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Global existence of analytic solutions to the Cauchy problem in a complex domain

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À la mémoire de Jean Leray

Abstract. We present results on the global existence of analytic solutions to the Cauchy problem in starshaped or convex complex domains. No growth conditions are imposed. Our results rely on a notion of solution-tube that we introduce.

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0. Introduction

In 1990, in a note published in the Comptes rendus de l'Académie des sciences de Paris [1], results on the global existence of an analytic solution to the following system of differential equations in a complex domain were presented:

$$\begin{aligned} &u'(z) = f(z, u(z)), \quad z \in \Omega, \\ &u(0) = u_0 \in \mathbb{C}^n. \end{aligned} \tag{0.1}$$

That note was presented by Jean Leray. Thanks to him, the results in that note and their presentation were much nicer than they were in the first version of that paper. Following that note, Jean Leray [5] established results on the analytic extension of the solution of a nonlinear analytic differential system using the method developed in a paper of Hamada, Leray and Takeuchi [4]. This special volume dedicated to Jean Leray leads me to reconsider this problem.

In [1], the results concerned analytic maps f satisfying a growth condition of Wintner type (see [8]). They generalized O'Regan's results [7].

In this paper, we establish the existence of an analytic solution to (0.1) on a starshaped domain without assuming any growth condition on f. We introduce a notion of solution-tube for this problem. This notion is inspired by the notion of solution-tube introduced in [2] for real second order systems of differential equations, and in [6] for real first order systems of differential equations (see also [3]). This concept permits us to get a priori bounds for the solutions of a suitable family of systems of differential equations. Our existence result relies on the Leray–Schauder degree theory.

Finally, we consider other initial conditions, namely $u(0) = w_0$ (resp. $u(\zeta) = w_0$) and we deduce the existence of a solution $(z, w_0) \mapsto u(z, w_0)$ (resp. $(z, w_0, \zeta) \mapsto u(z, w_0, \zeta)$) analytic on $\Omega \times \mathcal{O}$ for a suitable open set $\mathcal{O}$.

1. Preliminaries

Let $\langle \cdot, \cdot \rangle$ be the sesquilinear form inducing the usual norm $\| \cdot \|$ in $\mathbb{C}^n$. For $\Omega \subset \mathbb{C}$ a complex domain, we let $A(\overline{\Omega}, \mathbb{C}^n)$ (resp. $A^1(\overline{\Omega}, \mathbb{C}^n)$) be the Banach space of maps $u : \overline{\Omega} \to \mathbb{C}^n$ analytic on Ω, and continuous on $\overline{\Omega}$ (resp. with u' continuous on $\overline{\Omega}$) endowed with the norm $\|u\|_0 := \max\{\|u(z)\| : z \in \overline{\Omega}\}$ (resp. $\|u\|_1 := \max\{\|u\|_0, \|u'\|_0\}$).

The following lemma will be used to get a priori bounds.

Lemma 1.1. *Let $u \in A^1(\overline{\Omega}, \mathbb{C}^n)$ and $z, \zeta \in \overline{\Omega}$ with $z \neq \zeta$ and $[\zeta, z] \subset \overline{\Omega}$. Then for all $t \in [0, 1]$,*

$$\frac{d}{dt}\|u(\zeta + t(z - \zeta))\|^2 = 2\,\mathrm{Re}((z - \zeta)\langle u'(\zeta + t(z - \zeta)), u(\zeta + t(z - \zeta))\rangle).$$

In what follows, we consider a complex domain Ω starshaped at 0 (i.e. $tz \in \Omega$ for all $(t, z) \in [0, 1] \times \Omega$). We let $DD(\overline{\Omega}, \mathbb{R})$ be the space of directionally differentiable maps, i.e.

$$DD(\overline{\Omega}, \mathbb{R}) := \{M : \overline{\Omega} \to \mathbb{R} \text{ continuous} : \forall z \in \overline{\Omega} \setminus \{0\}, \text{ the map } t \mapsto M(tz) \text{ defined on } [0, 1] \text{ is differentiable at } t = 1\}.$$

For $M \in DD(\overline{\Omega}, \mathbb{R})$, we define

$$D_z M(z) := \frac{d}{dt} M(tz)|_{t=1}.$$

2. Main result

Let $\Omega \subset \mathbb{C}$ be a starshaped domain and $f : \overline{\Omega} \times \mathbb{C}^n \to \mathbb{C}^n$. We introduce the notion of solution-tube for Problem (0.1).

Definition 2.1. We say that $(v, M) \in A^1(\overline{\Omega}, \mathbb{C}^n) \times DD(\overline{\Omega}, \mathbb{R})$ is a *solution-tube of* (0.1) if

(i) $M(z) > 0$ for all $z \in \overline{\Omega}$, and $\|u_0 - v(0)\| < M(0)$;

(ii) for all $(z, u) \in (\overline{\Omega} \setminus \{0\}) \times \mathbb{C}^n$ such that $\|u - v(z)\| = M(z)$,
$$\mathrm{Re}(z\langle f(z, u) - v'(z), u - v(z)\rangle) < M(z) D_z M(z);$$

(iii) there exists $c \in \mathbb{C}$ such that $M(z)\,\mathrm{Re}(cz) + D_z M(z) \geq 0$ for all $z \in \overline{\Omega} \setminus \{0\}$.

Remark 2.2. In the particular case where $M > \|u_0\|$ is a real constant, $(0, M)$ is a solution-tube of (0.1) if and only if

$$\mathrm{Re}(z\langle f(z,u),u\rangle) < 0 \qquad \forall z \in \overline{\Omega} \setminus \{0\} \text{ and } \|u\| = M.$$

The notion of solution-tube leads to the following global existence result.

Theorem 2.3. *Let the map $f : \overline{\Omega} \times \mathbb{C}^n \to \mathbb{C}^n$ be analytic on $\Omega \times \mathbb{C}^n$ and continuous on $\overline{\Omega} \times \mathbb{C}^n$. Assume that there exists a solution-tube $(v, M) \in A^1(\overline{\Omega}, \mathbb{C}^n) \times DD(\overline{\Omega}, \mathbb{R})$ of* (0.1)*. Then Problem* (0.1) *has a solution $u \in A^1(\overline{\Omega}, \mathbb{C}^n)$ such that $\|u(z) - v(z)\| < M(z)$ for all $z \in \overline{\Omega}$.*

Proof. For $\lambda \in [0,1]$, we consider the family of problems

$$\begin{aligned} u'(z) &= f_\lambda(z, u(z)), \qquad z \in \Omega, \\ u(0) &= \lambda u_0 + (1-\lambda) v(0); \end{aligned} \tag{P1$_\lambda$}$$

where

$$f_\lambda(z,u) := \lambda f(z,u) + (1-\lambda)(v'(z) - c(u - v(z))),$$

where c is given in Definition 2.1.

We claim that

$$\|u(z) - v(z)\| < M(z) \quad \forall z \in \overline{\Omega}, \tag{2.1}$$

for all solutions u of (P1$_\lambda$) for any $\lambda \in [0,1]$. Indeed, assume that u is a solution of (P1$_\lambda$) such that there exists $z \in \overline{\Omega}$ for which $\|u(z) - v(z)\| \geq M(z)$. Obviously $\lambda > 0$ and $z \neq 0$ by Definition 2.1(i) and the initial value condition. We can assume without loss of generality that $\|u(tz) - v(tz)\| < M(tz)$ for all $t \in [0,1[$. Therefore

$$\frac{d}{dt}(\|u(tz) - v(tz)\|^2 - M^2(tz))\bigg|_{t=1} \geq 0.$$

By Lemma 1.1, this means that

$$0 \leq \mathrm{Re}(z\langle u'(z) - v'(z), u(z) - v(z)\rangle) - M(z) D_z M(z).$$

It follows from Definition 2.1 that

$$\begin{aligned} 0 \leq{} & \mathrm{Re}(z\langle f_\lambda(z,u(z)) - v'(z), u(z) - v(z)\rangle) - M(z) D_z M(z) \\ &= \lambda(\mathrm{Re}(z\langle f(z,u(z)) - v'(z), u(z) - v(z)\rangle) - M(z) D_z M(z)) \\ &\quad + (\lambda - 1)(M^2(z)\,\mathrm{Re}(cz) + M(z) D_z M(z)) \\ &< 0, \end{aligned}$$

a contradiction.

Now, define $H : A(\overline{\Omega}, \mathbb{C}^n) \times [0,1] \to A(\overline{\Omega}, \mathbb{C}^n)$ by

$$H(u,\lambda)(z) := \lambda u_0 + (1-\lambda) v(0) + \int_0^z f_\lambda(\zeta, u(\zeta))\, d\zeta.$$

This operator is continuous and completely continuous since $(z,u,\lambda) \mapsto f_\lambda(z,u,\lambda)$ is continuous on $\overline{\Omega} \times \mathbb{C}^n \times [0,1]$, and analytic on $\Omega \times \mathbb{C}^n$ with respect to (z,u). Notice that fixed points of $H(\cdot, \lambda)$ are solutions to (P1$_\lambda$).

Define $\mathcal{U} := \{u \in A(\overline{\Omega}, \mathbb{C}^n) : \|u(z) - v(z)\| < M(z)\ \forall z \in \overline{\Omega}\}$. From (2.1), $H(\cdot, \lambda)$ has no fixed point on $\partial\mathcal{U}$ for any $\lambda \in [0, 1]$. The Leray–Schauder degree theory implies that

$$\deg(I - H(\cdot, 1), \mathcal{U}, 0) = \deg(I - H(\cdot, 0), \mathcal{U}, 0).$$

Here is a simple argument to show that this degree is one. Consider the family of problems for $\lambda \in [0, 1]$,

$$\begin{aligned} &u'(z) = v'(z) - \lambda c(u(z) - v(z)), \qquad z \in \Omega,\\ &u(0) = v(0), \end{aligned} \tag{P2$_\lambda$}$$

and define $T : A(\overline{\Omega}, \mathbb{C}^n) \times [0, 1] \to A(\overline{\Omega}, \mathbb{C}^n)$ by

$$T(u, \lambda)(z) := v(0) + \int_0^z (v'(\zeta) - \lambda c(u(\zeta) - v(\zeta)))\, d\zeta.$$

Observe that $T(\cdot, 1) = H(\cdot, 0)$. Also v is the unique solution of (P2$_\lambda$), i.e. the unique fixed point of $T(\cdot, \lambda)$, for every $\lambda \in [0, 1]$. Since $I - T(\cdot, 0) = I - v$, and $v \in \mathcal{U}$,

$$1 = \deg(I - T(\cdot, 0), \mathcal{U}, 0) = \deg(I - T(\cdot, 1), \mathcal{U}, 0) = \deg(I - H(\cdot, 1), \mathcal{U}, 0).$$

Therefore, (P1$_1$) and hence (0.1) have a solution $u \in \mathcal{U} \cap A^1(\overline{\Omega}, \mathbb{C}^n)$. □

Example 2.4. Let Ω be the open unit ball in $\mathbb{C}$. Consider the problem

$$\begin{aligned} &u'(z) = \alpha + \beta z^k + e^{z\langle\gamma, u\rangle} u/5, \qquad z \in \Omega,\\ &u(0) = 0, \end{aligned} \tag{2.2}$$

with $k \in \mathbb{N}$ and $\alpha, \beta, \gamma \in \mathbb{C}^n$ such that $\|\alpha\| \le 1$, $\|\beta\| \le 1$, and $\|\gamma\| \le 1/4$. Obviously, the map $M : \overline{\Omega} \to \mathbb{R}$ defined by $M(z) := 1 + 3\|z\|$ is in $DD(\overline{\Omega}, \mathbb{R})$. It is easy to verify that $(0, M)$ is a solution-tube of (2.2) with $c = 0$. Therefore, Theorem 2.3 implies the existence of a solution $u \in A^1(\overline{\Omega}, \mathbb{C}^n)$ such that $\|u(z)\| < M(z)$ for all $z \in \overline{\Omega}$.

Now, consider the Cauchy problem with a different initial value:

$$\begin{aligned} &u'(z) = f(z, u(z)), \quad z \in \Omega,\\ &u(0) = w_0. \end{aligned} \tag{2.3}$$

Observe that if (v, M) is a solution-tube of (0.1), it is also a solution-tube of (2.3) for every

$$w_0 \in \mathcal{W} := \{w_0 \in \mathbb{C}^n : \|w_0 - v(0)\| < M(0)\}.$$

This remark leads to the following result.

Theorem 2.5. *Let the map $f : \overline{\Omega} \times \mathbb{C}^n \to \mathbb{C}^n$ be analytic on $\Omega \times \mathbb{C}^n$ and continuous on $\overline{\Omega} \times \mathbb{C}^n$. Assume that there exists a solution-tube $(v, M) \in A^1(\overline{\Omega}, \mathbb{C}^n) \times DD(\overline{\Omega}, \mathbb{R})$ of (0.1). Then there exists $u : \overline{\Omega} \times \mathcal{W} \to \mathbb{C}^n$ analytic on $\Omega \times \mathcal{W}$ such that for every $w_0 \in \mathcal{W}$, $u(\cdot, w_0) \in A^1(\overline{\Omega}, \mathbb{C}^n)$ is a solution of (2.3) such that $\|u(z, w_0) - v(z)\| < M(z)$ for all $z \in \overline{\Omega}$.*

Again, we want to consider a different initial condition, and also a different domain. Consider

$$\begin{aligned} u'(z) &= f(z, u(z)), \qquad z \in \Omega, \\ u(\zeta) &= w_0, \end{aligned} \tag{2.4}$$

where $\Omega \subset \mathbb{C}$ is a convex domain. For $\zeta \in \Omega$, we define

$$DD_\zeta(\overline{\Omega}, \mathbb{R}) := \{M : \overline{\Omega} \to \mathbb{R} \text{ continuous} : \forall z \in \overline{\Omega} \setminus \{\zeta\}, \text{ the map } t \mapsto M(\zeta + t(z-\zeta)) \text{ defined on } [0,1] \text{ is differentiable at } t = 1\}.$$

For $M \in DD_\zeta(\overline{\Omega}, \mathbb{R})$, we set

$$D_{z,\zeta} M(z) := \frac{d}{dt} M(\zeta + t(z-\zeta))\Big|_{t=1}.$$

Using Lemma 1.1 and arguing as in the proof of Theorem 2.3, we can prove the following result.

Theorem 2.6. *Let Ω be a convex domain in $\mathbb{C}$ and $\Gamma \subset \Omega$ an open subset. Let the map $f : \overline{\Omega} \times \mathbb{C}^n \to \mathbb{C}^n$ be analytic on $\Omega \times \mathbb{C}^n$ and continuous on $\overline{\Omega} \times \mathbb{C}^n$. Assume that there exist continuous maps $v : \overline{\Omega} \times \Gamma \to \mathbb{C}^n$ and $M : \overline{\Omega} \times \Gamma \to]0, \infty[$ such that for every $\zeta \in \Gamma$,*

(i) *$v(\cdot, \zeta) \in A^1(\overline{\Omega}, \mathbb{C}^n)$ and $M(\cdot, \zeta) \in DD_\zeta(\overline{\Omega}, \mathbb{R})$;*
(ii) *for all $(z, u) \in (\overline{\Omega} \setminus \{\zeta\}) \times \mathbb{C}^n$ such that $\|u - v(z, \zeta)\| = M(z, \zeta)$,*
$$\mathrm{Re}\left((z-\zeta) \left\langle f(z,u) - \frac{\partial v}{\partial z}(z,\zeta), u - v(z,\zeta) \right\rangle \right) < M(z,\zeta) D_{z,\zeta} M(z,\zeta);$$
(iii) *there exists $c_\zeta \in \mathbb{C}$ such that $M(z,\zeta)\,\mathrm{Re}(c_\zeta(z-\zeta)) + D_{z,\zeta}M(z,\zeta) \geq 0$ for all $z \in \overline{\Omega} \setminus \{\zeta\}$.*

Let
$$\mathcal{O} := \{(w_0, \zeta) \in \mathbb{C}^n \times \Gamma : \|w_0 - v(\zeta,\zeta)\| < M(\zeta,\zeta)\}.$$
Then there exists $u : \overline{\Omega} \times \mathcal{O} \to \mathbb{C}^n$ analytic on $\Omega \times \mathcal{O}$ such that for every $(w_0, \zeta) \in \mathcal{O}$, $u(\cdot, w_0, \zeta) \in A^1(\overline{\Omega}, \mathbb{C}^n)$ is a solution of (2.4) such that $\|u(z, w_0, \zeta) - v(z,\zeta)\| < M(z,\zeta)$ for all $z \in \overline{\Omega}$.

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