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A NOTE ON THE CAUCHY PROBLEM FOR DIFFERENTIAL INCLUSIONS

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Dedicated to the memory of Karol Borsuk

In this note, we shall be concerned with the existence of global solutions to the initial value problem for systems of differential inclusions of the type

$$(\star)\qquad \begin{cases} y^{(k)}(t) \in F(t, y(t), \dots, y^{(k-1)}(t)) & \text{a.e. } t \in [0, T], \\ y^{(i)}(t_i) = r_i, & i = 0, \dots, k-1, \end{cases}$$

where $F : [0, T] \times \mathbb{R}^{kn} \to \mathbb{R}^n$ is a multifunction with nonempty compact values satisfying some conditions of measurability, and upper or lower semi-continuity; $t_i \in [0, T]$ and $r_i \in \mathbb{R}^n$, $i = 0, \dots, k-1$.

For $k = 1$, the above Cauchy problem was treated in our Notes [2,3]. However, for $k > 1$, the established results are new even in the case where F is a single valued Carathéodory or a continuous function.

1. Preliminaries

(a) Generalities

We denote by $C^k([0, T], \mathbb{R}^n)$ the Banach space of functions k-times continuously differentiable on $[0, T]$ with the norm: $\|y\|_k = \max\{\|y\|_0, \dots, \|y^{(k)}\|_0\}$, where $\|y\|_0 = \max\{\|y(t)\| : t \in [0, T]\}$. The Banach space of functions y such that

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$\|y\|$ is Lebesgue integrable on $[0,T]$ is denoted by $L^1([0,T],\mathbb{R}^n)$ and is provided with the usual norm: $\|y\|_{L^1} = \int_0^T \|y(t)\|dt$. We denote by $W^{k,1}([0,T],\mathbb{R}^n)$ the class of function y in $C^{k-1}([0,T],\mathbb{R}^n)$ such that $y^{(k-1)}$ is an absolutely continuous function. By a *solution* to $(\star)$ we shall always mean a solution in the Carathéodory sense, i.e. a function y in $W^{k,1}([0,T],\mathbb{R}^n)$, which satisfies $(\star)$ almost everywhere.

Let E_1, E_2 be two Banach spaces, X a closed subset of E_1, and S a measurable space (resp. $S = I \times \mathbb{R}^n$ where I is a real interval, and $A \subset S$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $N \times D$, where N is Lebesgue measurable in I, and D is Borel measurable in $\mathbb{R}^n$). Let $F : X \to E_2$ and $G : S \to E_2$ be two multifunctions with nonempty closed values. The function G is *measurable* (resp. $\mathcal{L}\otimes\mathcal{B}$ *measurable*) if the set $\{t \in S : G(t)\cap B \neq \emptyset\}$ is measurable for any closed set B in S. The function F is *lower semi-continuous* (*l.s.c.*) (resp. *upper semi-continuous* (u.s.c.)) if the set $\{x \in X : F(x) \cap B \neq \emptyset\}$ is open (resp. closed) for any open (resp. closed) set B in E_2. If F is l.s.c. and u.s.c. then F is *continuous*.

A subset A of $L^1([0,T],\mathbb{R}^n)$ is *decomposable* if for all $u, v \in A$ and $N \subset [0,T]$ measurable the function $u\chi_N + v\chi_{[0,T]\setminus N} \in A$.

(b) Two basic types of multivalued maps

In what follows, $F : [0,T] \times \mathbb{R}^{kn} \to \mathbb{R}^n$ will be a multifunction with nonempty, compact values. We assign to F two multivalued operators

$$\mathcal{F} : C^{k-1}([0,T],\mathbb{R}^n) \to L^1([0,T],\mathbb{R}^n)$$

and

$$\mathcal{N} : C^{k-1}([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R}^n)$$

by letting:

$$\mathcal{F}(y) = \{v \in L^1([0,T],\mathbb{R}^n) : v(t) \in F(t, y(t), \dots, y^{(k-1)}(t)) \text{ a.e. } t \in [0,T]\}.$$

$$\mathcal{N}(y) = \{w \in C([0,T],\mathbb{R}^n) : w(t) = \int_0^t v(s)ds \text{ with } v \in \mathcal{F}(y)\}.$$

The operator $\mathcal{F}$ (resp. $\mathcal{N}$) is called the *Niemytzki* (resp. the *Carathéodory*) *operator* associated to F.

Using the above terminology we can describe two basic types of multivalued maps:

DEFINITION 1.1. *The multivalued function F is said to be of:*

(i) *the upper semi-continuous type (u.s.c. type) if its associated Caratheodory operator $\mathcal{N} : C^{k-1}([0,T],\mathbb{R}^n) \to C([0,T],\mathbb{R}^n)$ is upper semi-continuous and has nonempty compact and convex values;*

(ii) *the lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator* $\mathcal{F} : C^{k-1}([0,T],\mathbb{R}^n) \to L^1([0,T],\mathbb{R}^n)$ *is lower semi-continuous and has nonempty closed and decomposable values.*

Consider the following properties of the map F:

(H1) (i) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
(ii) $x \mapsto F(t,x)$ is lower semi-continuous for a.e. $t \in [0,T]$;

(H1′) (i) $t \mapsto F(t,x)$ is measurable for every $x \in \mathbb{R}^{kn}$,
(ii) $x \mapsto F(t,x)$ is continuous for a.e. $t \in [0,T]$;

(H1″) (i) $t \mapsto F(t,x)$ is measurable for every $x \in \mathbb{R}^{kn}$,
(ii) $x \mapsto F(t,x)$ is upper semi-continuous for a.e. $t \in [0,T]$;

(H2) for each $r > 0$, there exists a function $h_r \in L^1(0,T)$ such that $\|F(t,x)\| \le h_r(t)$ a.e. $t \in (0,T)$, and for every $x \in \mathbb{R}^{kn}$ with $\|x\| \le r$.

We formulate two results describing the two classes of multivalued maps introduced above.

PROPOSITION 1.2. (cf. [4]) *If F has compact convex values and satisfies* (H1″), (H2), *then F is of the u.s.c. type.*

PROPOSITION 1.3. (cf. [2]) *If F satisfies* (H2) *and either* (H1) *or* (H1′), *then F is of the l.s.c. type.*

REMARK 1.4. Assume that a single valued function $f : [0,T] \times \mathbb{R}^{kn} \to \mathbb{R}^n$ is a *Carathéodory function*, that is: (i) the map $t \mapsto f(t,x)$ is measurable for all $x \in \mathbb{R}^{kn}$; (ii) the map $x \mapsto f(t,x)$ is continuous for a.e. $t \in [0,T]$; (iii) for each $r > 0$, there exists a function $h_r \in L^1(0,T)$ such that $\|f(t,x)\| \le h_r(t)$ for a.e. $t \in (0,T)$, and for every $x \in \mathbb{R}^{kn}$ with $\|x\| \le r$. Then by letting $F(t,x) = \{f(t,x)\}$, we obtain a multifunction F of the u.s.c. and the l.s.c. type.

(c) The existence principles

Given $t_i \in [0,T]$ and $r_i \in \mathbb{R}^n$, $i = 0,\dots,k-1$, we consider the Cauchy problem $(\star)$ and the associated family of problems:

$$(\star)_\lambda \qquad \begin{cases} y^{(k)}(t) \in \lambda F(t, y(t), \dots, y^{(k-1)}(t)) & \text{a.e. } t \in [0,T], \\ y^{(i)}(t_i) = r_i, & i = 0,\dots,k-1, \end{cases}$$

where $\lambda \in [0,1]$.

The existence principles for the two basic types of differential inclusions are collected together in the following theorem and proved in a slightly less general form in [2] and [4].

Theorem 1.5. *Let F be a multifunction of the u.s.c. type or of the l.s.c. type. Suppose there exists a constant M such that for any λ in $[0,1]$ and any solution y to $(\star)_\lambda$, we have $\|y\|_{k-1} \leq M$. Then $(\star)$ has a solution.*

(d) The change of variables formula

In our discussion, we shall need a version of the change of variables formula for the integral (established in [3]), which relaxes the standard monotonicity assumption.

Lemma 1.6. *Let $g : [a,b] \to [A,B]$ and $h : [A,B] \to \mathbb{R}$, where g is absolutely continuous, h is measurable and $(h \circ g)g'$ is Lebesgue integrable on $[a,b]$. Then h is integrable on the interval with end points $g(a)$ and $g(b)$ and*

$$\int_{g(a)}^{g(b)} h(s)ds = \int_a^b h(g(t))g'(t)dt.$$

2. Main result

Consider the following conditions:

(H3) there exists a Borel measurable function $\alpha : (\|r_{k-1}\|,\infty) \to (0,\infty)$ such that $\|F(t,y_1,\dots,y_k)\| \leq \alpha(\|y_k\|)$ for a.e. $t \in [0,T]$ and for all $(y_1,\dots,y_k) \in \mathbb{R}^{kn}$ with $\|y_k\| > \|r_{k-1}\|$;

(H4) there exists a Borel measurable function $\alpha : [\|R\|,\infty) \to [0,\infty)$ such that $\|F(t,y_1,\dots,y_k))\| \leq \alpha(\|(y_1,\dots,y_k)\|)$ for a.e. $t \in [0,T]$ and for all $(y_1,\dots,y_k) \in \mathbb{R}^{kn}$ with $\|y_1,\dots,y_k)\| > \|R\|$, where $R = (r_0,\dots,r_{k-1})$.

We are able now to formulate our main result:

Theorem 2.1. *Assume that F is of the u.s.c. type or of the l.s.c. type and satisfies the condition* (H3). *If*

$$T < T_\infty = \int_{\|r_{k-1}\|}^{\infty} \frac{dx}{\alpha(x)},$$

the problem $(\star)$ has a solution.

Proof. In view of Theorem 1.5, we need only find a priori bounds on solutions y to the family of problems $(\star)_\lambda$ with λ in $[0,1]$.

Fix $T < T_\infty$ and let y be a solution to $(\star)_\lambda$ for some λ in $[0,1]$. Assuming that the function $t \mapsto \|y^{(k-1)}(t)\|$ takes its maximum at $\tau \in [0,T]$ and $\|y^{(k-1)}(\tau)\| > \|r_{k-1}\|$, we obtain the existence of an interval $(a,\tau]$ (or $[\tau,a)$) on which $\|y^{(k-1)}(t)\| >$

$\|r_{k-1}\|$ and $\|y^{(k-1)}(a)\| = \|r_{k-1}\|$. Since the function $t \mapsto \|y^{(k-1)}(t)\|$ is absolutely continuous on $(a, \tau]$ and

$$\|y^{(k-1)}(t)\|' \leq \|y^{(k)}(t)\| \quad \text{a.e. } t \in (a, \tau],$$

by (H3), we get

$$\|y^{(k-1)}(t)\|' \leq \alpha(\|y^{(k-1)}(t)\|) \quad \text{a.e. } t \in (a, \tau].$$

Dividing by $\alpha(\|y^{(k-1)}(t)\|)$, integrating from a to τ, and applying the change of variables formula (Lemma 1.6), we get

$$\int_{\|r_{k-1}\|}^{\|y^{(k-1)}(\tau)\|} \frac{dx}{\alpha(x)} = \int_a^\tau \frac{\|y^{(k-1)}(t)\|'}{\alpha(\|y^{(k-1)}(t)\|)} dt \leq T < T_\infty = \int_{\|r_{k-1}\|}^{\infty} \frac{dx}{\alpha(x)}.$$

It follows that there exists a constant $M_0 \geq \|r_{k-1}\|$ such that

$$\|y^{(k-1)}\|_0 \leq M_0 \tag{2.1}$$

for any solution y to $(\star)_\lambda$ for some λ in $[0,1]$.

Taking into account the inequality (2.1) and the initial conditions, it is easy to show the existence of a constant M such that $\|y\|_{k-1} \leq M$ for any solution y to $(\star)_\lambda$ for some λ in $[0, 1]$. □

COROLLARY 2.2. *Let $f : [0, T] \times \mathbb{R}^{kn} \to \mathbb{R}^n$ be a single valued Carathéodory function. Assume there exists $\alpha : (\|r_{k-1}\|, \infty) \to (0, \infty)$ a Borel measurable function such that $\|f(t, y_1, \ldots, y_k)\| \leq \alpha(\|y_k\|)$ a.e. $t \in [0, T]$ for all y_k with $\|y_k\| > \|r_{k-1}\|$. If*

$$T < T_\infty = \int_{\|r_{k-1}\|}^{\infty} \frac{dx}{\alpha(x)},$$

then the following problem has a solution

$$(\star\star) \qquad \begin{cases} y^{(k)}(t) \in f(t, y(t), \ldots, y^{(k-1)}(t)) & \text{a.e. } t \in [0, T], \\ y^{(i)}(t_i) = r_i, & i = 0, \ldots, k-1. \end{cases}$$

COROLLARY 2.3. *Let F be a multifunction of the u.s.c. type or of the l.s.c. type. Assume that the condition* (H4) *is satisfied. If $t_i = t_0 \in [0, T]$ for all $i = 0, \ldots, k-1$, and*

$$T < \tilde{T}_\infty = \int_{\|R\|}^{\infty} \frac{dx}{\sqrt{x^2 + \alpha(x)^2}}, \quad \text{where } R = (r_0, \ldots, r_{k-1}),$$

then the problem $(\star)$ has a solution.

Proof. Set $Y = (y_0, \dots, y_{k-1}) \in \mathbb{R}^{kn}$ and let $G : [0, T] \times \mathbb{R}^{kn} \to \mathbb{R}^{kn}$ be defined by $G(t, Y) = (y_1, \dots, y_{k-1}, F(t, Y))$. The multifunction G satisfies the condition (H3) with $\alpha_0(x) = \sqrt{x^2 + (\alpha(x))^2}$. By applying Theorem 2.1 to the problem

$$\begin{cases} Y'(t) \in G(t, Y(t)) & \text{a.e. } t \in [0, T], \\ Y(t_0) = R, \end{cases}$$

we get a solution $Y = (y_0, \dots, y_{k-1})$. It is clear that y_0 is a solution to $(\star)$. $\square$

Remark 2.6. In general, we cannot replace the condition (H3) by (H4) in Theorem 2.1. However, when both Theorem 2.1 and Corollary 2.3 apply, Theorem 2.1 furnishes best results in general.

3. Examples

Example 3.1. The problem

$$\begin{cases} y''(t) = y'(t) + \sin(y(t)) + 3 & \text{a.e. } t \in [0, T], \\ y(0) = 0, \; y'(1) = 1, \end{cases}$$

has a solution on $[0, T]$ for any $1 \le T < \infty$. The condition (H3) is satisfied with $\alpha(x) = x + 4$, so $T_\infty = \infty$.

Example 3.2. The problem

$$\begin{cases} y''(t) = y(t) - e^{-t} & \text{a.e. } t \in [0, T], \\ y(0) = 1, \; y'(0) = 0, \end{cases}$$

has a solution on $[0, T]$ for any $T < \infty$. Indeed, the condition (H4) is satisfied with $\alpha(x) = x + 1$.

Example 3.3. The problem

$$\begin{cases} y''(t) = y'(t)^2 - 2|y'(t)| + 2 & \text{a.e. } t \in [0, T], \\ y(0) = 0, y'(0) = 0, \end{cases}$$

has a solution on $[0, T]$ for any $T < 3\pi/4$. The condition (H3) is satisfied with $\alpha(x) = x^2 - 2x + 2$ and $T_\infty = 3\pi/4$. On the other hand, (H4) is satisfied with

$$\alpha_0(x) = \begin{cases} 2, & \text{if } 0 < x \le 2, \\ x^2 - 2x + 2, & \text{if } x > 2, \end{cases}$$

and

$$\tilde{T}_\infty = \int_0^\infty \frac{dx}{\sqrt{x^2 + (\alpha_0(x))^2}} \approx 1.57 < 3\pi/4 = T_\infty.$$

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