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Boundary Value Problems for Systems of Implicit Differential Equations*

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0. INTRODUCTION

We study the existence of solutions $y: [0, T] \to \mathbb{R}^n$ of implicit differential equations of the type

$$F(t, y(t), y'(t), ..., y^{(k)}(t)) = 0, \qquad t \in [0, T], \tag{0.1}$$

under various boundary conditions. There is little known about the existence of solutions to such boundary value problems, unless it is possible to solve the equation

$$F(t, y_0, y_1, ..., y_k) = 0, \qquad t \in [0, T],\ y_i \in \mathbb{R}^n,\ i = 0, 1, ..., k, \tag{0.2}$$

for the last variable y_k and so to reduce (0.1) to a quasi-linear equation

$$y^{(k)}(t) = f(t, y(t), y'(t), ..., y^{(k-1)}(t)), \qquad t \in [0, T], \tag{0.3}$$

where a variety of existence results based on the Leray–Schauder theory is available. In the case of a Cauchy problem (i.e., (0.1) subject to conditions $y^{(i)}(0) = r_i$, $i = 0, 1, ..., k-1$), the local solvability of (0.2) with respect to y_k about a point $(0, r_0, r_1, ..., r_k)$ on the graph of the equation is sufficient. In the case of a boundary value problem, however, one needs a global continuous solvability in y_k which is very rarely available.

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The approach taken in this paper is to look for solutions with kth order derivative defined almost everywhere and integrable, i.e., solutions in the Sobolev space $W^{k,p}([0,T],\mathbb{R}^n)$. We make use of recent developments in the theory of multivalued maps and differential inclusions such as a topological selection theorem due to Bielawski and Gorniewicz [1], existence results for differential inclusions due to Frigon and Granas [3], and, implicitly, the Fryszkowski selection theorem on maps with decomposable values recently generalized to a noncompact case by Bressan and Colombo [2]. Those developments permit reducing boundary value problems for (0.1) to those for a differential inclusion of the type

$$y^{(k)}(t) \in \Psi(t, y(t), y'(t), ..., y^{(k-1)}(t)), \qquad t \in [0, T], \tag{0.4}$$

with a "well-behaved" multifunction Ψ, and then concluding the solvability of (0.4). Some fractional results and examples using this method were presented in [7].

In Section 1, the main abstract theorem is proved and an example showing that even the smoothness of F does not imply the existence of C^k solutions (so that the use of Sobolev space techniques most likely is unavoidable) is presented. A series of special cases of the Main Theorem and illustrating examples follows in Sections 2, 3, and 4.

As a final remark, let us recall that boundary value problems for semilinear differential equations such as $y''=f(t, y, y', y'')$ have motivated many authors to study k-set contractive, condensing, or A-proper mappings. Those equations are particular cases of ours and the method developed in this paper allows us to abolish some restrictive Lipschitz conditions on the behavior of f in y'', as illustrated by Example 4.5.

1. Preliminaries and the Main Theorem

Let $T>0$, $1 \leqslant p \leqslant \infty$, and let $k, n \in \mathbb{N}$. We work with the following spaces: the space $L^p([0,T],\mathbb{R}^n)$ of functions $y\colon [0,T] \to \mathbb{R}^n$ which are integrable with exponent p (essentially bounded if $p=\infty$) with the standard norm $\|y\|_p$; the space $C^{k-1}([0,T],\mathbb{R}^n)$ of $k-1$ times continuously differentiable functions from $[0,T]$ to $\mathbb{R}^n$ with the max-sup norm $\|y\|_{k-1,\infty} := \max\{\|y^{(j)}\|_\infty\colon\ j=0,,1,...,k-1\}$; and the Sobolev space $W^{k,p}([0,T],\mathbb{R}^n)$ of functions whose weak derivatives of order up to k belong to $L^p([0,T],\mathbb{R}^n)$. We use abbreviations L^p, $W^{k,p}$, C^{k-1}, whenever the domain and codomain of considered functions are clear.

The euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$. If $A \subset \mathbb{R}^n$, then $|A| := \sup\{|a| : a \in A\}$. By dim A, we mean the topological dimension of a set A (see, e.g., [6]) and by deg(f, U) the topological degree of a mapping f (see, e.g., [8]).

Given k, m, $n \in \mathbb{N}$, we consider two functions $a\colon [0, T] \times \mathbb{R}^{nk} \to \mathbb{R}^m$, and $F\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, where a is continuous with respect to the variable in $\mathbb{R}^{nk}$ and measurable (by measurable we mean measurable with respect to the product of Lebesgue measure on $[0, T]$ and Borel measure on $\mathbb{R}^{nk}$), and F is continuous.

The existence of solutions $y \in W^{k,p}$ to the following boundary value problem is studied,

$$\begin{aligned} &F(a(t, y(t), y'(t), ..., y^{(k-1)}(t)), y^{(k)}(t)) = 0, \qquad \text{a.e.} \quad t \in [0, T], \\ &y \in \mathscr{B} \end{aligned} \qquad (\mathscr{P})$$

where $y \in \mathscr{B}$ stands for linear boundary or initial conditions which will be explicitly stated in examples to follow. At this instant, we only assume that $W^{k,p}_{\mathscr{B}} := W^{k,p} \cap \mathscr{B}$ is an affine subspace of $W^{k,p}$ of finite codimension (a linear subspace if the boundary conditions are homogeneous) so that the operator $\mathscr{L}\colon W^{k,p}_{\mathscr{B}} \to L^p$, $\mathscr{L}y := y^{(k)} - \varepsilon y$, is invertible either for $\varepsilon = 0$ or for a sufficiently small $\varepsilon > 0$.

We let the variable $a \in \mathbb{R}^m$ stand for $a(t, y(t), y'(t), ..., y^{(k-1)}(t))$ and $x \in \mathbb{R}^m$ for $y^{(k)}(t)$. Given $a \in \mathbb{R}^m$, we define $F_a\colon \mathbb{R}^n \to \mathbb{R}^n$ by $F_a(x) := F(a, x)$ and we let $F_a^{-1}(0) = \{x \in \mathbb{R}^n : F(a, x) = 0\}$. The problem $(\mathscr{P})$ can be embedded as $(\mathscr{P}) = (\mathscr{P}_1)$ into the family of problems

$$\begin{aligned} &F(a(t, y(t), y'(t), ..., y^{(k-1)}(t)), \mu y^{(k)}(t) + (1-\mu)\, \varepsilon y(t)) = 0, \\ &\qquad \text{a.e.} \quad t \in [0, T], \\ &y \in \mathscr{B} \end{aligned} \qquad (\mathscr{P}_\mu)$$

parametrised by $\mu \in [1, \infty)$ which, in turn, is equivalent to the family of problems

$$\begin{aligned} &y^{(k)}(t) \in \lambda F^{-1}_{a(t, y(t), y'(t), ..., y^{(k-1)}(t))}(0) + (1-\lambda)\, \varepsilon y(t), \qquad \text{a.e.} \quad t \in [0, T], \\ &y \in \mathscr{B} \end{aligned} \qquad (\mathscr{P}_\lambda)$$

parametrised by $\lambda = 1/\mu \in [0, 1]$ (the case $\lambda = 0$ is trivial since, due to the choice of ε, the unique solution of $(\mathscr{P}_0)$ is $y = \mathscr{L}^{-1}(0)$).

MAIN THEOREM 1.1. *Let F and a be two functions as before. Suppose that the following conditions are verified:*

(1) *for any $a \in \mathbb{R}^m$, there exists an open bounded $U_a \subset \mathbb{R}^n$ such that $\partial U_a \cap F_a^{-1}(0) = \varnothing$ and $\deg(F_a, U_a) \neq 0$; moreover, $\dim F_a^{-1}(0) = 0$;*

(2) *there exists a locally bounded function $g\colon \mathbb{R}^m \to \mathbb{R}$ such that $|F_a^{-1}(0)| \leqslant g(a)$ for all $a \in \mathbb{R}^m$;*

(3) *for any $r > 0$, there exists a function $h_r \in L^p([0, T], [0, \infty))$ such that $|g(a(t, y_0, ..., y_{k-1}))| \leqslant h_r(t)$ for a.e. $t \in [0, T]$, and for all $(y_0, y_1, ..., y_{k-1}) \in \mathbb{R}^{nk}$ with $|(y_0, y_1, ..., y_{k-1})| \leqslant r$;*

(4) *there exists* $M>0$ *such that for any* $\mu>1$ *and any possible solution* $y\in W^{k,p}$ *of the problem* $(\mathscr{P}_\mu)$, *we must have* $\|y-\mathscr{L}^{-1}(0)\|_{k-1,\infty}<M$.

Then the problem $(\mathscr{P})$ *has a solution* $y\in W^{k,p}$.

Proof. The conditions (1), (2) and Theorem 2.5 in [1] imply that $F_a^{-1}(0)$, as a multifunction of a, has a multivalued l.s.c. selector $\Phi:\mathbb{R}^m\to\mathbb{R}^n$ with nonempty compact values, i.e., $\Phi(a)\subset F_a^{-1}(0)$ for all a. We define $\Psi:[0,T]\times\mathbb{R}^{nk}\to\mathbb{R}^n$ by $\Psi(t,y_0,y_1,\ldots,y_{k-1})=\Phi(a(t,y_0,y_1,\ldots,y_{k-1}))$. It is clear that Ψ is measurable and l.s.c. in $(y_0,y_1,\ldots,y_{k-1})$ for a.e. t, so this and (3) imply that Ψ is of "l.s.c. type" in the sense of [3]. Any solution of the problem

$$\begin{aligned} y^{(k)}(t)\in\lambda\Psi(t,y(t),y'(t),\ldots,y^{(k-1)}(t)) \\ +(1-\lambda)\,\varepsilon y(t), \qquad \text{a.e.}\quad t\in[0,T], \\ y\in\mathscr{B} \end{aligned}$$

with $\lambda\in(0,1]$ also is a solution of $(\mathscr{P}_\mu)$ with $\mu=1/\lambda$. If $p=1$ and $\varepsilon=0$, the conclusion follows from Theorem 1 in [3] and if $\varepsilon>0$, it follows from Theorem 2 in [3] (the homogeneity of the boundary conditions in that theorem is not essential). If $1<p\leqslant\infty$, the previous arguments guarantee solutions $y\in W^{k,1}$, since $L^p\subset L^1$. Due to (2) and (3), any such solution is in $W^{k,p}$. ∎

Remark 1.2. If $a(t,y_0,y_1,\ldots,y_{k-1})$ is essentially bounded for $(y_0,y_1,\ldots,y_{k-1})$ in any bounded subset of $\mathbb{R}^{nk}$, the condition (3) is automatically satisfied with a constant h_r, due to the continuity of F. We therefore obtain solutions in $W^{k,\infty}$. If a is continuous in all variables, we may eliminate that function from the formulation of $(\mathscr{P})$ and simply consider the problem

$$\begin{aligned} &F(t,y(t),y'(t),\ldots,y^{(k)}(t))=0, \qquad \text{a.e.}\quad t\in[0,T], \\ &y\in\mathscr{B}, \end{aligned} \tag{1.1}$$

where $F:[0,T]\times\mathbb{R}^{n(k+1)}\to\mathbb{R}^n$ is continuous. A natural question is to ask whether or not the continuity in t implies the existence of C^k solutions. The answer is negative as illustrated by

EXAMPLE 1.3. Consider the periodic problem of the first order

$$\begin{aligned} &2(y'(t)-y(t))^3-9(y'(t)-y(t))^2 \\ &\quad+12(y'(t)-y(t))=t, \qquad t\in[0\,6], \\ &y(0)=y(6). \end{aligned} \tag{1.2}$$

It is easy to verify that the assumptions of Theorem 1.1 are satisfied (in fact, Ψ from the proof of the theorem, here is a bounded multifunction of t only) so the problem has a solution in $W^{1,\infty}$. However, elementary calculus and sketching the graph of $t = 2s^3 - 9s^2 + 12s$ shows that there is no continuous $s(t)$ solving that equation for $t \in [0, T]$ unless $T \leqslant 5$. Consequently, (1.2) has no solutions $y \in C^1([0, 6], \mathbb{R})$.

Remark 1.4. The differential equation in (1.2) has smooth solution curves implicitly given by an equation of the type $G(t, y) = c$. Unfortunately, we are unable to prove the existence of such solution curves satisfying $(\mathscr{P})$ under the general assumptions of our theorem.

2. Polynomial Differential Equation in the Complex Plane

A differential equation in the complex plane $\mathbb{C}$ can be interpreted as a system of two real equations (the case $n = 2$ of the previous section). We are concerned in this section with the general boundary value problem $z \in \mathscr{B}$ for the equation

$$\sum_{j=0}^{m} a_j(t, z(t), z'(t), ..., z^{(k-1)}(t))(z^{(k)}(t) - \varepsilon z(t))^j = 0. \tag{2.1}$$

As previously, we assume that $a = (a_0, a_1, ..., a_m)$: $[0, T] \times \mathbb{C}^k \to \mathbb{C}^{m+1}$ is measurable and that $a(t, \cdot)$ is continuous. Our F here is a complex polynomial of order m with coefficients $a_0, a_1, ..., a_m$, and $\varepsilon \geqslant 0$ is such that $\mathscr{L}z = z^{(k)} - \varepsilon z$ is invertible from $W^{k,p}_{\mathscr{B}}$ to L^p. Let c be the Lipschitz constant of $\mathscr{L}^{-1}$ considered as an operator from L^p to C^{k-1}.

Theorem 2.1. *Suppose that there is* $g \in L^p([0, T], [0, \infty))$, *and a constant* $A > 0$ *with* $cA(T)^{1/p} < 1$ *such that*

$$\left|\frac{ma_j(t, z_0, ..., z_{k-1})}{a_m(t, z_0, ..., z_{k-1})}\right|^{1/(m-j)} \leqslant g(t) + A \max\{|z_i| : i = 0, 1, ..., k-1\}, \tag{2.2}$$

for all $j = 0, 1, ..., m-1$, *and a.e.* $t \in [0, T]$.

Then Eq. (2.1) *has a solution* $z \in W^{k,p}_{\mathscr{B}}([0, T], \mathbb{C})$.

Proof. The verification of (1) in Theorem 1.1 is evident since any complex polynomial has finitely many roots and its topological degree on an open set containing all the roots is equal to its algebraic degree m. If w is any root of $a_m w^m + \cdots + a_1 w + a_0 = 0$, then the inequality

$$|w| \leqslant \max\left\{\left|\frac{ma_j}{a_m}\right|^{1/(m-j)} : j = 0, 1, ..., m-1\right\} \tag{2.3}$$

is easily verified and it implies, together with (2.2), the conditions (2) and (3) of Theorem 1.1. The condition (4) is verified with a constant

$$M > \frac{c\|g\|_p + \|\mathscr{L}^{-1}(0)\|_{k-1,\infty}}{1 - cAT^{1/p}}.$$

Indeed, if $z(t)$ is any solution of $(\mathscr{P}_\mu)$ then, due to (2.2) and (2.3),

$$|z^{(k)}(t) - \varepsilon z(t)| \leqslant \lambda(g(t) + A \max_j |z^{(j)}(t)|),$$

for a.e. t and for $\lambda = 1/\mu \leqslant 1$. By integrating both sides and applying $\mathscr{L}^{-1}$ we get

$$\|z - \mathscr{L}^{-1}(0)\|_{k-1,\infty} \leqslant c(\|g\|_p + AT^{1/p}\|z\|_{k-1,\infty}),$$

and it follows that $\|z - \mathscr{L}^{-1}(0)\|_{k-1,\infty} < M$. ∎

3. Interval of Existence of Solutions for Interpolation Problems

We are concerned with the following interpolation problem

$$\begin{aligned} &F(a(t, y(t), y'(t), ..., y^{(k-1)}(t)), y^{(k)}(t)) = 0, \qquad \text{a.e. } t, \\ &y^{(j)}(t_j) = r_j, \qquad j = 0, 1, ..., k-1, \end{aligned} \tag{3.1}$$

where $t_j \in [0, \infty)$ and $r_j \in \mathbb{R}^n$. Our problem is a Cauchy problem if $t_j = 0$ for all j. We want to determine T for which this problem has a solution on the interval $[0, T]$. We assume that $F\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $a\colon [0, \infty) \times \mathbb{R}^{nk} \to \mathbb{R}^m$ is continuous with respect to the variable in $\mathbb{R}^{nk}$ and measurable, and that, for any $T > 0$, F and the restriction of a to $[0, T] \times \mathbb{R}^{nk}$ satisfy the conditions (1), (2), and (3) of Theorem 1.1.

THEOREM 3.1. *Let F and a be two functions satisfying the previous conditions. Suppose in addition that the following condition is verified:*

(5) *there exists $M \geqslant |r_{k-1}|$ and a Borel measurable function $\alpha\colon (M, \infty) \to (0, \infty)$ such that*

$$|F^{-1}_{a(t, y_0, y_1, ..., y_{k-1})}(0)| \leqslant \alpha(|y_{k-1}|)$$

for all $(y_0, y_1, ..., y_{k-1}) \in \mathbb{R}^{nk}$ with $|y_{k-1}| > M$ and a.e. $t \in [0, T_\infty)$, where

$$T_\infty = \int_M^\infty \frac{ds}{\alpha(s)}.$$

If $\max\{t_0, t_1, ..., t_{k-1}\} \leqslant T < T_\infty$, *then the problem* (3.1) *has a solution* $y \in W^{k,p}([0, T], \mathbb{R}^n)$.

Proof. Let Ψ be defined as in the proof of Theorem 1.1. Any solution of the differential inclusion

$$y^{(k)}(t) \in \Psi(t, y(t), y'(t), ..., y^{(k-1)}(t)), \qquad \text{a.e.} \quad t \in [0, T],$$
$$y^{(j)}(t_j) = r_j, \qquad j = 0, 1, ..., k-1,$$

is a solution of (3.1). As shown in the proof of Theorem 1.1, the conditions (1), (2), and (3) imply that Ψ is of l.s.c. type in the sense of [3]. The existence of a solution follows essentially from Theorem 2.1 in [5]. ▮

It is natural to expect that solutions obtained in Theorem 3.1 extend to C^{k-1} solutions defined on $[0, T_\infty)$ but it is not so evident, because we do not have the uniqueness of solution. We present this result below.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, *the problem* (3.1) *has a solution* y *in* $W^{k,p}_{\text{loc}}([0, T_\infty), \mathbb{R}^n)$ (*the localised Sobolev space*).

Proof. Let $T_0 = 0$, $M_0 = M$, where M is as in Theorem 3.1; and let $T_1 > \max\{t_0, ..., t_{k-1}\}$ be such that

$$\int_{M_0}^{M_0+1} \frac{ds}{\alpha(s)} \leqslant T_1 < \int_{M_0}^{\infty} \frac{ds}{\alpha(s)}.$$

By Theorem 3.1, (3.1) has a solution $y_1 \in W^{k,p}([T_0, T_1], \mathbb{R}^n)$. Define $M_1 = \max\{M_0 + 1, |y_1^{(k-1)}(T_1)|\}$. It is easy to show that

$$T_1 \geqslant \int_{M_0}^{M_1} \frac{ds}{\alpha(s)}.$$

We now consider the problem

$$F(a(t, y(t), y'(t), ..., y^{(k-1)}(t)), y^{(k)}(t)) = 0, \qquad \text{a.e. } t \in [T_1, T_2],$$
$$y^{(j)}(T_1) = y_1^{(j)}(T_1), \qquad j = 0, 1, ..., k-1. \tag{3.2}$$

Let T_2 be such that

$$\int_{M_1}^{M_1+1} \frac{ds}{\alpha(s)} \leqslant T_2 - T_1 < \int_{M_1}^{\infty} \frac{ds}{\alpha(s)}.$$

By Theorem 3.1, (3.2) has a solution $y_2 \in W^{k,p}([T_1, T_2], \mathbb{R}^n)$. We define $M_2 = \max\{M_1 + 1, |y_2^{(k-1)}(T_2)|\}$ and observe that

$$T_2 - T_1 \geqslant \int_{M_1}^{M_2} \frac{ds}{\alpha(s)}.$$

Continuing that way, we construct a sequence of solutions $y_i \in W^{k,p}([T_{i-1}, T_i], \mathbb{R}^n)$ with $y_i^{(j)}(T_i) = y_{i+1}^{(j)}(T_i)$, $j = 0, 1, ..., k-1$, where $\{T_i\}$ is an increasing sequence of constants. We note that $\lim_{i\to\infty} T_i \geq T_\infty$ since

$$\begin{aligned} T_i &= (T_i - T_{i-1}) + \cdots + (T_2 - T_1) + T_1 \\ &\geq \int_{M_{i-1}}^{M_i} \frac{ds}{\alpha(s)} + \cdots + \int_{M_0}^{M_1} \frac{ds}{\alpha(s)} \\ &= \int_{M_0}^{M_i} \frac{ds}{\alpha(s)} \to T_\infty \end{aligned}$$

as $i \to \infty$. It remains to define the solution y by $y(t) = y_i(t)$ if $T_{i-1} \leq t < T_i$, $i = 0, 1, ...$. ∎

EXAMPLE 3.3. Consider the initial value problem

$$\begin{aligned} &3(y''(t))^3 + (y''(t))^2 + ((y'(t))^2 - 2|y'(t)| + 2)^3 = 0, \\ &y(0) = y'(0) = 0. \end{aligned} \tag{3.3}$$

In this case $a\colon [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is given by $a(t, y_0, y_1) = (y_1)^2 - 2|y_1| + 2$ and $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $F(a, x) = 3x^3 + x^2 + a^3$. Then

$$|F^{-1}_{a(t, y_0, y_1)}(0)| \leq (y_1)^2 - 2|y_1| + 2.$$

The condition (5) is verified with $\alpha(s) = s^2 - 2s + 2$. The problem (3.3) has a solution $y \in W^{2,\infty}_{\text{loc}}([0, 3\pi/4), \mathbb{R})$.

4. SECOND ORDER BOUNDARY VALUE PROBLEMS

In this section, we consider $k = 2$, and $y \in \mathscr{B}$ stands for either Dirichlet or periodic boundary conditions,

$$\begin{aligned} &F(a(t, y(t), y'(t)), y''(t)) = 0, \quad \text{a.e. } t \in [0, 1], \\ &\qquad y(0) = r_0, \qquad y(1) = r_1; \end{aligned} \tag{4.1}$$

$$\begin{aligned} &F(a(t, y(t), y'(t)), y''(t)) = 0, \quad \text{a.e. } t \in [0, 1], \\ &\qquad y(0) = y(1), \qquad y'(0) = y'(1); \end{aligned} \tag{4.2}$$

where $F\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and $a\colon [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^m$ satisfy assumptions of Section 1. We introduce the notion of upper and lower solutions of (4.1) and (4.2).

DEFINITION 4.1. *Let* $y \in W^{2,1}([0, 1], \mathbb{R})$ and consider the following conditions:

(a) $F(a(t, y(t), y'(t)), x) \neq 0$ for any $x < y''(t)$ and a.e. $t \in [0, 1]$ (resp. $F(a(t, y(t), y'(t)), x) \neq 0$ for any $x > y''(t)$ and a.e. $t \in [0, 1]$);

(b1) $y(0) \geq r_0$, $y(1) \geq r_1$ (resp. $y(0) \leq r_0$, $y(1) \leq r_1$);

(b2) $y(0) = y(1)$, $y'(1) \geq y'(0)$ (resp. $y(0) = y(1)$, $y'(1) \leq y'(0)$).

If the conditions (a) and (b_i) are satisfied, we say that y is an *upper solution* (resp. *lower solution*) *of* (4.i), $i = 1, 2$.

Remark 4.2. The condition (a) can be rewritten as

$$F^{-1}_{a(t, y(t), y'(t))}(0) \subset [y''(t), \infty) \qquad \text{a.e.} \quad t \in [0, 1],$$

$$(\text{resp. } F^{-1}_{a(t, y(t), y'(t))}(0) \subset (-\infty, y''(t)] \qquad \text{a.e.} \quad t \in [0, 1]).$$

THEOREM 4.3. *Suppose, in addition to* (1), (2), *and* (3) *of Theorem* 1.1, *the following conditions*:

(6) *there exist* $\alpha \leq \beta$ *in* $W^{2,1}([0, 1], \mathbb{R})$ *respectively lower and upper solutions of* (4.i);

(7) *there exist constants* $k_1, k_2 \geq 0$ *such that*

$$|F^{-1}_{a(t, y_0, y_1)}(0)| \leq k_1(y_1)^2 + k_2$$

for a.e. $t \in [0, 1]$ *and for any* $y_0 \in \mathbb{R}$ *with* $\alpha(t) \leq y_0 \leq \beta(t)$.

Then the problem (4.i), *where* $i = 1, 2$, *has a solution* y *such that* $\alpha(t) \leq y(t) \leq \beta(t)$ *for all* $t \in [0, 1]$.

Proof. Any solution of the differential inclusion $y''(t) \in \Psi(t, y(t), y'(t))$ with the boundary condition as in (4.i), where Ψ is as in the proof of Theorem 1.1, is a solution of (4.i), $i = 1, 2$. It is clear that there exists $v \in L^1$ such that $v(t) \in \Psi(t, \alpha(t), \alpha'(t))$ and $v(t) \leq \alpha''(t)$ (resp. $v(t) \in \Psi(t, \beta(t), \beta'(t))$ and $v(t) \geq \beta''(t)$. Therefore α and β are respectively lower and upper solutions of the above differential inclusion in the sense of [4]. The conclusion directly follows from Theorem 2.2 (resp. Theorem 3.2) of [4]. ▮

Remark 4.4. (i) We can consider other boundary value conditions, for example, Neumann or Sturm–Liouville conditions.

(ii) The condition (7) is called a Bernstein growth condition. It can be generalised to a Bernstein–Nagumo growth condition (cf. [4]).

EXAMPLE 4.5. The following example was given by Petryshyn in [9]:

$$\begin{aligned} &y''(t) = h(t) + y(t)^3 + y'(t)^2 + k \sin(y''(t)), \qquad t \in [0, 1] \\ &y(0) = y(1) = 0. \end{aligned} \tag{4.3}$$

Petryshyn proved (with the use of A-proper mapping theory) that (4.3) has a solution $y \in C^2([0, 1])$ provided $0 \leqslant k < 1$. Using the method developed in this paper, Kaczynski proved in [7] that (4.3) has solutions $y \in W^{2,1}([0, 1])$ for any real constant k and it has C^2 solutions if $|k| \leqslant 1$. An alternative way of proving that is to use Theorem 4.3 with $\beta(t) = -\alpha(t) = (|k| + \|h\|_\infty)^{1/3}$. The corresponding periodic problem can be treated in the same way.

EXAMPLE 4.6. Let us consider the problem

$$\begin{aligned} y''(t)^3 + y''(t)^2\,(1 - 3y'(t)) + y''(t)(y'(t)^2 - 1) & \\ + y'(t)^4 - 1 = 0, \qquad t \in [0, 1] & \qquad (4.4) \\ y(0) = y(1) = 0. & \end{aligned}$$

We have $F(a_0, a_1, x) = x^3 + x^2(1 - 3a_1) + x((a_1)^2 - 1) + (a_1)^4 - 1$, and $|F^{-1}_{(a_0, a_1)}(0)| \leqslant \max\{|3(1 - 3a_1)|, |3((a_1)^2 - 1)|^{1/2}, |3((a_1)^4 - 1)|^{1/3}\}$. So the assumptions (1), (2), (3), and (7) are clearly satisfied. For the condition (6), we verify that $\alpha(t) = -t$ and $\beta(t) = t$ are respectively lower and upper solutions of (4.4). By Theorem 4.3, the problem has a solution y with $|y(t)| \leqslant t$. Let us remark that there exist no constant lower and upper solutions of (4.4), and that the assumptions of Theorem 2 in [7] are not satisfied. ∎

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