(t) \equiv 0,\ \ a = 1,\ \text{ and } l(t) = 3.$$

From Theorem 4.2, the system (4.2) has at least one solution x such that $\|x(t)\| \le t$ and $\|x'(t)\| \le 1$ for all $t \in [0,1]$.

5. NAGUMO-WINTNER TYPE GROWTH CONDITION

Now, we want to establish the existence of a solution of (1.1) when the right member satisfies a standard Nagumo-Wintner growth condition. To this aim, we will use the theory of differential inclusions. We will assume the following hypothesis.

(H6) There exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and a function $\omega \in L^1([0,1],\mathbb{R})$ such that
- (i) $\|f(t,x,y,z)\| \le \phi(\|z\|)(\omega(t) + \|z\|)$ for almost every $t \in [0,1]$ and for all $(x,y,z) \in \mathbb{R}^{3n}$ such that $\|x - v(t)\| \le M(t)$ and $\|y - v'(t)\| \le M'(t)$;
- (ii) for all $c \ge 0$, $\int_c^\infty \frac{ds}{\phi(s)} = \infty$.

For $\epsilon, \lambda \in [0,1]$, we define the multivalued mapping $S^\epsilon_\lambda : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ by $S^\epsilon_\lambda(t,x,y,z) := \widehat{f}^\epsilon_\lambda(t,x,y,z) + G_\lambda(t,x,y,z)$ where the function $\widehat{f}^\epsilon_\lambda : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ is defined by

$$\widehat{f}^\epsilon_\lambda(t,x,y,z) = \begin{cases} \lambda\Big(\frac{M'(t)}{\|y-v'(t)\|} f_1(t,x,\hat{y},\tilde{z}) - \epsilon\hat{y}\Big) & \\ \quad -\epsilon(1-\lambda)v'(t), & \text{if } \|y - v'(t)\| > M'(t) > 0, \\ \lambda(f_1(t,x,y,z) - \epsilon y) - \epsilon(1-\lambda)v'(t), & \text{if } \|y - v'(t)\| \le M'(t), \\ & \quad \text{and } M'(t) > 0, \\ v'''(t) - \epsilon v'(t), & \text{if } M'(t) = 0; \end{cases}$$

and the multivalued function $G_\lambda : [0,1] \times \mathbb{R}^{3n} \to 2^{\mathbb{R}^n}$ is defined by

$$G_\lambda(t,x,y,z)$$
$$= \begin{cases} \Bigg(\Big(1 - \frac{\lambda M'(t)}{\|y-v'(t)\|}\Big)\Big(M'''(t) + \frac{\langle y-v'(t), v'''(t)\rangle}{\|y-v'(t)\|}\Big) & \\ \quad + \frac{(1-\lambda)\big(M''(t)^2 - \|\tilde{z}-v''(t)\|^2\big)}{\|y-v'(t)\|}\Bigg)^+ \frac{(y-v'(t))}{\|y-v'(t)\|}, & \text{if } \|y - v'(t)\| > M'(t) > 0, \\ [0,(1-\lambda)]\Bigg(M'''(t) + \frac{\langle y-v'(t), v'''(t)\rangle}{\|y-v'(t)\|} & \\ \quad + \frac{M''(t)^2 - \|\tilde{z}-v''(t)\|^2}{\|y-v'(t)\|}\Bigg)^+ \frac{(y-v'(t))}{\|y-v'(t)\|}, & \text{if } \|y - v'(t)\| = M'(t) > 0, \\ 0, & \text{if } \|y - v'(t)\| < M'(t) \\ & \quad \text{or } M'(t) = 0; \end{cases}$$

where (v, M) is a solution-tube of (1.1) given by (H1), f_1, $\hat{y}$, $\tilde{z}$ are defined in (3.2), (3.4) and (3.5) respectively, and the superscript $+$ means the positive part, i.e. $(a)^+ = \max\{a, 0\}$.

Proposition 5.1. *Under* (H1)*, the multivalued function* $G : [0,1] \times \mathbb{R}^{3n} \times [0,1] \to \mathbb{R}^n$ *defined by*

$$G(t, x, y, z, \lambda) = G_\lambda(t, x, y, z)$$

is Carathéodory.

Proof. It is easy to see that $G(t, x, y, z, \lambda)$ has compact, convex and nonempty values and that $t \mapsto G(t, x, y, z, \lambda)$ is measurable for all $(x, y, z, \lambda) \in \mathbb{R}^{3n} \times [0,1]$.

We now prove that for almost every $t \in [0,1]$, $(x, y, z, \lambda) \mapsto G_\lambda(t, x, y, z)$ is upper semi-continuous. On the set $t \in \{t \in [0,1] : M'(t) = 0\}$, the statement is clear. For the other cases, we want to show that the set

$$\widetilde{A} = \{(x, y, z, \lambda) \in \mathbb{R}^{3n} \times [0,1] : G(t, x, y, z, \lambda) \cap A \neq \emptyset\}$$

is closed for a closed set $A \subset \mathbb{R}^n$. Consider $\{u_n\}_{n \in \mathbb{N}}$, a sequence in $\widetilde{A}$ converging to an element $u = (x, y, z, \lambda)$. If $\|y - v'(t)\| > M'(t) > 0$, for n sufficiently large, we have $\|y_n - v'(t)\| > M'(t) > 0$ and clearly, $G(t, x_n, y_n, z_n, \lambda_n) \to G(t, x, y, z, \lambda) \in A$ since A is closed. Now suppose $\|y - v'(t)\| = M'(t) > 0$. If there exists a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ such that $\|y_{n_k} - v'(t)\| > M'(t) > 0$, we argue as before noticing that

$$\frac{\lambda_n M'(t)}{\|y_{n_k} - v'(t)\|} \to \lambda.$$

If there exists a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ such that $\|y_{n_k} - v'(t)\| < M'(t)$, just remark that $G(t, u_n) = 0 \in G(t, x, y, z, \lambda)$. Otherwise, for n sufficiently large, $\|y_n - v'(t)\| = M'(t)$ and there exists $\gamma_n \in [0, (1 - \lambda_n)] \subset [0,1]$ such that

$$\begin{aligned} w_n = \gamma_n \Big(M'''(t) + \frac{\langle y_n - v'(t), v'''(t) \rangle}{\|y_n - v'(t)\|} + \frac{M''(t)^2 - \|\tilde{z_n} - v''(t)\|^2}{\|y_n - v'(t)\|} \Big)^+ \frac{(y_n - v'(t))}{\|y_n - v'(t)\|} \\ \in G(t, u_n) \cap A. \end{aligned}$$

Since $[0,1]$ is compact, there exists a subsequence $\{\gamma_{n_k}\}_{k \in \mathbb{N}}$ converging to an element $\tilde{\gamma} \in [0, (1 - \lambda)]$. Thus, $w_{n_k} \to w$, where

$$w = \tilde{\gamma} \Big(M'''(t) + \frac{\langle y - v'(t), v'''(t) \rangle}{\|y - v'(t)\|} + \frac{M''(t)^2 - \|\tilde{z} - v''(t)\|^2}{\|y - v'(t)\|} \Big)^+ \frac{(y - v'(t))}{\|y - v'(t)\|} \in G(t, u).$$

Thus, $u \in \widetilde{A}$.

Now, remark that if $\|y - v'(t)\| > M'(t) > 0$ and $\|G(t, x, y, z, \lambda)\| > 0$, then using (H1) we have

$$\|G(t, x, y, z, \lambda)\| = \Big(1 - \frac{\lambda M'(t)}{\|y - v'(t)\|} \Big) \Big(M'''(t) + \frac{\langle y - v'(t), v'''(t) \rangle}{\|y - v'(t)\|} \Big)$$

$$
\begin{aligned}
&+(1-\lambda)\Big(\frac{M''(t)^2-\|\widetilde{z}''(t)-v''(t)\|^2}{\|y-v'(t)\|}\Big)\\
&=(1-\lambda)\bigg(\frac{\langle M'(t)(y-v'(t)),v'''(t)\rangle}{\|y-v'(t)\|^2}\\
&\quad+\frac{M'''(t)M'(t)+M''(t)^2-\|\widetilde{z}-v''(t)\|^2}{\|y-v'(t)\|}\bigg)\\
&\quad+\Big(1-\frac{M'(t)}{\|y-v'(t)\|}\Big)\Big(\frac{\langle y-v'(t),v'''(t)\rangle}{\|y-v'(t)\|}+M'''(t)\Big)\\
&\le(1-\lambda)\frac{\langle\frac{M'(t)(y-v'(t))}{\|y-v'(t)\|},f(t,\overline{x},\widehat{y},\widetilde{z})\rangle}{\|y-v'(t)\|}+\|v'''(t)\|+|M'''(t)|\\
&\le\|f(t,\overline{x},\widehat{y},\widetilde{z})\|+\|v'''(t)\|+|M'''(t)|.
\end{aligned}
$$

Moreover, if $\|y-v'(t)\|=M'(t)>0$ and $\|G(t,x,y,z,\lambda)\|>0$, since $M''(t)^2\le\|\widetilde{z}-v''(t)\|^2$, we have

$$
\begin{aligned}
\|G(t,x,y,z,\lambda)\|&=\big(M'''(t)+\frac{\langle y-v'(t),v'''(t)\rangle}{\|y-v'(t)\|}\big)\\
&\quad+\big(\frac{M''(t)^2-\|\widetilde{z}''(t)-v''(t)\|^2}{\|y-v'(t)\|}\big)\le\|v'''(t)\|+|M'''(t)|.
\end{aligned}\tag{5.1}
$$

This leads us to conclude that G is a Carathéodory function. □

Now, let us define the multivalued operator

$$\mathcal{S}^\epsilon=\widehat{\mathcal{F}}^\epsilon+\mathcal{G}:C^2([0,1],\mathbb{R}^n)\times[0,1]\to L^1([0,1],\mathbb{R}^n)$$

where $\widehat{\mathcal{F}}^\epsilon$ and $\mathcal{G}$ are respectively defined by

$$\widehat{\mathcal{F}}^\epsilon(x,\lambda)(t)=\widehat{f}^\epsilon_\lambda(t,x(t),x'(t),x''(t)),$$

$$\mathcal{G}(x,\lambda):=\{u\in L^1([0,1],\mathbb{R}^n):u(t)\in G_\lambda(t,x(t),x'(t),x''(t))\text{ a.e. }t\in[0,1]\}.$$

To $\mathcal{S}^\epsilon:C^2([0,1],\mathbb{R}^n)\times[0,1]\to L^1([0,1],\mathbb{R}^n)$, we now associate the multivalued operator

$$N_{\mathcal{S}^\epsilon}:C^2([0,1],\mathbb{R}^n)\times[0,1]\to C_0([0,1],\mathbb{R}^n)$$

defined by

$$N_{\mathcal{S}^\epsilon}(x,\lambda)(t)=\{w:w(t)=\int_0^t u(s)ds\ \text{ with }\ u\in\mathcal{S}^\epsilon(x,\lambda)\}.$$

Proposition 5.2. *Let $f:[0,1]\times\mathbb{R}^{3n}\to\mathbb{R}^n$ be a Carathéodory function satisfying* (H1)*. Then, $N_{\mathcal{S}^\epsilon}$ is upper semi-continuous and completely continuous, with closed, convex and nonempty values.*

Proof. If we argue as in Proposition 3.4, we can show that the operator $\widehat{\mathcal{F}}^\epsilon$ is continuous and integrably bounded on bounded. It follow from the previous proposition and the Kuratowski-Ryll-Nardzewski selection Theorem (Theorem 2.4) that $N_{\mathcal{S}^\epsilon}$ has nonempty and convex values.

Let $B \subset C^2([0,1],\mathbb{R}^n)$ be a bounded set. Since $\widehat{\mathcal{F}}^\epsilon$ is integrably bounded on bounded and G is a multivalued Carathéodory function, $N_{\mathcal{S}^\epsilon}(B)$ is uniformly bounded and equicontinuous. Thus, $N_{\mathcal{S}^\epsilon}(B)$ is relatively compact by the Arzela-Ascoli Theorem. So, $N_{\mathcal{S}^\epsilon}$ is completely continuous. We can follow the ideas of the proof of Lemma 2.3 in [9] to prove that $N_{\mathcal{S}^\epsilon}$ is upper semi-continuous and has compact values. $\square$

Let us consider the family of differential inclusions

$$\begin{aligned} &x'''(t) - \epsilon x'(t) \in S^\epsilon_\lambda(t, x(t), x'(t), x''(t)) \quad \text{a.e. } t \in [0,1],\\ &x(0) = x_0,\ x' \in (BC); \end{aligned} \tag{5.2}$$

where $\lambda, \epsilon \in [0,1]$. Let us show that solutions of (5.2) are in $T(v, M)$.

Lemma 5.3. *Let $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function satisfying* (H1)*. If x is a solution of* (5.2)*, then $x \in T(v, M)$.*

Proof. Let x be a solution of (5.2). Almost every where on $\{t \in [0,1] : \|x'(t) - v'(t)\| > M'(t) > 0\}$, we have as before (3.7), (3.8) and (3.9).

From (H1),

$$\begin{aligned}
&\frac{\langle x'(t) - v'(t), x'''(t) - v'''(t)\rangle + \|x''(t) - v''(t)\|^2}{\|x'(t) - v'(t)\|}\\
&\quad - \frac{\langle x'(t) - v'(t), x''(t) - v''(t)\rangle^2}{\|x'(t) - v'(t)\|^3} - \epsilon\|x'(t) - v'(t)\|\\
&= \frac{\langle x'(t) - v'(t), \frac{\lambda M'(t)}{\|x'(t)-v'(t)\|}(f_1(t, x(t), \widehat{x'}(t), \widetilde{x''}(t)) - \epsilon(x'(t) - v'(t)))\rangle}{\|x'(t) - v'(t)\|}\\
&\quad + \frac{\langle x'(t) - v'(t), G_\lambda(t, x(t), x'(t), x''(t))\rangle}{\|x'(t) - v'(t)\|} + \frac{\|\widetilde{x''}(t) - v''(t)\|^2 - (M''(t))^2}{\|x'(t) - v'(t)\|}\\
&\quad - \frac{\langle x'(t) - v'(t), v'''(t)\rangle}{\|x'(t) - v'(t)\|} - \epsilon\|x'(t) - v'(t)\|\\
&= \frac{\lambda(\langle \widehat{x'}(t) - v'(t), f_1(t, x(t), \widehat{x'}(t), \widetilde{x''}(t))\rangle + \|\widetilde{x''}(t) - v''(t)\|^2 - (M''(t))^2)}{\|x'(t) - v'(t)\|}\\
&\quad + \frac{(1-\lambda)(\|\widetilde{x''}(t) - v''(t)\|^2 - (M''(t))^2)}{\|x'(t) - v'(t)\|} - \frac{\langle x'(t) - v'(t), v'''(t)\rangle}{\|x'(t) - v'(t)\|}\\
&\quad + \|G_\lambda(t, x(t), x'(t), x''(t))\| - \lambda\epsilon M'(t)\\
&\geq \frac{\lambda(M'(t)M'''(t) + \langle \widehat{x}'(t) - v'(t), v'''(t)\rangle)}{\|x'(t) - v'(t)\|} - \frac{\langle x'(t) - v'(t), v'''(t)\rangle}{\|x'(t) - v'(t)\|}\\
&\quad - \frac{(1-\lambda)(\|\widetilde{x''}(t) - v''(t)\|^2 - (M''(t))^2)}{\|x'(t) - v'(t)\|} + \|G_\lambda(t, x(t), x'(t), x''(t))\| - \epsilon M'(t).
\end{aligned}$$

Set

$$\xi(t)=\frac{\lambda(M'(t)M'''(t)+\langle\widehat{x}'(t)-v'(t),v'''(t)\rangle)}{\|x'(t)-v'(t)\|}-\frac{\langle x'(t)-v'(t),v'''(t)\rangle}{\|x'(t)-v'(t)\|}\\ -\frac{(1-\lambda)(\|\widetilde{x}''(t)-v''(t)\|^2-(M''(t))^2)}{\|x'(t)-v'(t)\|}.$$

One can check that $\|G_\lambda(t,x(t),x'(t),x''(t))\|=(M'''(t)-\xi(t))^+$. Thus, we have

$$\begin{aligned}&\frac{\langle x'(t)-v'(t),x'''(t)-v'''(t)\rangle+\|x''(t)-v''(t)\|^2}{\|x'(t)-v'(t)\|}\\ &\qquad\qquad-\frac{\langle x'(t)-v'(t),x''(t)-v''(t)\rangle^2}{\|x'(t)-v'(t)\|^3}-\epsilon\|x'(t)-v'(t)\|\\ &\qquad\geq\xi(t)+(M'''(t)-\xi(t))^+-\epsilon M'(t)\\ &\qquad\geq M'''(t)-\epsilon M'(t).\end{aligned}$$

Observe that $x'''(t)=v'''(t)+\epsilon(x'(t)-v'(t))$ almost everywhere on the set $\{t\in[0,1]:\|x'(t)-v'(t)\|>M'(t)=0\}$, and hence,

$$\begin{aligned}&\frac{\langle x'(t)-v'(t),x'''(t)-v'''(t)\rangle+\|x''(t)-v''(t)\|^2}{\|x'(t)-v'(t)\|}\\ &\qquad\qquad-\frac{\langle x'(t)-v'(t),x''(t)-v''(t)\rangle^2}{\|x'(t)-v'(t)\|^3}-\epsilon\|x'(t)-v'(t)\|\\ &\qquad\geq 0=M'''(t)-\epsilon M'(t)\end{aligned}$$

by Lemma 2.3. Thus, by Lemma 3.3, for all solutions of (5.2), we have $\|x'(t)-v'(t)\|\leq M'(t)$, for every $t\in[0,1]$ and hence, $\|x(t)-v(t)\|\leq M(t)$ for all $t\in[0,1]$. □

In order, to obtain an a priori bound on the second order derivative of solutions of (5.2), we will use the following lemmas.

Lemma 5.4. *Let $f:[0,1]\times\mathbb{R}^{3n}\to\mathbb{R}^n$ be a Carathéodory function. Assume* (H1) *and* (H6). *Then, every solutions x of* (5.2) *satisfies*

$$\|x'''(t)\|\leq 2\phi(\|x''(t)\|)(\omega(t)+\|x''(t)\|)+\epsilon\|M'\|_0 \ \textit{a.e. } t\in[0,1].$$

Proof. Let x be a solution of (5.2). We know that $x\in T(v,M)$ by the previous lemma. So,

$$\|x'''(t)\|\leq\|f(t,x(t),x'(t),x''(t))\|+\epsilon\|x'(t)-v'(t)\|+\|G_\lambda(t,x(t),x'(t),x''(t))\|.$$

Almost everywhere on $\{t\in[0,1]:\|G_\lambda(t,x(t),x'(t),x''(t))\|>0\}$, we have by (H1) and Lemma 2.3,

$$\|G_\lambda(t,x(t),x'(t),x''(t))\|=\Big(M'''(t)+\frac{\langle x'(t)-v'(t),v'''(t)\rangle}{\|x'(t)-v'(t)\|}\Big)\\ +\Big(\frac{M''(t)^2-\|\widetilde{x}''(t)-v''(t)\|^2}{\|x'(t)-v'(t)\|}\Big)$$

$$\leq \frac{\langle x'(t) - v'(t), f(t, x(t), x'(t), \widetilde{x}''(t))\rangle}{\|x'(t) - v'(t)\|}$$
$$\leq \|f(t, x(t), x'(t), \widetilde{x}''(t))\| = \|f(t, x(t), x'(t), x''(t))\|.$$

Then, using (H6), we obtain

$$\|x'''(t)\| \leq 2\phi(\|x''(t)\|)(\omega(t) + \|x''(t)\|) + \epsilon\|M'\|_0.$$

□

Lemma 5.5. *Let* $f : [0,1] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ *be a Carathéodory function. Assume* (H1) *and* (H4). *Then, there exist* $b_0 > 0$, $c_0 \geq 0$ *and a function* $\delta_0 \in L^1([0,1], \mathbb{R})$ *such that for all solutions* x *of* (5.2),

$$(b_0 + c_0\|x'(t)\|)\sigma_0(t, x') + \frac{c_0\langle x'(t), x''(t)\rangle^2}{\|x'(t)\|\,\|x''(t)\|} \geq \|x'(t)\| - \delta_0(t)$$

almost everywhere on $\{t \in [0,1] : \|x''(t)\| \geq r\}$ *where* r *is given in* (H4) *and* σ_0 *is defined in Lemma* 4.4.

Proof. Let x be a solution of (5.2). By (5.1) and Lemma 5.3,

$$\|G_\lambda(t, x(t), x'(t), x''(t))\| \leq |M'''(t)| + \|v'''(t)\|. \tag{5.3}$$

Moreover, there exists a function $u \in \mathcal{G}(x, \lambda)$ such that

$$x'''(t) - \epsilon x'(t) = \widehat{f}^{\epsilon}_{\lambda}(t, x(t), x'(t), x''(t)) + u(t) \quad \text{a.e. } t \in [0,1].$$

We have

$$\begin{aligned}
&(b+c\|x'(t)\|)\sigma_0(t, x') \\
&\quad = \lambda(b + c\|x'(t)\|)\sigma(t, x(t), x'(t), x''(t)) + (1-\lambda)(b + c\|x'(t)\|)\|x''(t)\| \\
&\qquad + (b + c\|x'(t)\|)\Bigg(\frac{\langle x'(t), u(t)\rangle}{\|x''(t)\|} - \frac{\langle x''(t), u(t)\rangle\langle x'(t), x''(t)\rangle}{\|x''(t)\|^3} \\
&\qquad + \epsilon(1-\lambda)\Big(\frac{\langle x'(t), x'(t) - v'(t)\rangle}{\|x''(t)\|} - \frac{\langle x''(t), x'(t) - v'(t)\rangle\langle x'(t), x''(t)\rangle}{\|x''(t)\|^3}\Big)\Bigg).
\end{aligned}$$

Using (H4), we have

$$\begin{aligned}
&(b + c\|x'(t)\|)\sigma_0(t, x') \\
&\quad \geq (\lambda + b(1-\lambda))\|x''(t)\| - \lambda h(t) \\
&\qquad - 2(b + c(\|x'(t)\|)\Big(\epsilon\frac{\|x'(t)\|(\|x'(t) - v'(t)\| + \|u(t)\|)}{r}\Big) \\
&\quad \geq (\lambda + b(1-\lambda))\|x''(t)\| - \lambda h(t) \\
&\qquad - \frac{2}{r}(b + c(\|v'(t)\| + M'(t)))(\epsilon(\|v'(t)\| + M'(t))(M'(t) + \|v'''(t)\| + |M'''(t)|)).
\end{aligned}$$

Set $\nu = \min_{\lambda\in[0,1]}\{\lambda + (1-\lambda)b\}$, $b_0 = b/\nu$, $c_0 = c/\nu$ and

$$\begin{aligned}\nu\delta_0(t) &= -h(t)\\ &\quad - \frac{2}{r}(b + c(\|v'(t)\| + M'(t))(\epsilon(\|v'(t)\| + M'(t))(M'(t) + \|v'''(t)\| + |M'''(t)|)).\end{aligned}$$

Therefore, almost everywhere on $\{t \in [0,1] : \|x''(t)\| \geq r\}$, $(b_0 + c_0\|x'(t)\|)\sigma_0(t,x') \geq \|x'(t)\| - \delta_0(t)$, and hence

$$(b_0 + c_0\|x'(t)\|)\sigma_0(t,x') + \frac{c_0\langle x'(t), x''(t)\rangle^2}{\|x'(t)\|\,\|x''(t)\|} \geq \|x'(t)\| - \delta_0(t).$$

□

Lemma 5.6. *Let $f : [0,1]\times\mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1) *and* (H5). *Then, there exists a function $m_0 \in L^1([0,1],\mathbb{R})$ such that for all solutions x of* (5.2),

$$\|x'''(t)\| \leq a(\langle x'(t), x'''(t)\rangle + \|x''(t)\|^2) + m_0(t)$$

for almost every $t \in [0,1]$ where a is given in (H5).

Proof. Let x be a solution of (5.2). Using (H5), (5.1) and Lemma 5.3, we have

$$\begin{aligned}\|x'''(t)\| &\leq \lambda\|f(t,x(t),x'(t),x''(t))\| + \epsilon(1-\lambda)\|x'(t) - v'(t)\| + \|v'''(t)\| + |M'''(t)|\\ &\leq a(\langle x'(t), \lambda f(t,x(t),x'(t),x''(t))\rangle + \|x''(t)\|^2) + l(t)\\ &\qquad + \epsilon M'(t) + \|v'''(t)\| + |M'''(t)|\\ &\leq a(\langle x'(t), x'''(t)\rangle + \|x''(t)\|^2) + l(t) + \epsilon M'(t) + \|v'''(t)\| + |M'''(t)|\\ &\qquad + a\|x'(t)\|(\epsilon M'(t) + \|v'''(t)\| + |M'''(t)|).\end{aligned}$$

Then, it is easy to check that the proof is complete. □

Now, we can prove the main theorem of this section.

Theorem 5.7. *Let $f : [0,1]\times\mathbb{R}^{3n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1), (H6), *and* (H4) *or* (H5). *Then, there exists $x \in T(v,M)\cap W^{3,1}([0,1],\mathbb{R}^n)$ a solution of* (1.1).

Proof. To prove the existence of a constant $K > 0$ such that $\|x''\|_0 < K$, we will apply Lemma 4.3 to $x' \in W^{2,1}([0,1],\mathbb{R}^n)$. If we use (H4), conditions (i) and (ii) of Lemma 4.3 are satisfied from Lemmas 4.4, 5.3 and 5.5. If we use (H5), from Lemmas 4.5, 5.3 and 5.6, we deduce that conditions (i) and (ii) of Lemma 4.3 are satisfied. So, we have to check condition (iii). From Lemma 5.4, we have,

$$|\langle x''(t), x'''(t)\rangle| \leq \|x''(t)\|\big(2\phi(\|x''(t)\|) + \epsilon\|M'\|_0\big)\big(\omega(t) + \|x''(t)\| + 1\big)$$

almost everywhere on $[0,1]$. From (H6), we can choose $\epsilon \in [0,1]$ sufficiently small to have

$$\int_r^\infty \frac{sds}{s(2\phi(s)+\epsilon\|M'\|_0)}$$

as large as needed to apply Lemma 4.3 with $\psi(s) = s(2\phi(s)+\epsilon\|M'\|_0)$ and $m(t) = \omega(t)+1$.

A solution of (5.2) is a fixed point of the multivalued operator

$$D^{-1}\circ L_\epsilon^{-1}\circ N_{\mathcal{S}^\epsilon} : C^2([0,1],\mathbb{R}^n)\times[0,1] \to C^2_{x_0,B}([0,1],\mathbb{R}^n)\subset C^2([0,1],\mathbb{R}^n)$$

which is fixed point free on ∂U with

$$U = \{x\in C^2([0,1],\mathbb{R}^n) : \|x^{(i)}\|_0 < \|v^{(i)}\|_0 + \|M^{(i)}\|_0 + 1; i = 0,1; \|x''\|_0 < K\}.$$

By using the multivalued version of Leray-Schauder degree for upper semi-continuous, compact map with nonempty, compact and convex values and arguing as in the proof of Theorem 3.5, we deduce the existence of a solution of (5.2) and hence, the existence of a solution of (1.1). □

Example 5.8. Consider the system

$$\begin{aligned} x'''(t) &= \phi(\|x''(t)\|)\langle x(t),x'(t)\rangle^2 x''(t)\\ x(0) &= 0,\ x'(0)=0,\ x''(1)=r_1, \end{aligned} \tag{5.4}$$

where $\|r_1\| > 0$ and $\phi : [0,\infty[\to\ ]0,\infty[$ is a Borel measurable function such that for all $c\geq 0$, $\int_c^\infty \frac{ds}{\phi(s)} = \infty$. Verify that with $v(t)\equiv 0$, $M(t) = \|r_1\|t^2/2$, (v,M) is a solution-tube of (5.4). Also, assumption (H4) is satisfied with $b=1$, $c=0$ and $h(t)\equiv 0$. Since it is easy to check (H6), from Theorem 5.7, the system (5.4) has at least one solution x such that $\|x(t)\|\leq \|r_1\|t^2/2$ and $\|x'(t)\|\leq\|r_1\|t$ for all $t\in[0,1]$.

Example 5.9. Consider the system

$$\begin{aligned} x'''(t) &= \|x''(t)\|^2\langle x(t),x'(t)\rangle^2 x'(t)\\ x(0) &= 0,\ x'(0)=0,\ x''(1)=r_1, \end{aligned} \tag{5.5}$$

where $\|r_1\|>0$. Verify that with $v(t)\equiv 0$, $M(t)=\|r_1\|t^2/2$, (v,M) is a solution-tube of (5.5). Assumption (H5) is satisfied with $a = \|r_1\|^3$, $l(t)\equiv 0$, and it is easy to check (H6). From Theorem 5.7, the system (5.5) has at least one solution x such that $\|x(t)\|\leq\|r_1\|t^2/2$ and $\|x'(t)\|\leq\|r_1\|t$ for all $t\in[0,1]$.

Remark 5.10. In the scalar case where $x_0 = 0$ and $f : [0,1]\times\mathbb{R}^3\to\mathbb{R}$ is continuous, Grossinho and Minhós [12] obtained the existence of a solution to (1.1), (1.2) assuming (C) (stated in Remark 3.2) and

(D) there exist $a>0$ and a continuous function $h : [0,\infty[\to[a,\infty[$ such that
   - (i) $|f(t,x,y,z)|\leq h(|z|)$ for all $t\in[0,1]$ and for every $(x,y,z)\in\mathbb{R}^3$ such that $\alpha(t)\leq x\leq\beta(t)$ and $\alpha'(t)\leq y\leq\beta'(t)$;

(ii) $\int_0^\infty \frac{sds}{h(s)} = \infty$.

This assumption is a particular case of (H6). Since (H4) is trivially satisfied in the scalar case and (H1) generalizes (C), Theorem 5.7 generalizes their result not only for systems but also in the scalar case in the particular case where $A_0 \geq 0$ and $A_1 \geq 0$.

Remark 5.11. The same authors, see [13], have recently used ideas developed in [12] to prove an existence theorem for a m^{th} order equation ($m \geq 2$) where $x^{(i)}(0) = 0$ for $i = 0, 1, \ldots, m-3$ and where $x^{(m-2)}$ satisfies (1.2). It could be easy to extend the last theorem to m^{th} order systems with $x^{(i)}(0) = x_i \in \mathbb{R}^n$ for $i = 0, 1, \ldots, m-3$ and $x^{(m-2)}$ satisfying (1.2) or (1.3). We decided to present only the case $m = 3$ to avoid heavy notations. The case $m = 2$ is treated in [8].

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