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EXISTENCE PRINCIPLES FOR CARATHÉODORY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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(*Submitted by A. Granas*)

Dedicated to the memory of Karol Borsuk

1. Introduction and Preliminaries

In [4], the topological transversality theorem of A. Granas [3] was used to establish existence principles for systems of differential equations in $\mathbb{R}^n$. The results developed in [4] extend rather easily to the case of Banach space-valued solutions. For initial results along these lines see [6]. The present paper extends the general existence results in the papers above in two directions. First, we do not ask that the right member of the differential equation be completely continuous. Instead, we introduce a new property, called K-Carathéodory, that is automatically satisfied when the right member is Carathéodory or continuous and the Banach space is finite dimensional and implies condition $(*)$ in [6] in the infinite dimensional case. With this new property, the basic existence principles have the same formulation in a Banach space as in $\mathbb{R}^n$. Second, we enlarge substantially the class of admissible boundary conditions considered in [4] and [6]. In effect, the results formulated below allow any linear boundary forms which together with the differential operator in question determine an invertible operator. In the papers above, the boundary conditions were required to

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satisfy a particular factorization property. This property is satisfied by typical multipoint boundary conditions and natural integral boundary forms, but is not satisfied by matrix boundary conditions such as $Au(0) + Bu'(0) = c \in \mathbb{R}^n$. The present treatment includes all these boundary conditions and their Banach space analogues.

Throughout E is a real Banach space with norm $|\cdot|$. In case $E = H$ a Hilbert space, we denote the inner product by $\cdot$ so that $|h|^2 = h \cdot h$ for $h \in H$. We denote by $C^m = C^m([a,b],E)$ the Banach space of functions $u : [a,b] \to E$ such that $u^{(m)}$ is continuous and equipped with the norm

$$|u|_m = \max\{|u|_0, |u'|_0, \dots, |u^{(m)}|_0\},$$

where $|\nu|_0 = \max\{|\nu(t)| : t \in [a,b]\}$ for any $\nu \in C^0([a,b],E) = C([a,b],E)$. We set $C_0 = C_0([a,b],E) = \{u \in C([a,b],E) : u(a) = 0\}$.

Let $u : [a,b] \to E$ be a measurable function. By $\int_a^b u(t)\,dt$ we mean the Bochner integral of u, assuming it exists. See [1,7] for properties of the Bochner integral mentioned below. A measurable function $u : [a,b] \to E$ is Bochner integrable if and only if $|u|$ is Lebesgue integrable. Moreover, if $u : [a,b] \to E$ is measurable, $|u(t)| \le g(t)$ almost everywhere, and $g(t)$ is integrable, then $u(t)$ is integrable. Let $u : [a,b] \to E$ be integrable and set $\nu(t) = \int_a^t u(s)\,ds$. The function $\nu : [a,b] \to E$ is absolutely continuous, ν is differentiable almost everywhere, and $\nu'(t) = u(t)$ almost everywhere in $[a,b]$. Finally, let $u : [a,b] \to E$ be integrable and $T : E \to E_1$ be a bounded linear operator, where E_1 is also a Banach space. Then $Tu : [a,b] \to E_1$ is integrable and $\int_N Tu(t)\,dt = T\int_N u(t)\,dt$ for each measurable subset $N \subset [a,b]$.

As usual, $L^p[a,b] = L^p([a,b],E)$ for $1 \le p < \infty$ denotes the Banach space of measurable functions $u : [a,b] \to E$ such that $|u|^p$ is Lebesgue integrable with $\|u\|_p = (\int_a^b |u(t)|^p\,dt)^{1/p}$. The space $L^\infty[a,b]$ is defined in the usual way and equipped with the essential supremum norm $\| \ \|_\infty$. In our context, $[a,b]$ is a bounded interval and Hölder's inequality and an earlier remark imply that each L^p function is Bochner integrable. We define the Sobolev classes $W^{k,p}[a,b] = W^{k,p}([a,b],E)$ inductively as follows. A function $u \in W^{1,p}[a,b]$ if it is continuous and there exists $\nu \in L^p[a,b]$ such that $u(t) - u(a) = \int_a^t \nu(s)\,ds$, for all t in $[a,b]$. Then for $k > 1, u \in W^{k,p}[a,b]$ if $u, u' \in W^{k-1,p}[a,b]$. Notice that if $u \in W^{1,p}[a,b]$ then u is differentiable almost everywhere on $[a,b], u' \in L^p[a,b]$, and $u(t) - u(a) = \int_a^t u'(s)\,ds$, for t in $[a,b]$.

We are concerned with solutions to initial or boundary value problems of the form

$$y^{(k)}(t) = f(t, y(t), \dots y^{(k-1)}(t)), \qquad y \in \mathrm{B},$$

where $y : [a, b] \to E$ and the differential equation is to hold either everywhere or almost everywhere according as f is continuous or is a Carathéodory function. The admissible boundary conditions $\mathbf{B}$ will be described next.

Boundary Conditions. For $i = 1, 2, \ldots, k$, let $U_i : C^{k-1}([a, b], E) \to E$ be continuous linear operators. Let $z_1, \ldots, z_k$ be real-valued and a basis of solutions for the scalar equation $\Lambda_0 z \equiv z^{(k)} = 0$. As usual, E^k is the k-fold cartesian product of E. Define

$$U : E^k \to E^k$$

by

$$U(c_1, \ldots, c_k) = (U_1(c_1 z_1 + \cdots + c_k z_k), \ldots, U_k(c_1 z_1 + \cdots + c_k z_k)).$$

We are concerned with the invertibility of U. It is easy to check that if U is invertible for one basis of solutions $z_1, \ldots, z_k$ it is invertible for all such bases. Thus, any convenient basis can be used.

We say a function $u \in C^m([a, b], E)$ with $m \geq k - 1$ satisfies the boundary conditions $\mathbf{B}$ if $U_i(u) = \gamma_i$ for $i = 1, 2, \ldots, k$, and where $\gamma_i \in E$ are given data. We write $u \in \mathbf{B}$ when u satisfies the boundary conditions $\mathbf{B}$. When $\gamma_i = 0$ for each i we write $\mathbf{B} = \mathbf{B}_0$. Thus, $u \in \mathbf{B}_0$ means that $U_i(u) = 0$ for $i = 1, 2, \ldots, k$. It is convenient to define $C^m_{\mathbf{B}} = C^m_{\mathbf{B}}([a, b], E) = \{u \in C^m([a, b], E) : u \in \mathbf{B}\}$. Define

$$\Lambda : C^k_{\mathbf{B}_0} \to C, \qquad \Lambda y = y^{(k)},$$
$$L : C^{k-1}_{\mathbf{B}} \to C_0, \qquad Ly(t) = y^{(k-1)}(t) - y^{(k-1)}(a).$$

Note that Λ is linear and L is affine. We say a continuous bijection is invertible if it has a continuous inverse.

LEMMA 1.1. *Let U, Λ, and L be as above. The following statements are equivalent.*

(a) $U : E^k \to E^k$ *is invertible;*
(b) $\Lambda : C^k_{\mathbf{B}_0} \to C$ *is invertible;*
(c) $L : C^{k-1}_{\mathbf{B}} \to C_0$ *is invertible.*

PROOF. Consider (a) and (b). Let $g \in C$ and define

$$G(t) = \begin{cases} \displaystyle\int_a^t g(t_0)\, dt_0, & k = 1; \\ \displaystyle\int_a^t \int_a^{t_{k-1}} \cdots \int_a^{t_1} g(t_0)\, dt_0 \ldots dt_{k-1}, & k \geq 2. \end{cases}$$

The differential equation $y^{(k)} = g$ has general solution $y = c_1 z_1(t) + \cdots + c_k z_k(t) + G(t)$, where $c_i \in E$ are arbitrary. Consequently, $\Lambda y = g$ holds precisely when y is given by the previous expression and $c_1, \ldots, c_k$ satisfy

$$U_i(y) = 0 \quad \text{for} \quad i = 1, \ldots, k \quad \Longleftrightarrow \quad U(c_1, \ldots, c_k) = -(U_1(G), \ldots, U_k(G)).$$

The equivalence of (a) and (b) is now clear. Next, consider (a) and (c). Let $g \in C_0$ and define $G_1(t) = G'(t)$ with G as above. Since $g(a) = 0$, it is easy to check that G_1 is a particular solution to $y^{(k-1)}(t) - y^{(k-1)}(a) = g(t)$. It follows that the general solution of this equation is $y = c_1 z_1(t) + \cdots + c_k z_k(t) + G_1(t)$, where $c_i \in E$ are arbitrary. Now, reason as above to see that (a) and (c) are equivalent. □

REMARK. We can give an elementary extension of Lemma 1.1: Define $\Lambda_0 z = a_k z^{(k)} + \cdots + a_1 z' + a_0 z$, where $a_j \in \mathbb{R}$ and $a_k \neq 0$. Let $z_1, \ldots, z_k$ be real-valued and a basis of solutions for the scalar equation $\Lambda_0 z = 0$. Define $U : E^k \to E^k$ as above and define

$$\Lambda : C^k_{\mathbf{B}_0} \to C, \qquad \Lambda y = a_k y^{(k)} + \cdots + a_1 y' + a_0 y,$$

$$L : C^{k-1}_{\mathbf{B}} \to C_0, \qquad Ly(t) = \sum_{j=1}^{k} a_j (y^{(j-1)}(t) - y^{(j-1)}(a)) + a_0 \int_a^t y(s)\,ds.$$

Then Lemma 1.1 holds and the proof is virtually the same: use variation of parameters to construct $G(t)$ such that $\Lambda G(t) = g(t)$ and $G(a) = G'(a) = \cdots = G^{(k-1)}(a) = 0$. Now, reason as before.

The boundary conditions in [4], where $E = \mathbb{R}^n$, are of the type considered here but with the added restriction: for each $i = 1, \ldots, k$ there is a scalar form $\tilde{U}_i : C^{k-1}([a,b], \mathbb{R}) \longrightarrow \mathbb{R}$ such that $U_i(\varphi(t)c) = \tilde{U}_i(\varphi(t))c$ for each $(k-1)$ times differentiable scalar function $\varphi(t)$ and each vector $c \in E$. For such forms,

$$\begin{aligned} U(c_1, \ldots, c_k) &= (\tilde{U}_1(z_1)c_1 + \cdots + \tilde{U}_1(z_k)c_k, \ldots, \tilde{U}_k(z_1)c_1 + \cdots + \tilde{U}_k(z_k)c_k) \\ &= [\tilde{U}_i(z_j)]c, \qquad \text{where } c = (c_1, \ldots, c_k)^T. \end{aligned}$$

Thus, U is invertible if and only if $\det[\tilde{U}_i(z_j)] \neq 0$, which is the principal invertibility condition used in [4].

Important boundary conditions that do not have the special factorization property above and which are treated here include the separated boundary conditions

$$U_1(y) \equiv A_0 y(0) - B_0 y'(0) = r, \quad U_2(y) \equiv A_1 y(1) + B_1 y'(1) = s,$$

where A_i and B_i are particular bounded linear operators and r, $s \in E$. For use later, we prove

LEMMA 1.2. *Let U_1 and U_2 be separated boundary forms as above and* $\mathbf{B}$ *denote the corresponding boundary conditions. The operator $\Lambda : C^2_{\mathbf{B}_0} \to C$ is invertible if either $A_0 + B_0$ and $M = (A_1 + B_1) - B_1(A_0 + B_0)^{-1}B_0$ are invertible, or $A_1 + B_1$ and $N = (A_0 + B_0) - B_0(A_1 + B_1)^{-1}B_1$ are invertible.*

PROOF. We use Lemma 1.1. In this context, it is convenient to choose the basis $z_1(t) = t$ and $z_2(t) = 1 - t$. Then the boundary operator U is given by

$$U(c_1, c_2) = ((A_0 + B_0)c_2 - B_0c_1, (A_1 + B_1)c_1 - B_1c_2).$$

Assume that $A_0 + B_0$ and M are invertible. The other case follows by symmetry. A short calculation shows that the equation $U(c_1, c_2) = (d_1, d_2)$ is equivalent to

$$c_2 = (A_0 + B_0)^{-1}[B_0c_1 + d_1],$$
$$[(A_1 + B_1) - B_1(A_0 + B_0)^{-1}B_0]c_1 = d_2 + B_1(A_0 + B_0)^{-1}d_1.$$

By assumption the second equation uniquely determines c_1 and then the first equation uniquely determines c_2. Clearly c_1 and c_2 depend continuously on (d_1, d_2), so U is invertible. □

In some applications given in Sec. 3, we shall assume that $E = H$ is a Hilbert space and that the boundary conditions **B** are Sturm-Liouville, SL-*boundary conditions*: that is,

$$U_1(y) \equiv A_0y(0) - B_0y'(0) = r, \quad U_2(y) \equiv A_1y(1) + B_1y'(1) = s,$$

where the bounded linear operators A_0, B_0, A_1, and B_1 satisfy the following conditions:

(i) $B_0 = 0$ or I; $B_1 = 0$ or I.

(ii) A_0 and A_1 are bounded linear operators on H such that there exists $\alpha_0, \alpha_1 \geq 0$ with $A_0x \cdot x \geq \alpha_0|x|^2$ and $A_1x \cdot x \geq \alpha_1|x|^2$.

(iii) α_0 and B_0 are not both zero; α_1 and B_1 are not both zero.

Thus, A_0 (resp., A_1) is positive definite or nonnegative definite according as α_0 (resp., α_1) is positive or zero. Notice that we do not require A_0 or A_1 to be symmetric. Evidently, the SL-boundary conditions generalize the familiar Sturm-Liouville conditions for the classical case when $E = H = \mathbb{R}$.

We shall use the following elementary facts. A short proof is included for completeness.

LEMMA 1.3. *If $C\colon H \to H$ is a positive definite, bounded linear operator such that $Cx \cdot x \geq \gamma|x|^2$ with $\gamma > 0$, then C is invertible and the operator norm $|C^{-1}| \leq 1/\gamma$.*

PROOF. The relations $\gamma|x|^2 \leq Cx \cdot x = x \cdot C^*x$ imply that C and C^* are one-to-one and $\gamma|x| \leq |Cx|$. Then $\overline{R(C)} = N(C^*)^\perp = H$. Consequently, if $y \in H$ there exists $Cx_n \to y$. Then $\gamma|x_n - x_m| \leq |Cx_n - Cx_m|$ implies that $\{x_n\}$ is Cauchy, $x_n \to x \in H, Cx_n \to Cx$; therefore, $y = Cx$ and C is onto. So C^{-1} exists. Finally, set $x = C^{-1}y$ in $\gamma|x| \leq |Cx|$ to obtain $|C^{-1}y| \leq (1/\gamma)|y|$ and $|C^{-1}| \leq 1/\gamma$. □

LEMMA 1.4. *Let* **B** *denote the Sturm-Liouville,* SL*-boundary conditions. Then the linear operator* $\Lambda : C^2_{\mathbf{B}_0}([0,1],H) \to C([0,1],H)$ *is invertible if* $\alpha_0 + \alpha_1 > 0$.

PROOF. The desired conclusion follows from Lemmas 1.2 and 1.3 because $A_0 + B_0$ and $M = A_1 + B_1 - B_1(A_0 + B_0)^{-1}B_0$ are positive definite and hence invertible. This is clear for $A_0 + B_0$ because α_0 or B_0 are not both zero. Likewise if $B_0 = 0$ or $B_1 = 0$, then $M = A_1 + B_1$ is positive definite. If $B_0 = B_1 = I$ then $M = A_1 + I - (A_0 + I)^{-1}$ and

$$Mx \cdot x = A_1 x \cdot x + x \cdot x - (A_0 + I)^{-1} x \cdot x \geq (\alpha_1 + 1)|x|^2 - \left|(A_0 + I)^{-1}\right| |x|^2.$$

By Lemma 1.3, $\left|(A_0 + I)^{-1}\right| \leq 1/(\alpha_0 + 1)$. Therefore,

$$Mx \cdot x \geq \left(\alpha_1 + 1 - \frac{1}{\alpha_0 + 1}\right) |x|^2 = \frac{\alpha_0\alpha_1 + \alpha_0 + \alpha_1}{\alpha_0 + 1} |x|^2$$

and $\alpha_0 + \alpha_1 > 0$. □

Carathéodory and K-Carathéodory Functions. Let E_1 and E_2 be Banach spaces. A function $f : [a,b] \times E_1 \to E_2$ is an *L^p-Carathéodory function* if:

(1) the map $t \to f(t,z)$ is measurable for each z in E_1;
(2) the map $z \to f(t,z)$ is continuous for almost all t in $[a,b]$;
(3) for each $r > 0$ there exists $h_r \in L^p([a,b],\mathbb{R})$ such that $|z| \leq r$ implies $|f(t,z)| \leq h_r(t)$, for almost all t in $[a,b]$.

By a *Carathéodory function* we mean an L^1-Carathéodory function.

Now, we introduce a new property that replaces a customary complete continuity assumption made in treating nonlinear differential equations in Banach spaces.

A function $f : [a,b] \times E_1 \to E_2$ is said to be *K-Carathéodory* if it satisfies properties (1) and (2) of a Carathéodory function and also has the following property, called *property-K*:

> for each $r > 0$ there exist a nonnegative function $\eta_r \in L^p([a,b],\mathbb{R})$ and a compact set K_r in E_2 such that $|z| \leq r$ implies $f(t,z) \in \eta_r(t)K_r$ for almost all t in $[a,b]$.

It is clear that a K-Carathéodory function is a Carathéodory function. Also, it is easy to check that if f is completely continuous (that is, maps bounded sets into relatively compact sets) then f has property-K with $\eta_r(t) \equiv 1$ for each $r > 0$. Thus, in the finite dimensional context, a continuous or Carathéodory function has property-K and, in fact, condition (3) in the definition of a Carathéodory function implies property-K. In other words, Carathéodory and K-Carathéodory are equivalent in finite dimensions. This is not true in infinite dimensions.

A Carathéodory function $f : [a,b] \times E_1 \to E_2$ induces an associated operator $\bar{N}_f : C([a,b], E_1) \to C_0([a,b], E_2)$ defined by $\bar{N}_f u(t) = \int_a^t f(s, u(s))\, ds$. We use the following properties of $\bar{N}_f$.

THEOREM 1.5. *Let $f : [a,b] \times E_1 \to E_2$ be a K-Carathéodory function. Then the operator $\bar{N}_f : C([a,b], E_1) \to C_0([a,b], E_2)$ is continuous and completely continuous.*

PROOF. Since f is a Carathéodory function, the reasoning used in [4] in $\mathbb{R}^n$ can be applied in the Banach space context to establish that $\bar{N}_f$ is continuous and that $\bar{N}_f S$ is bounded and equicontinuous for each bounded set $S \subset C([a,b], E_1)$. Thus, by the general Arzela-Ascoli Theorem, $\bar{N}_f S$ will be relatively compact, and therefore $\bar{N}_f$ will be completely continuous, if for each t in $[a,b]$ the set

$$\{\bar{N}_f u(t) : u \in S\} = \left\{ \int_a^t f(s, u(s))\, ds : u \in S \right\}$$

is relatively compact in E_2. To prove this, let $r > 0$ be such that $|u|_0 \le r$ for all $u \in S$. By property-K there is a function $\eta \in L^1([a,b], [0,\infty))$ and a compact set K in E_2 such that $f(t, u(t)) \in \eta(t) K$ for almost all t in $[a,b]$. Let $b_2^* \in E_2^*$ and suppose K is contained in the half-space, where $b_2^* \le c$; that is, $x \in K$ implies $b_2^*(x) \le c$. For each $u \in S$ and almost all s in $[a,b]$,

$$f(s, u(s)) = \eta(s) k_{u(s)}, \qquad \text{for some } k_{u(s)} \in K.$$

Then, for almost all s,

$$b_2^*(f(s, u(s))) = b_2^*(\eta(s) k_{u(s)}) = \eta(s) b_2^*(k_{u(s)}) \le \eta(s) c.$$

Now, suppose $\int_a^t \eta(s)\, ds > 0$. Then

$$b_2^*\left(\frac{1}{\int_a^t \eta(s)\, ds} \int_a^t f(s, u(s))\, ds \right) = \frac{1}{\int_a^t \eta(s)\, ds} \int_a^t b_2^*(f(s, u(s)))\, ds \le c.$$

Since the intersection of all half-spaces that contain K is its closed convex hull, we find that

$$\int_a^t f(s, u(s))\, ds \in \left(\int_a^t \eta(s)\, ds \right) \overline{co}(K) = K_1,$$

which is compact by a theorem of Mazur. Finally, if $\int_a^t \eta(s)\, ds = 0$, then $\eta(s) = 0$ almost everywhere on $[0,t]$, $f(s, u(s)) = \eta(s) k_{u(s)} = 0$ almost everywhere on $[0,t]$, and $\int_a^t f(s, u(s))\, ds = 0$. Therefore, $\int_a^t f(s, u(s))\, ds \in K_1$ for each $u \in S$. As noted above, this completes the proof that $\bar{N}_f$ is completely continuous. □

The map

$$j : C^{k-1}([a,b], E) \longrightarrow C([a,b], E^k), \quad ju = \left(u, u', \ldots, u^{(k-1)}\right)$$

is continuous and maps bounded sets to bounded sets. Therefore, if $f : [a,b] \times E^k \to E$ is a K-Carathéodory function, the operator

$$N_f = \bar{N}_f \circ j : C^{k-1}([a,b],E) \longrightarrow C_0([a,b],E)$$

inherits all the properties in Theorem 1.5. Thus,

COROLLARY 1.6. *Let $f : [a,b] \times E^k \to E$ be a K-Carathéodory function and N_f its associated Carathéodory operator,*

$$N_f u(t) = \int_a^t f(s, u(s), \dots, u^{(k-1)}(s))\, ds.$$

Then N_f is continuous and completely continuous.

2. Existence Principles

Let $U_i : C^{k-1}([a,b],E) \to E$ be continuous linear operators as in Sec. 1 and fix $\gamma_i \in E$ for $i = 1, \dots, k$. Let $\mathbf{B}$ specify the boundary conditions $U_i(u) = \gamma_i$ for $i = 1, \dots, k$. Consider the k^{th} order problem

$$(\mathcal{P}) \qquad y^{(k)} = f(t, y, \dots, y^{(k-1)}), \quad y \in \mathbf{B},$$

where $f : [a,b] \times E^k \to E$ is a K-Carathéodory function, and the related problems

$$(\mathcal{P}_\lambda) \qquad y^{(k)} = \lambda f(t, y, \dots, y^{(k-1)}), \quad y \in \mathbf{B},$$

where $\lambda \in [0,1]$. If f is continuous, a *solution* y to $(\mathcal{P}_\lambda)$ is a *classical solution*: that is, $y \in C^k([a,b],E)$ and satisfies $(\mathcal{P}_\lambda)$ for all t in $[a,b]$. If f is an L^p - Carathéodory function, a solution y to $(\mathcal{P}_\lambda)$ is a *Carathéodory solution*; that is, $y \in W^{k,p}([a,b],E)$ and satisfies the differential equation in $(\mathcal{P}_\lambda)$ for almost all t in $[a,b]$.

The main result of this paper is the following existence principle.

THEOREM 2.1. *Assume that the linear operator $\Lambda : C^k_{\mathbf{B}_0} \to C$ is invertible and that $f : [a,b] \times E^k \to E$ is a K-Carathéodory function. Assume there exists a bounded open set $G \subset C^{k-1}_{\mathbf{B}}([a,b],E)$ such that for any $\lambda \in [0,1)$ and any solution y to $(\mathcal{P}_\lambda)$ we have $y \in G$. Then the problem $(\mathcal{P})$ has a solution $y \in \bar{G}$.*

PROOF. It follows easily from the properties of the Bochner integral mentioned in Sec. 1 that y is a solution to $(\mathcal{P}_\lambda)$ if and only if $y \in C^{k-1}_{\mathbf{B}}([a,b],E)$ and

$$y^{(k-1)}(t) - y^{(k-1)}(a) = \lambda \int_a^t f(s, y(s), \dots, y^{(k-1)}(s))\, ds.$$

Equivalently,

$$Ly = \lambda N_f y,$$

where $L : C^{k-1}_{\mathbf{B}} \to C_0$ and $N_f : C^{k-1}_{\mathbf{B}} \to C_0$ are the operators introduced in Sec.1. Recall that N_f is continuous and completely continuous (Corollary 1.6) and that L is an invertible, affine operator (Lemma 1.1). Thus, y is a solution to $(\mathcal{P}_\lambda)$ if and only if

$$y = \lambda L^{-1}N_f(y) + (1-\lambda)L^{-1}(0).$$

Observe that the operator $L^{-1}N_f$ is continuous and completely continuous, $L^{-1}(0) \in G$, and there is no $y \in \partial G$ such that $y = \lambda L^{-1}N_f(y) + (1-\lambda)L^{-1}(0)$ for some $\lambda \in (0,1)$. With these observations, the conclusion of Theorem 2.1 follows from the nonlinear alternative (Th. 2.3 in [4]), which is a variation on the well-known Leray-Schauder alternative. □

REMARK. The more refined existence principles stated in [4] for $\mathbb{R}^n$ and when Λ is invertible hold in a Banach space under the assumptions of Theorem 2.1. The proofs are the same, once property-K is available.

Similar reasoning leads to existence results for certain problems with nonlinear boundary conditions. To this end, let $V_i : C^{k-1}([a,b],E) \to E$ be continuous and completely continuous and let U_i be continuous linear operators as in Sec. 1. Denote by $\tilde{\mathbf{B}}$ the nonlinear boundary conditions given by $U_i(u) = V_i(u)$ for $i = 1, \ldots, k$. Consider the k^{th} order problem

$$(\tilde{\mathcal{P}}) \qquad y^{(k)} = f(t, y, \ldots, y^{(k-1)}), \qquad y \in \tilde{\mathbf{B}},$$

and the related family of problems

$$(\tilde{\mathcal{P}}_\lambda) \qquad y^{(k)} = \lambda f(t, y, \ldots, y^{(k-1)}), \quad U_i(y) = \lambda V_i(y),$$

where $\lambda \in [0,1]$ and $i = 1, 2, \ldots, k$.

THEOREM 2.2. *Assume that the linear operator $\Lambda : C^k_{\mathbf{B}_0} \to C$ is invertible and that $f : [a,b] \times E^k \to E$ is a K-Carathéodory function. Assume there exists a bounded open set $G \subset C^{k-1}([a,b],E)$ such that for any $\lambda \in [0,1)$ and any solution y to $(\tilde{\mathcal{P}}_\lambda)$ we have $y \in G$. Then the problem $(\tilde{\mathcal{P}})$ has a solution $y \in \bar{G}$.*

PROOF. Define $\tilde{L}, \tilde{N}_f : C^{k-1}([a,b],E) \to C_0([a,b],E) \times E^k$ by

$$\tilde{L}y = (y^{(k-1)} - y^{(k-1)}(a), U_1(y), \ldots, U_k(y)),$$
$$\tilde{N}_f y = (N_f y, V_1(y), \ldots, V_k(y)).$$

It follows from Lemma 1.1 that the operator $\tilde{L}$ is invertible. Consequently, y is a solution to $(\tilde{\mathcal{P}}_\lambda)$, equivalently, $\tilde{L}y = \lambda \tilde{N}_f y$, if and only if

$$y = \lambda \tilde{L}^{-1}\tilde{N}_f(y).$$

The rest of the proof goes as in Theorem 2.1. □

REMARK. In [4], existence principles in $\mathbb{R}^n$ are established for $(\mathcal{P})$ and $(\tilde{\mathcal{P}})$ by a translation technique when the natural differential operator is not invertible.

The resulting translate of f is not K-Carathéodory in the Banach space setting. We will develop existence principles for this situation in a forthcoming paper.

In most applications, an *a priori* bound r with $|y|_{k-1} < r$ is established for solutions y to $(\mathcal{P}_\lambda)$. Then Theorem 2.1, applied with $G = \{y \in C_{\mathbf{B}}^{k-1}([a,b], E) : |y|_{k-1} < r\}$, yields a solution y to $(\mathcal{P})$ with $|y|_{k-1} \leq r$.

3. Applications

This section contains several applications of the existence results given in Sec.2. The examples extend and refine corresponding results in the literature, as noted below.

Cauchy Problems. We begin with an extension of Theorem 5.1 in [4] from $\mathbb{R}^n$ to a Hilbert space. We also generalize the growth condition modestly. Consider the Cauchy problem

$$(\mathcal{I}) \qquad \begin{aligned} &y' = f(t,y), \qquad 0 \leq t \leq T, \\ &y(0) = r, \end{aligned}$$

and the related family of problems

$$(\mathcal{I}_\lambda) \qquad \begin{aligned} &y' = \lambda f(t,y), \qquad 0 \leq t \leq T, \\ &y(0) = r, \end{aligned}$$

where $r \in H$ and $\lambda \in [0,1]$.

Theorem 3.1. *Let $f : [0,T] \times H \to H$ be a K-Carathéodory function such that*

$$y \cdot f(t,y) \leq \alpha(t)\varphi(|y|), \qquad \textit{for a.e. } t \in [0,T] \textit{ and } y \in H,$$

where $\varphi : [0,\infty) \to (0,\infty)$ is Borel measurable, $\alpha \in L^1([0,T],[0,\infty))$ and

$$\int_0^T \alpha(t)\,dt < \int_{|r|}^\infty \frac{x}{\varphi(x)}\,dx.$$

Then $(\mathcal{I})$ has a solution.

Proof. Let $\mathbf{B}$ denote the initial condition $y(0) = r$. Evidently, the operator $\Lambda : C^1_{\mathbf{B}_0} \to C$, $\Lambda y = y'$, is invertible. So by Theorem 2.1, it suffices to establish an *a priori* bound in $C[0,T]$ on solutions y to $(\mathcal{I}_\lambda)$ for $\lambda \in [0,1)$. Such a bound follows immediately from the next lemma, applied with $t_0 = 0, R = |r|, z = |y|$, and $\psi(x) = \varphi(x)/x$. □

Lemma 3.2. *Let $R \geq 0$, $\psi\colon [0,\infty) \to (0,\infty)$ be a Borel function and $\alpha \in L^1([0,T],[0,\infty))$ be such that*

$$\int_0^T \alpha(t)\,dt < \int_R^\infty \frac{1}{\psi(x)}\,dx.$$

Then there exists a constant M, dependent only upon α, ψ, and R, such that for any z in $W^{1,1}([0,T],[0,\infty))$ such that $z'(t) \le \alpha(t)\psi(z(t))$ for a.e. $t \in [0,T]$ and $z(t_0) \le R$ for some $t_0 \in [0,T]$, we have $z(t) \le M$ for all $t \in [t_0, T]$.

REMARK. If in the previous lemma the inequality $z'(t) \le \alpha(t)\psi(z(t))$ is replaced by $-z'(t) \le \alpha(t)\psi(z(t))$, then the conclusion is replaced by $z(t) \le M$ for all $t \in [0, t_0]$.

PROOF. Let $z \in W^{1,1}([0,T],[0,\infty))$ be such a function. Suppose $z(t) > R$ for some $t \in [t_0, T]$. Then there exists $\tau \in [t_0, t)$ such that $z(\tau) = R$ and $z(s) > R$ on $(\tau, t]$. Now,

$$z'(s) \le \alpha(s)\psi(z(s)).$$

Divide by ψ, integrate from τ to t, and change of variables (Lemma 2 in [2]) to obtain

$$\int_R^{z(t)} \frac{dx}{\psi(x)} = \int_\tau^t \frac{z'(s)\,ds}{\psi(z(s))} \le \int_0^T \alpha(s)\,ds < \int_R^\infty \frac{dx}{\psi(x)},$$

by assumption. Thus, there exists $M \ge R$ independent of z such that $z(t) \le M$ in $[t_0, T]$. □

A useful corollary of Theorem 3.1 follows.

COROLLARY 3.3. *Let $f : [0,T] \times H \to H$ be a K-Carathéodory function such that*

$$|f(t,y)| \le \alpha(t)\psi(|y|) \text{ for a.e. } t \in [0,T],$$

where $\psi : [0,\infty) \to (0,\infty)$ is Borel measurable, $\alpha \in L^1([0,T],[0,\infty))$ and

$$\int_0^T \alpha(t)\,dt < \int_{|r|}^\infty \frac{dx}{\psi(x)}.$$

Then $(\mathcal{I})$ has a solution.

The analogous result holds in a Banach space when ψ is nondecreasing.

THEOREM 3.4. *Let $f : [0,T] \times E \to E$ be a K-Carathéodory function. Assume that there exists a nondecreasing, Borel function $\psi : [0,\infty) \to (0,\infty)$ and $\alpha \in L^1([0,T],[0,\infty))$ such that*

$$|f(t,y)| \le \alpha(t)\psi(|y|) \qquad \text{for a.e. } t \in [0,T] \text{ and } y \in E.$$

Then $(\mathcal{I})$ has a solution provided

$$\int_0^T \alpha(t)\,dt < \int_{|r|}^\infty \frac{dx}{\psi(x)}.$$

PROOF. As in Theorem 3.1, it suffices to find an *a priori* bound in $C[0,T]$ for solutions $y \in W^{1,1}[0,T]$ to $(\mathcal{I}_\lambda)$ for $\lambda \in [0,1)$. For such a y, $y(t) = y(0) + \int_0^t y'(s)\,ds$ so that

$$|y(t)| \le |r| + \int_0^t |y'(s)|\,ds \equiv \rho(t).$$

Clearly, $\rho(t)$ is absolutely continuous and $\rho'(t) = |y'(t)|$ almost everywhere. Since ψ is nondecreasing,

$$\rho'(t) = |y'(t)| \leq \alpha(t)\psi(|y(t)|) \leq \alpha(t)\psi(\rho(t))$$

almost everywhere. Thus, Lemma 3.2 applied with $t_0 = 0, R = |r|$, and $z = \rho$ gives the existence of a constant M, independent of λ, such that $|y(t)| \leq \rho(t) \leq M$ for $t \in [0,T]$. Thus, $|y|_0 \leq M$ is the required *a priori* bound. □

REMARK. Let $f(t,y) = \eta(t)g(t,y)$, where $\eta \in L^q([0,T],\mathbb{R})$ and $g(t,y)$ is a completely continuous, Carathéodory function. Under these assumptions, it is easy to check that $f = \eta g$ is K-Carathéodory; consequently, Theorem 3.4 implies Theorem 2.1 in [6].

Sturm-Liouville Problems. Let H be a Hilbert space. Consider the boundary value problem

$$(\mathcal{SL}) \qquad \begin{aligned} &y'' = f(t,y,y'), \qquad 0 \leq t \leq 1,\\ &A_0y(0) - B_0y'(0) = r\\ &A_1y(1) + B_1y'(1) = s, \end{aligned}$$

where $f : [0,1]\times H^2 \to H$, $r,s \in H$, and $A_0, B_0, A_1, B_1 : H \to H$ are bounded linear operators that generate Sturm-Liouville, SL-boundary conditions; see Sec. 1. Consider the related family of problems

$$(\mathcal{SL}_\lambda) \qquad \begin{aligned} &y'' = \lambda f(t,y,y'), \qquad 0 \leq t \leq 1,\\ &A_0y(0) - B_0y'(0) = r,\\ &A_1y(1) + B_1y'(1) = s, \end{aligned}$$

for $\lambda \in [0,1]$.

THEOREM 3.5. *Suppose the* SL-*boundary conditions are such that* $\alpha_0 + \alpha_1 > 0$ *and that* B_0 *and* B_1 *are not both zero. Assume:* $f : [0,1] \times H^2 \to H$ *is continuous and completely continuous; there is a constant* $M > 0$ *such that*

$$\alpha_0 M \geq |r|, \quad \alpha_1 M \geq |s|,$$

and

$$|y| = M \quad \text{and} \quad y \cdot p = 0 \Longrightarrow y \cdot f(t,y,p) + |p|^2 > 0;$$

there is a Borel function $\psi : [0,\infty) \to (0,\infty)$ *such that* $|f(t,y,p)| \leq \psi(|p|)$ *for* $(t,|y|) \in [0,1] \times [0,M]$ *and*

$$\int_c^\infty \frac{dx}{\psi(x)} > 1, \qquad \text{where } c = \max\left\{\frac{|A_0|M + |r|}{|B_0|}, \frac{|A_1|M+|s|}{|B_1|}\right\}$$

with the convention that when $|B_0| = 0$ *or* $|B_1| = 0$ *the corresponding term in the right member is deleted. Then the Sturm-Liouville problem* $(\mathcal{SL})$ *has a solution* $y \in C^2([0,1], H)$.

REMARK. Theorem 3.5 extends Theorem 3.1 in [6]. In the present theorem, the conditions on f are somewhat less restrictive. The boundary data in [6] corresponds to the present ones further restricted so that A_0 and A_1 are multiples of I. The boundary conditions in Theorem 3.5 exclude pure Dirichlet data at both ends, exclude pure Neumann data at both ends, and require any pure Neumann condition to be homogeneous. We shall consider homogeneous Neumann conditions at both ends in a forthcoming paper. The proof given here does not apply when f is K-Carathéodory. That case will be treated in a separate paper.

PROOF. Since $\alpha_0 + \alpha_1 > 0$ in the SL-boundary conditions, the operator $\Lambda : C^2_{\mathbf{B}_0} \to C$ is invertible by Lemma 1.4. Consider the auxiliary problem

$$(\mathcal{SL}_{\lambda,1}) \qquad \begin{aligned} &y'' = \lambda f_1(t, y, y'), \qquad 0 \le t \le 1,\\ &A_0 y(0) - B_0 y'(0) = r,\\ &A_1 y(1) + B_1 y'(1) = s, \end{aligned}$$

where $\lambda \in [0,1]$ and

$$f_1(t,y,p) = \begin{cases} \dfrac{M}{|y|} f\Big(t, M\dfrac{y}{|y|}, p\Big), & |y| \ge M;\\ f(t,y,p), & |y| \le M. \end{cases}$$

It is easy to confirm that f_1 is continuous and has property-K. Observe that $y \cdot f + |p|^2 > 0$ implies that $y \cdot \lambda f + |p|^2 > 0$ for $\lambda \in (0,1]$. Consequently, for $\lambda \in (0,1]$,

$$(3.1) \qquad |y| \le M \quad \text{and} \quad y \cdot p = 0 \Longrightarrow y \cdot \lambda f_1(t,y,p) + |p|^2 > 0.$$

Indeed, if $|y| \ge M$ and $y \cdot p = 0$ then

$$y \cdot \lambda f_1(t,y,p) + |p|^2 = \frac{My}{|y|} \cdot \lambda f\left(t, \frac{My}{|y|}, p\right) + |p|^2 > 0,$$

because $|My/|y|| = M$ and $(My/|y|) \cdot p = 0$.

We shall prove that $(\mathcal{SL}_{1,1})$ has a solution $y \in C^2$ with $|y|_0 \le M$. Evidently, such a y also solves $(\mathcal{SL})$. The existence of a solution to $(\mathcal{SL}_{1,1})$ follows from Theorem 2.1 once we establish an *a priori* bound in C^1 on solutions $y \in C^2$ to $(\mathcal{SL}_{\lambda,1})$ for $\lambda \in [0,1)$. Lemmas 3.6 and 3.7, that follow, establish that there is a constant M_1 independent of λ such that $|y|_1 \le \max\{M, M_1\}$ for any solution $y \in C^2$ to $(\mathcal{SL}_{\lambda,1})$ for $\lambda \in [0,1)$. These lemmas complete the proof of Theorem 3.5. □

LEMMA 3.6. *If y solves $(\mathcal{SL}_{\lambda,1})$ for some $\lambda \in [0,1)$, then $|y|_0 \le M$.*

PROOF. The problem $(\mathcal{SL}_{0,1})$ has the unique solution $y = 0$. So without loss in generality assume $\lambda \in (0,1)$. Let $r(t) = \frac{1}{2}|y(t)|^2$. Then

$$\begin{aligned} r'(t) &= y(t) \cdot y'(t), \\ r''(t) &= y(t) \cdot \lambda f_1(t, y(t), y'(t)) + |y'(t)|^2. \end{aligned} \tag{3.2}$$

Now, $|y(t_0)| = |y|_0$ for some $t_0 \in [0,1]$. If $t_0 \in (0,1)$ then $r'(t_0) = 0$ and $r''(t_0) \le 0$. In view of (3.1) and (3.2), we find that $|y(t_0)| < M$. Suppose next that $t_0 = 0$. Since $r(0)$ is a maximum, $r'(0) = y(0) \cdot y'(0) \le 0$. We consider two cases: $\alpha_0 > 0$ and $\alpha_0 = 0$. First, suppose that $\alpha_0 > 0$. Since $B_0 = 0$ or I and $\alpha_0 M \ge |r|$, we find that

$$\begin{aligned} 0 \ge y(0) \cdot B_0 y'(0) = y(0) \cdot A_0 y(0) - y(0) \cdot r \\ \ge \alpha_0 |y(0)|^2 - |r||y(0)| \\ \ge |y(0)|(\alpha_0|y(0)| - \alpha_0 M). \end{aligned}$$

So

$$|y(0)| \le M.$$

Second, suppose that $\alpha_0 = 0$. Then $r = 0, B_0 = I$, the boundary condition at $t = 0$ is $A_0 y(0) - y'(0) = 0$, and, hence,

$$0 \ge r'(0) = y(0) \cdot y'(0) = y(0) \cdot A_0 y(0) \ge 0.$$

Then $r'(0) = 0$ and $r''(0) \le 0$ for a maximum at $t_0 = 0$ and reasoning as in the case when $t_0 \in (0,1)$, we find that $|y(0)| \le M$. Finally, if $t_0 = 1$, we find $|y(1)| \le M$ by reasoning as for $t_0 = 0$. □

LEMMA 3.7. *If y solves $(\mathcal{SL}_{\lambda,1})$ for $\lambda \in [0,1)$ then $|y'|_0 \le M_1$ for some constant independent of λ and y.*

PROOF. From Lemma 3.6, $|y|_0 \le M$. Either B_0 or B_1 is nonzero. If $B_0 = I$, then

$$y'(0) = A_0 y(0) - r \Longrightarrow |y'(0)| \le |A_0||y(0)| + |r| \le |A_0|M + |r|.$$

Likewise, if $B_1 = I$ then $|y'(1)| \le |A_1|M + |s|$. Consequently, there exists $\tau \in [0,1]$ such that $|y'(\tau)| \le c$ with c as in Theorem 3.5. On the other hand,

$$|y'(t) \cdot y''(t)| \le |y'(t)|\psi(|y'(t)|), \qquad \text{for } t \in [0,1].$$

Thus, by Lemma 3.2 and the remark following it, there is a constant M_1 such that

$$|y'|_0 \le M_1.$$

□

Our last example extends results in [5] for differential equations in $\mathbb{R}^n$ and with f continuous to a Hilbert space H and with f a K-Carathéodory function. In addition, the boundary conditions permitted here are more general than in [5] even when $H = \mathbb{R}^n$. Finally, the principal growth restriction on f in [5] is relaxed somewhat: $K = K(t)$ is constant in [5].

THEOREM 3.8. *Let $f : [0,1] \times H^2 \to H$ be a K-Carathéodory function and assume* SL*-boundary conditions with $\alpha_0 + \alpha_1 > 0$ and $r = s = 0$. Assume there is a constant $\sigma > 0$ and a function $K \in L^1([0,1],[0,\infty))$ such that*

$$y \cdot f(t,y,p) + |p|^2 \geq -K(t)(1 + |y| + |y \cdot p|) + \sigma |f(t,y,p)|$$

for almost all $t \in [0,1]$ and y, $p \in H$. Then $(\mathcal{SL})$ has a solution.

PROOF. It is sufficient to find an *a priori* bound in $C^1[0,1]$ on solutions y to $(\mathcal{SL}_\lambda)$ for $\lambda \in [0,1)$. It turns out that two key norm estimates in [5] extend directly to any Hilbert space H. We record these results as Lemmas 3.9 and 3.10. The reader is referred to [5] for the proofs, which can be modified to accomodate the variability of K.

LEMMA 3.9. *Let γ be a nonnegative constant and $K \in L^1([0,1],[0,\infty))$. Let u be a nonnegative function in $W^{2,1}([0,1],\mathbb{R})$ that satisfies the inequalities*

$$u'(0) \geq 0, \quad u'(1) \leq 0,$$
$$u(0) \leq \gamma u'(0) \quad \text{or} \quad u(1) \geq -\gamma u'(1),$$
$$u''(t) \geq -K(t)[1 + (2u(t))^{1/2} + |u'(t)|], \qquad \text{for a.e. } t \in [0,1].$$

Then, there exists a constant M_0 (depending only on K and γ) such that

$$|u(t)| \leq M_0, \quad |u'(t)| \leq M_0, \qquad \text{for } t \in [0,1].$$

LEMMA 3.10. *Let $\theta \in L^1([0,1],[0,\infty))$, and let M_0, M_1, and $\sigma > 0$ be nonnegative constants. Let $y \in W^{2,1}([0,1],H)$ satisfy*

$$|y(t)| \leq M_0, \ |y(t) \cdot y'(t)| \leq M_1, \qquad \text{for } t \in [0,1],$$
$$\frac{d^2}{dt^2}\Big(\frac{1}{2}|y(t)|^2\Big) \geq -\theta(t) + \sigma |y''(t)|, \qquad \text{for a.e. } t \in [0,1].$$

Then there exists a constant M_2 (depending only on M_0, M_1, σ, and θ) such that $|y'(t)| \leq M_2$ for all $t \in [0,1]$.

Now, let $y \in W^{2,1}([0,1],H)$ be a solution to $(\mathcal{SL}_\lambda)$ for some $\lambda \in [0,1)$. We claim that $u(t) = \frac{1}{2}|y(t)|^2$ satisfies the hypotheses in Lemma 3.9. Indeed,

$$u'(t) = y(t) \cdot y'(t), \quad u''(t) = y(t) \cdot \lambda f(t, y(t), y'(t)) + |y'(t)|^2.$$

From the inequality in Theorem 3.8 we deduce that

$$y \cdot f(t,y,p) + |p|^2 \geq -K(t)(1 + |y| + |y \cdot p|),$$

$$y \cdot \lambda f(t,y,p) + |p|^2 \geq -K(t)(1 + |y| + |y \cdot p|),$$

for $\lambda \in [0,1)$. Consequently, $u'' \geq -K(1+(2u)^{1/2}+|u''|)$ for almost all $t \in [0,1]$. Suppose $\alpha_0 = 0$. Then $B_0 = I$ and $u'(0) = y(0) \cdot y'(0) = y(0) \cdot A_0 y(0) \geq 0$. If $\alpha_0 > 0$ then A_0^{-1} exists and is nonnegative definite. Since $B_0 = 0$ or I,

$$u'(0) = y'(0) \cdot y(0) = y'(0) \cdot A_0^{-1} B_0 y'(0) \geq 0.$$

In the same way, we show that $u'(1) \leq 0$. Suppose again that $\alpha_0 > 0$. If $B_0 = 0$ then $A_0 y(0) = 0, 0 = y(0) \cdot A_0 y(0) \geq \alpha_0 |y(0)|^2$, and $y(0) = 0$. Then $u(0) = 0$ and $u'(0) = 0$. If $B_0 = I$ then

$$u'(0) = y(0) \cdot y'(0) = y(0) \cdot A_0 y(0) \geq \alpha_0 |y(0)|^2 = 2\alpha_0 u(0).$$

In either case, $u(0) \leq \gamma u'(0)$ for $\gamma = \alpha_0/2$. Now, if $\alpha_0 = 0$ then $\alpha_1 > 0$ and we find that $u(1) \leq -\gamma u'(1)$ for $\gamma = \alpha_1/2$, by similar reasoning. Therefore, $u = |y(t)|^2/2$ satisfies the hypotheses of Lemma 3.9 and there is a constant M_0 independent of λ such that

$$|y(t)| \leq M_0, \quad |y(t) \cdot y'(t)| \leq M_0, \qquad \text{for } t \in [0,1].$$

Then,

$$\begin{aligned}
\frac{d^2}{dt^2}\left(\tfrac{1}{2}|y(t)|^2\right) &= y(t) \cdot y''(t) + |y'(t)|^2 \\
&= y \cdot \lambda f(t, y(t), y'(t)) + |y'(t)|^2 \\
&\geq -K(t)(1 + |y(t)| + |y'(t) \cdot y(t)|) + \sigma |\lambda f(t, y(t), y'(t))| \\
&\geq -\theta(t) + \sigma |y''(t)|,
\end{aligned}$$

for $\theta(t) = (1 + 2M_0)K(t)$ and $\sigma > 0$. By Lemma 3.10 there is a constant $\tilde{M}$ depending only upon M_0, K, and σ such that $|y'|_0 \leq \tilde{M}$. Finally, $|y|_1 \leq \max\{M_0, \tilde{M}\}$, which is the required *a priori* bound. □

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