Nonlinear Stud. **15** (2008), 71 - 92.

MULTIPLICITY RESULTS FOR SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS

M. FRIGON AND E. MONTOKI

Abstract. Multiplicity results are obtained for systems of second order differential equations with periodic or Sturm-Liouville boundary conditions. Results rely on the notion of strict solution-tube. Different growth conditions of Wintner-Nagumo type are considered.

1. Introduction

In this paper, we establish multiplicity results for the following system of second order differential equations:

$$\begin{aligned} &x''(t) = f(t, x(t), x'(t)), \quad \text{a.e. } t \in [0,1], \\ &x \in \text{BC}. \end{aligned} \tag{1.1}$$

Here $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a Carathéodory function, and BC denotes the Strum-Liouville or the periodic boundary conditions:

$$\begin{aligned} A_0 x(0) - \beta_0 x'(0) &= r_0, \\ A_1 x(1) + \beta_1 x'(1) &= r_1; \end{aligned} \tag{1.2}$$

$$\begin{aligned} x(0) &= x(1), \\ x'(0) &= x'(1); \end{aligned} \tag{1.3}$$

where for $i = 0, 1$, $\beta_i \in \{0, 1\}$, and A_i is a $n \times n$ matrix such that there exists $\alpha_i \geq 0$ satisfying $\langle A_i x, x\rangle \geq \alpha_i \|x\|^2$, and $\alpha_i + \beta_i > 0$.

In the literature, there are very few multiplicity results for systems ($n > 1$) of second order differential equations. Let us mention results of Barutello, Capietto and Habets [1] where systems of two second order differential equations are considered. Multiplicity results with prescribed nodal structure were obtained for systems of second order differential equations by Capietto with Dalbono [2] and Dambrosio [4] (see also [3]). Multiplicity results for systems of superlinear second order equations were also obtained in [13] via a continuation theorem and computation of the degree associated to some scalar equations.

In the particular case of a boundary value problem for a single differential equation of second order, more results were obtained. Our results extend to systems the results of Henderson and Thompson [12] for classical upper and lower solutions. Other multiplicity results for a single equation can be found for instance in [10].

1991 *Mathematics Subject Classification.* Primary 34B15,

Key words and phrases. systems of differential equations, boundary value problems, multiplicity results, degree theory.

This work was partially supported by CRSNG Canada.

In order to obtain multiplicity results to (1.1), we will assume the existence of solution-tubes of (1.1). This notion was introduced in [6], and generalizes the notions of upper and lower solutions of a differential equation. Also, we introduce the notion of strict solution-tube. Different growth conditions of Wintner-Nagumo type will be imposed, and for which different arguments will be needed. Indeed, in section 4, using the Schauder degree theory, we will prove a multiplicity result (which slightly generalizes a result obtained in the Ph.D. thesis of the second author [14]) when f satisfies the growth condition:

> there exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I)$ such that

$$|\langle p, f(t,x,p)\rangle| \le \phi(\|p\|)(\gamma(t) + \|p\|) \quad \text{and} \quad \int_k^\infty \frac{s\,ds}{\phi(s)+s} = \infty \quad \forall k \ge 0.$$

On the other hand, in section 5, we will establish multiplicity results in the case where f satisfies the more standard growth condition:

> there exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I)$ such that

$$\|f(t,x,p)\| \le \phi(\|p\|)(\gamma(t) + \|p\|) \quad \text{and} \quad \int_k^\infty \frac{ds}{\phi(s)} = \infty \quad \forall k \ge 0.$$

The proof of these multiplicity results will rely on degree theory for multi-valued compact upper semi-continuous maps with closed, convex values.

It is well known that for systems of second order differential equations, a Nagumo type growth condition is not sufficient to guarantee the existence of a priori bounds on the derivative of solutions. Two different types of hypothesis will be considered. The first one (see (H4)) is the well known condition introduced by Hartman [11]. The second one (see (H5) or (H8)) is a generalization of a condition introduced by the first author in [7]. It has the advantage of being trivially satisfied in the scalar case.

In what follows, we will use the following notations: $I = [0,1]$, $C^k(I,\mathbb{R}^n)$ is the space of k-times continuously differentiable functions endowed with the usual norm that we denote $\|\cdot\|_k$; $C^k_B(I,\mathbb{R}^n)$ is the subset of x in $C^k(I,\mathbb{R}^n)$ satisfying the boundary condition BC; $C_0(I,\mathbb{R}^n) = \{x \in C(I,\mathbb{R}^n) : x(0) = x(1) = 0\}$; $L^1(I,\mathbb{R}^n)$ is the space of integrable functions with the usual norm $\|\cdot\|_{L^1}$; $W^{2,1}(I,\mathbb{R}^n)$ is the Sobolev space $\{x \in C^1[0,1] : x' \text{ is absolutely continuous}\}$ endowed with the norm $\|x\|_{2,1} = \|x\|_{L^1} + \|x'\|_{L^1} + \|x''\|_{L^1}$; $W^{2,1}_B(I,\mathbb{R}^n) = W^{2,1}(I,\mathbb{R}^n) \cap C_B(I,\mathbb{R}^n)$.

We will denote single-valued maps with lower case letters while we will use capital letters for multi-valued maps. Let X and Y be topological spaces, we say that a multi-valued map $F : X \to Y$ is upper semi-continuous (u.s.c.) (resp. if $X = [0,1]$, F is measurable) if $\{x \in X : F(x) \cap B \neq \emptyset\}$ is closed (resp. measurable) for every closed set B of Y. We say that $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ a single-valued map (resp. $F : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ a multi-valued map with closed convex values) is Carathéodory if $t \mapsto f(t,x,p)$ (resp. $t \mapsto F(t,x,p)$) is measurable for all x, p; $(x,p) \mapsto f(t,x,p)$ is continuous (resp. $(x,p) \mapsto F(t,x,p)$ is u.s.c.) for a.e. $t \in I$; for every $k > 0$, there exists $d_k \in L^1(I)$ such that $f(t,B(0,k),B(0,k)) \subset B(0,d_k(t))$ (resp. $F(t,B(0,k),B(0,k)) \subset B(0,d_k(t))$) a.e. $t \in I$, where $B(0,r)$ is the closed ball of radius r centered at the origin.

2. Main Theorem

The notion of solution-tube will play a fundamental role in our main result. This notion was introduced in [6] and generalizes naturally to systems the well known notion of upper and lower solutions.

Definition 2.1. Let $v \in W^{2,1}(I,\mathbb{R}^n)$, and $\rho \in W^{2,1}(I,\mathbb{R})$. We say that (v,ρ) is a *solution-tube of* (1.1) if

(i) for a.e. $t \in I$, and every $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| = \rho(t)$ and $\langle x - v(t), p - v'(t)\rangle = \rho(t)\rho'(t)$,
$$\langle x - v(t), f(t,x,p) - v''(t)\rangle + \|p - v'(t)\|^2 \geq \rho(t)\rho''(t) + \rho'(t)^2;$$

(ii) $v''(t) = f(t, v(t), v'(t))$ a.e. on $\{t \in [0,1] : \rho(t) = 0\}$;

(iii) if BC denotes (1.2),
$$\|A_0 v(0) - \beta_0 v'(0) - r_0\| \leq \alpha_0\rho(0) - \beta_0\rho'(0),$$
$$\|A_1 v(1) + \beta_1 v'(1) - r_1\| \leq \alpha_1\rho(1) + \beta_1\rho'(1);$$
and if BC denotes (1.3),
$$\rho(0) = \rho(1),\ v(0) = v(1),\ \|v'(0) - v'(1)\| \leq \rho'(1) - \rho'(0).$$

We denote
$$T(v,\rho) = \{x \in C(I,\mathbb{R}^n) : \|x(t) - v(t)\| \leq \rho(t)\ \forall t \in I\}.$$

Our goal is to establish multiplicity results for the system of second order differential equation (1.1). To this aim, we introduce the notion of strict solution-tube.

Definition 2.2. Let $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,\mathbb{R})$. We say that (v,ρ) is a *strict solution-tube of* (1.1) if

(i) $\rho(t) > 0$ for all $t \in I$;

(ii) for every $t \in I$, there exist $\varepsilon > 0$ and V a neighborhood of t such that
$$\langle x - v(t), f(t,x,p) - v''(t)\rangle + \|p - v'(t)\|^2 - \rho(t)\rho''(t) - \rho'(t)^2 \geq 0$$
a.e. $t \in V$ and for every $(x,p) \in S_{t,\varepsilon}$, where
$$S_{t,\varepsilon} = \{(x,p) \in \mathbb{R}^{2n} : \rho(t) - \varepsilon \leq \|x - v(t)\| \leq \rho(t),$$
$$|\langle x - v(t), p - v'(t)\rangle - \rho(t)\rho'(t)| \leq \varepsilon\};$$

(iii) if BC denotes (1.2),
$$\|A_0 v(0) - \beta_0 v'(0) - r_0\| < \alpha_0\rho(0) - \beta_0\rho'(0),$$
$$\|A_1 v(1) + \beta_1 v'(1) - r_1\| < \alpha_1\rho(1) + \beta_1\rho'(1);$$
and if BC denotes (1.3),
$$\rho(0) = \rho(1),\ v(0) = v(1),\ \|v'(0) - v'(1)\| < \rho'(1) - \rho'(0).$$

In the particular case where $n = 1$, if (v,ρ) is a strict solution-tube of (1.1) then $v - \rho$ (resp. $v + \rho$) is a strict lower (resp. upper) solution of (1.1), see [5].

We will consider the following assumptions:

(H1) $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is Carathéodory;

(H2) there exist (v_0,ρ_0) a solution-tube of (1.1) and $(v_1,\rho_1), (v_2,\rho_2)$ two strict solution-tubes of (1.1) such that $T(v_1,\rho_1) \cap T(v_2,\rho_2) = \emptyset$, and $T(v_i,\rho_i) \subset T(v_0,\rho_0)$, $i = 1,2$;

(H3) there exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I)$ such that
$$|\langle p, f(t,x,p)\rangle| \leq \phi(\|p\|)(\gamma(t) + \|p\|)$$
a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \leq \rho_0(t)$, and
$$\int_k^\infty \frac{sds}{\phi(s)+s} = \infty \quad \forall k \geq 0;$$

(H4) there exist $a \geq 0$ and $l \in L^1(I)$ such that
$$\|f(t,x,p)\| \leq a(\langle x, f(t,x,p)\rangle + \|p\|^2) + l(t)$$
a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \leq \rho_0(t)$;

(H5) there exist $R > 0$, $b > 0$, $c \geq 0$, and $h \in L^1(I)$ such that
$$(b + c\|x\|)\sigma(t,x,p) \geq \|p\| - h(t),$$
for a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \leq \rho_0(t)$, $\|p\| \geq R$, where
$$\sigma(t,x,p) = \frac{\langle x, f(t,x,p)\rangle + \|p\|^2}{\|p\|} - \frac{\langle p, f(t,x,p)\rangle\langle x, p\rangle}{\|p\|^3}.$$

Observe that (H5) is trivially satisfied in the scalar case i.e. when $n = 1$.

Now, we can state our main result.

Theorem 2.3 (Main Theorem). *Assume that* (H1)–(H3), and (H4) or (H5) are satisfied then the problem (1.1) has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0,1,2$, $j = 1,2$, $i \neq j$.

3. Auxiliary results

As in the method of upper and lower solutions, the assumptions on the existence of solution-tubes will permit to obtain a priori bounds on the solutions. First of all, we recall a result of [7] (see [7, Lemma 3.2]).

Lemma 3.1. *Let $\varepsilon \geq 0$, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,\mathbb{R})$ a solution-tube of* (1.1)*. If $x \in W^{2,1}_B(I,\mathbb{R}^n)$ satisfies*

$$\begin{aligned}(3.1)\qquad &\frac{\langle x(t)-v(t), x''(t)-v''(t)\rangle + \|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} \\ &\quad - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3} - \varepsilon\|x(t)-v(t)\| \geq \rho''(t) - \varepsilon\rho(t)\end{aligned}$$

for a.e. $t \in \{t \in I : \|x(t)-v(t)\| > \rho(t)\}$, then $x \in T(v,\rho)$.

Next, we want to show that if x is a solution of (1.1) lying in $T(v,\rho)$ for (v,ρ) a strict solution-tube of (1.1), then x belongs to the interior of $T(v,\rho)$.

Lemma 3.2. *Let $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,\mathbb{R})$ be a strict solution-tube of* (1.1)*. If $x \in W^{2,1}_B(I,\mathbb{R}^n) \cap T(v,\rho)$ is a solution of* (1.1)*, then $\|x(t) - v(t)\| < \rho(t)$ for all $t \in I$.*

Proof. We claim that
$$(3.2)\qquad \|x(0) - v(0)\| < \rho(0) \quad \text{and} \quad \|x(1) - v(1)\| < \rho(1).$$

Indeed, if BC denotes (1.2) and if $\|x(0)-v(0)\| = \rho(0)$, then by Definition 2.2 and since $\|x(t)-v(t)\| \le \rho(t)$ for all t,

$$\begin{aligned}
0 &\ge \frac{\beta_0}{2}\frac{d}{dt}(\|x(t)-v(t)\|^2-\rho(t)^2)|_{t=0}\\
&= \beta_0\big(\langle x(0)-v(0), x'(0)-v'(0)\rangle - \rho(0)\rho'(0)\big)\\
&> \beta_0\langle x(0)-v(0), x'(0)-v'(0)\rangle + \rho(0)\|A_0v(0)-\beta_0v'(0)-r_0\| - \alpha_0\rho(0)^2\\
&\ge \beta_0\langle x(0)-v(0), x'(0)-v'(0)\rangle - \langle x(0)-v(0), A_0v(0)-\beta_0v'(0)-r_0\rangle\\
&\quad - \alpha_0\|x(0)-v(0)\|^2\\
&= \langle A_0(x(0)-v(0)), x(0)-v(0)\rangle - \alpha_0\|x(0)-v(0)\|^2\\
&\ge 0,
\end{aligned}$$

contradiction. Similarly, we show that $\|x(1)-v(1)\| < \rho(1)$.

If BC denotes (1.3), then

$$\|x(0)-v(0)\| < \rho(0) \quad \text{if and only if} \quad \|x(1)-v(1)\| < \rho(1).$$

Therefore, if the claim is false, by Definition 2.2, $\|x(0)-v(0)\| = \rho(0) = \rho(1) = \|x(1)-v(1)\|$, and

$$\begin{aligned}
0 &\le \frac{1}{2}\Big(\frac{d}{dt}(\|x(t)-v(t)\|^2-\rho(t)^2)|_{t=1} - \frac{d}{dt}(\|x(t)-v(t)\|^2-\rho(t)^2)|_{t=0}\Big)\\
&= \langle x(1)-v(1), x'(1)-v'(1)\rangle - \langle x(0)-v(0), x'(0)-v'(0)\rangle\\
&\quad - \rho(1)\rho'(1) + \rho(0)\rho'(0)\\
&= \langle x(1)-v(1), v'(0)-v'(1)\rangle - \rho(1)\big(\rho'(1)-\rho'(0)\big)\\
&< \rho(1)\big(\rho'(1)-\rho'(0)\big) - \rho(1)\big(\rho'(1)-\rho'(0)\big);
\end{aligned}$$

which is a contradiction.

Now assume that $E = \{t \in\,]0,1[\, : \|x(t)-v(t)\| = \rho(t)\} \ne \emptyset$. Let $t_1 = \inf E$. By (3.2), $t_1 \in\,]0,1[$ and $\langle x(t_1)-v(t_1), x'(t_1)-v'(t_1)\rangle = \rho(t_1)\rho'(t_1)$. Definition 2.2 implies that there exist $\varepsilon > 0$ and V a neighborhood of t_1 such that

$$\langle y-v(t), f(t,y,p)-v''(t)\rangle + \|p-v'(t)\|^2 - \rho(t)\rho''(t) + \rho'(t)^2 \ge 0$$

a.e. $t \in V$ and for every $(y,p) \in S_{t_1,\varepsilon}$ which is defined in Definition 2.2. Since $x \in C^1$, we can find $O \subset V$ a neighborhood of t_1 such that $(x(t), x'(t)) \in S_{t_1,\varepsilon}$ for all $t \in O$. On the other hand, by definition of t_1, we can choose $t_0 \in O \cap [0, t_1[$ such that

$$\frac{d}{dt}(\|x(t)-v(t)\|^2-\rho(t)^2)|_{t=t_0} > 0.$$

Therefore,

$$\begin{aligned}
0 &> \frac{1}{2}\int_{t_0}^{t_1}\frac{d^2}{dt^2}(\|x(t)-v(t)\|^2-\rho(t)^2)\,dt\\
&= \int_{t_0}^{t_1}\big(\langle x(t)-v(t), f(t,x(t),x'(t))-v''(t)\rangle + \|x'(t)-v'(t)\|^2\\
&\qquad\qquad - \rho(t)\rho''(t) - \rho'(t)^2\big)\,dt\\
&\ge 0,
\end{aligned}$$

and the proof is complete. □

In order to obtain a priori bounds on the derivative of the solutions, we recall the following results of [7]. The first one concerns a priori bounds that can be obtained under a Wintner-Nagumo growth condition (see [7, Lemma 3.4]).

Lemma 3.3. *Let $r, k \geq 0$, $m \in L^1(I)$, and $\psi : [0, \infty[\to]0, \infty[$ a Borel measurable function such that*

$$\int_r^\infty \frac{sds}{\psi(s)} > \|m\|_{L^1} + k.$$

Then there exists $K > 0$ such that $\|x'\|_0 < K$ for every $x \in W^{2,1}(I, \mathbb{R}^n)$ satisfying

(i) $\min_{t\in I} \|x'(t)\| \leq r$;
(ii) $\|x'\|_{L^1[t_0,t_1]} \leq k$ *for every* $[t_0, t_1] \subset \{t \in I : \|x'(t)\| \geq r\}$;
(iii) $|\langle x'(t), x''(t)\rangle| \leq \psi(\|x'(t)\|)(m(t) + \|x'(t)\|)$ *a.e. on* $\{t \in I : \|x'(t)\| \geq r\}$.

The previous result shows that to obtain an a priori bound of the derivative x' with respect to norm of the uniform convergence, we need to obtain an a priori bound of x' in the L^1-norm.

The following result generalizes and simplifies [7, Lemma 3.3].

Lemma 3.4. *Let $u \in W^{2,1}(I, \mathbb{R}^n)$, $\xi > 0$, $\zeta \geq 0$, $r > 0$, $m \in L^1(I)$. Then there exists $\omega : [0, \infty[\to [0, \infty[$ an increasing function such that we have for any interval $[t_0, t_1]$ on which $\|x'(t) - u'(t)\| \geq r$,*

$$\|x' - u'\|_{L^1[t_0,t_1]} \leq \omega(\|x - u\|_0),$$

and

$$\min_{t\in I} \|x'(t) - u'(t)\| \leq \max\{r, \omega(\|x - u\|_0)\}.$$

for every $x \in W^{2,1}(I, \mathbb{R}^n)$ satisfying almost everywhere on $\{t \in I : \|x'(t) - u'(t)\| \geq r\}$,

$$(\xi + \zeta\|x(t) - u(t)\|)\sigma_u(t, x) + \frac{\zeta\langle x(t) - u(t), x'(t) - u'(t)\rangle^2}{\|x(t) - u(t)\| \, \|x'(t) - u'(t)\|} \geq \|x'(t) - u'(t)\| - m(t),$$

where

$$\sigma_u(t, x) = \frac{\langle x(t) - u(t), x''(t) - u''(t)\rangle + \|x'(t) - u'(t)\|^2}{\|x'(t) - u'(t)\|} - \frac{\langle x'(t) - u'(t), x''(t) - u''(t)\rangle\langle x(t) - u(t), x'(t) - u'(t)\rangle}{\|x'(t) - u'(t)\|^3}.$$

Proof. First of all, observe that

$$\frac{d}{dt}\frac{(\xi + \zeta\|x(t) - u(t)\|)\langle x(t) - u(t), x'(t) - u'(t)\rangle}{\|x'(t) - u'(t)\|} = (\xi + \zeta\|x(t) - u(t)\|)\sigma_u(t, x) + \frac{\zeta\langle x(t) - u(t), x'(t) - u'(t)\rangle^2}{\|x(t) - u(t)\| \, \|x'(t) - u'(t)\|}. \tag{3.3}$$

Assume that $\|x'(t)-u'(t)\| \geq r$ on $[t_0,t_1]$ then the assumptions and (3.3) yield

$$\begin{aligned}
&\int_{t_0}^{t_1} \|x'(t)-u'(t)\|\,dt \\
&\quad \leq \|m\|_{L^1[0,1]} + \int_{t_0}^{t_1} \frac{d}{dt}\frac{(\xi+\zeta\|x(t)-u(t)\|)\langle x(t)-u(t), x'(t)-u'(t)\rangle}{\|x'(t)-u'(t)\|}\,dt \\
&\quad \leq \|m\|_{L^1[0,1]} + 2(\xi+\zeta\|x-u\|_0)\|x-u\|_0 \\
&\quad = \omega(\|x-u\|_0).
\end{aligned}$$

□

An a priori bound of x' in L^1-norm can also be obtained using a condition introduced by Hartman [11].

Lemma 3.5. *Let $k \geq 0$ and $m \in L^1(I)$. Then there exists $\omega : [0,\infty[\to]0,\infty[$ an increasing function such that for every $x \in W^{2,1}(I,\mathbb{R}^n)$ satisfying*

$$\|x''(t)\| \leq k\big(\langle x(t), x''(t)\rangle + \|x'(t)\|^2\big) + m(t) \quad a.e.\ t \in I,$$

we have $\|x'\|_{L^1} \leq \omega(\|x\|_0)$.

Proof. Let $x \in W^{2,1}(I,\mathbb{R}^n)$. Observe that

$$\frac{x'(t)}{2} = x(t+\frac{1}{2}) - x(t) - \int_t^{t+\frac{1}{2}} (t+\frac{1}{2}-s)x''(s)\,ds \quad \text{for } 0 \leq t \leq \frac{1}{2}. \tag{3.4}$$

So, for $t \in [0,1/2]$,

$$\begin{aligned}
\frac{\|x'(t)\|}{2} &\leq 2\|x\|_0 + \|m\|_{L^1} + k\int_t^{t+\frac{1}{2}} (t+\frac{1}{2}-s)\big(\langle x(s), x''(s)\rangle + \|x'(s)\|^2\big)\,ds \\
&= 2\|x\|_0 + \|m\|_{L^1} + \frac{k}{2}\int_t^{t+\frac{1}{2}} (t+\frac{1}{2}-s)\frac{d^2}{ds^2}\|x(s)\|^2\,ds.
\end{aligned}$$

Using (3.4) with $\|x\|^2$, we obtain

$$\frac{\|x'(t)\|}{2} \leq 2\|x\|_0 + \|m\|_{L^1} + k\|x\|_0^2 - \frac{k}{2}\langle x(t), x'(t)\rangle \quad \text{for } 0 \leq t \leq \frac{1}{2}. \tag{3.5}$$

Similarly, using the identity

$$\frac{x'(t)}{2} = x(t) - x(t-\frac{1}{2}) - \int_{t-\frac{1}{2}}^{t} (t-\frac{1}{2}-s)x''(s)\,ds \quad \text{for } \frac{1}{2} \leq t \leq 1, \tag{3.6}$$

we obtain

$$\frac{\|x'(t)\|}{2} \leq 2\|x\|_0 + \|m\|_{L^1} + k\|x\|_0^2 + \frac{k}{2}\langle x(t), x'(t)\rangle \quad \text{for } \frac{1}{2} \leq t \leq 1. \tag{3.7}$$

Combining (3.5) and (3.7), we deduce that

$$\begin{aligned}\|x'\|_{L^1} &= \int_0^{1/2} \|x'(t)\|\,dt + \int_{1/2}^1 \|x'(t)\|\,dt \\ &\leq 4\|x\|_0 + 2\|m\|_{L^1} + 2k\|x\|_0^2 - \frac{k}{2}\int_0^{1/2} \frac{d}{dt}\|x(t)\|^2\,dt \\ &\quad + \frac{k}{2}\int_{1/2}^1 \frac{d}{dt}\|x(t)\|^2\,dt \\ &\leq 4\|x\|_0 + 2\|m\|_{L^1} + 2k\|x\|_0^2 + \frac{k}{2}\big(\|x(1)\|^2 + \|x(0)\|^2 - 2\|x(1/2)\|^2\big) \\ &\leq 4\|x\|_0 + 2\|m\|_{L^1} + 3k\|x\|_0^2.\end{aligned}$$

The conclusion follows if we define $\omega(r) = 3kr^2 + 4r + 2\|m\|_{L^1}$. □

4. Proof of the Main Theorem

Fix $\varepsilon \geq 0$ such that the operator $l_\varepsilon : C^1_B(I,\mathbb{R}^n) \to C_0(I,\mathbb{R}^n)$ defined by

$$l_\varepsilon(x)(t) = x'(t) - x'(0) - \varepsilon \int_0^t x(s)\,ds$$

is invertible.

For $\lambda \in [0,1]$, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,[0,\infty[)$, we define

$$f^\lambda_{v,\rho} \;:\; I \times \mathbb{R}^{2n} \to \mathbb{R}^n \quad \text{by} \quad f^\lambda_{v,\rho} = \lambda \bar{f}_{v,\rho} + g^\lambda_{v,\rho} - \varepsilon(1-\lambda)v,$$

where

$$\bar{f}_{v,\rho}(t,x,p) = \begin{cases} \frac{\rho(t)}{\|x-v(t)\|} f(t,\widetilde{x}_{v,\rho},\widehat{p}_{v,\rho}) - \varepsilon\widetilde{x}_{v,\rho}, & \text{if } \|x-v(t)\| > \rho(t), \\ f(t,x,p) - \varepsilon x, & \text{if } \|x-v(t)\| \leq \rho(t) \end{cases}$$

$$g^\lambda_{v,\rho}(t,x,p) = \begin{cases} \Big(1 - \frac{\lambda\rho(t)}{\|x-v(t)\|}\Big)\Big(v''(t) + \frac{\rho''(t)}{\|x-v(t)\|}(x-v(t))\Big), & \text{if } \|x-v(t)\| \\ & > \rho(t), \\ (1-\lambda)\Big(v''(t) + \frac{\rho''(t)}{\rho(t)}(x-v(t))\Big), & \text{otherwise;} \end{cases}$$

with

$$\widetilde{x}_{v,\rho} = v(t) + \frac{\rho(t)}{\|x-v(t)\|}(x-v(t)),$$

$$\widehat{p}_{v,\rho} = p + \Big(\rho'(t) - \frac{\langle x-v(t), p-v'(t)\rangle}{\|x-v(t)\|}\Big)\Big(\frac{x-v(t)}{\|x-v(t)\|}\Big),$$

and where we mean $\rho''(t)(x-v(t))/\rho(t) = 0$ on $\{t \in [0,1] : \rho(t) = 0\}$.

Remark 4.1. For $(x,p) \in \mathbb{R}^{2n}$ such that $\|x-v(t)\| > 0$,

$$\|\widetilde{x}_{v,\rho} - v(t)\| = \rho(t), \quad \langle \widetilde{x}_{v,\rho} - v(t), \widehat{p}_{v,\rho} - v'(t)\rangle = \rho(t)\rho'(t),$$

$$\|\widehat{p}_{v,\rho} - v'(t)\|^2 = \|p - v'(t)\|^2 + (\rho'(t))^2 - \frac{\langle x(t)-v(t), p-v'(t)\rangle^2}{\|x(t)-v(t)\|^2};$$

and for $x \in C^1(I,\mathbb{R}^n)$,

$$(\widetilde{x(t)}_{v,\rho}, \widehat{x'(t)}_{v,\rho}) = (x(t),x'(t)) \quad \text{a.e. on } \{t \in I : \|x(t)-v(t)\| = \rho(t) > 0\}.$$

We consider for $i = 0, 1, 2$, and $\lambda \in [0, 1]$, the problem

$$(P_{i,\lambda}) \qquad \begin{aligned} & x''(t) - \varepsilon x(t) = f^{\lambda}_{v_i,\rho_i}(t, x(t), x'(t)) \quad \text{a.e. } t \in [0,1], \\ & x \in \mathrm{BC}. \end{aligned}$$

A priori bounds can be obtained for the solutions of $(P_{i,\lambda})$.

Proposition 4.2. *Assume that* (H1) *and* (H2) *are satisfied. Then*

(a) *every solution x of $(P_{i,\lambda})$ with $i = 0, 1, 2$, $\lambda \in [0, 1]$, is such that $x \in T(v_i, \rho_i)$;*
(b) *every solution x of $(P_{i,1})$ for $i = 1, 2$, satisfies $\|x(t) - v_i(t)\| < \rho_i(t)$ for all $t \in [0, 1]$.*

Proof. (a) For $i \in \{0, 1, 2\}$ and $\lambda \in [0, 1]$, let x be a solution of $(P_{i,\lambda})$. To simplify the notation, let v and ρ stand for v_i and ρ_i respectively, and $\tilde{x}(t)$ and $\hat{x}'(t)$ stand for $\widetilde{x(t)}_{v_i,\rho_i}$ and $\widehat{x'(t)}_{v_i,\rho_i}$. Then for almost every $t \in \{t \in I : \|x(t) - v(t)\| > \rho(t)\}$,

$$\begin{aligned}
&\frac{1}{\|x(t)-v(t)\|}\big(\langle x(t)-v(t), x''(t)-v''(t)\rangle + \|x'(t)-v'(t)\|^2\big) \\
&\quad - \frac{1}{\|x(t)-v(t)\|^3}\langle x(t)-v(t), x'(t)-v'(t)\rangle^2 - \varepsilon\|x(t)-v(t)\| \\
&= \frac{1}{\|x(t)-v(t)\|}\Big(\Big\langle x(t)-v(t), \frac{\lambda\rho(t)}{\|x(t)-v(t)\|}(f(t,\tilde{x}(t),\hat{x}'(t)) - v''(t))\Big\rangle \\
&\quad - \varepsilon\big\langle x(t)-v(t), \lambda\tilde{x}(t) + (1-\lambda)v(t) - x(t)\big\rangle\Big) \\
&\quad + \Big(1 - \frac{\lambda\rho(t)}{\|x(t)-v(t)\|}\Big)\rho''(t) \\
&\quad + \frac{\|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3} - \varepsilon\|x(t)-v(t)\| \\
&= \frac{\lambda}{\|x(t)-v(t)\|}\Big(\langle \tilde{x} - v(t), f(t,\tilde{x}(t),\hat{x}'(t)) - v''(t)\rangle + \|\hat{x}'(t)-v'(t)\|^2\Big) \\
&\quad + \frac{1}{\|x(t)-v(t)\|}\Big(\|x'(t)-v'(t)\|^2 - \lambda\|\hat{x}'(t)-v'(t)\|^2\Big) \\
&\quad - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3} + \Big(1 - \frac{\lambda\rho(t)}{\|x(t)-v(t)\|}\Big)\rho''(t) - \varepsilon\lambda\rho(t) \\
&\ge \frac{\lambda}{\|x(t)-v(t)\|}\big(\rho(t)\rho''(t) + \rho'(t)^2\big) + \Big(1 - \frac{\lambda\rho(t)}{\|x(t)-v(t)\|}\Big)\rho''(t) \\
&\quad + \frac{1}{\|x(t)-v(t)\|}\Big((1-\lambda)\|\hat{x}'(t)-v'(t)\|^2 - \rho'(t)^2\Big) - \varepsilon\lambda\rho(t) \\
&\ge \rho''(t) - \varepsilon\rho(t).
\end{aligned}$$

It follows from Lemma 3.1 that statement (a) holds.

(b) Observe that if x is a solution of $(P_{i,1})$ for some $i \in \{1, 2\}$, then by (a), x is a solution of (1.1). Lemma 3.2 yields to the conclusion. □

We associate to $f^{\lambda}_{v,\rho}$ an operator

$$\eta_{v,\rho} : [0,1] \times C^1(I, \mathbb{R}^n) \to C_0(I, \mathbb{R}^n)$$

defined by

$$\eta_{v,\rho}(\lambda, x) = \int_0^t f^{\lambda}_{v,\rho}(s, x(s), x'(s))\, ds.$$

The following result establishes some properties of $\eta_{v,\rho}$. The proof of this result is a direct consequence of [7, Proposition 3.5].

Proposition 4.3. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function, and $(v, \rho) \in W^{2,1}(I, \mathbb{R}^n) \times W^{2,1}(I, [0, \infty[)$ a solution-tube of* (1.1)*. Then the operator $\eta_{v,\rho}$ is continuous and completely continuous.*

In order to prove our main theorem, we first establish the following general result.

Theorem 4.4. *Assume that* (H1) *and* (H2) *are satisfied. Assume also that there exists a constant $K > 0$ such that every solution x of $(P_{i,\lambda})$ with $i = 0, 1, 2$, $\lambda \in [0, 1]$, satisfies $\|x'\|_0 < K$. Then the problem* (1.1) *has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0, 1, 2$, $j = 1, 2$, $i \neq j$.*

Proof. For $i = 0, 1, 2$, let us define

$$h_i, \overline{h_i} : [0, 1] \times C^1(I, \mathbb{R}^n) \to C^1(I, \mathbb{R}^n)$$

by $h_i(\lambda, x) = l_\varepsilon \circ \eta_{v_i,\rho_i}(\lambda, x)$, and $\overline{h_i}(\lambda, x) = \lambda h_i(0, x)$ respectively. Proposition 4.3 implies that h_i, and hence $\overline{h_i}$, are continuous and completely continuous. Observe that $\overline{h_i}$ is bounded, so we can find an open bounded set $W \subset C^1(I, \mathbb{R}^n)$ such that $\overline{h_i}([0, 1] \times C^1(I, \mathbb{R}^n)) \subset W$. Therefore,

$$1 = \deg(id, W, 0) = \deg(id - \overline{h_i}(1, \cdot), W, 0) = \deg(id - h_i(0, \cdot), W, 0). \tag{4.1}$$

We denote

$$U_i = \{x \in C^1(I, \mathbb{R}^n) : \|x(t) - v_i(t)\| < \rho_i(t) + 1, \|x'(t)\| < K \ \forall t \in I\},$$
$$V_i = \{x \in C^1(I, \mathbb{R}^n) : \|x(t) - v_i(t)\| < \rho_i(t), \|x'(t)\| < K \ \forall t \in I\}.$$

Without lost of generality, we can assume that $U_i \subset W$. It follows from Proposition 4.2(a), the excision property, and the homotopic invariance of the degree, and (4.1) that

$$1 = \deg(id - h_i(1, \cdot), U_i, 0). \tag{4.2}$$

Also, for $i = 1, 2$, we have by Proposition 4.2(b), and the excision property of the degree that

$$1 = \deg(id - h_i(1, \cdot), V_i, 0). \tag{4.3}$$

On the other hand, since for $i = 1, 2$, $T(v_i, \rho_i) \subset T(v_0, \rho_0)$,

$$h_i(1, x) = h_0(1, x) \quad \text{for all } x \in \overline{V_i},$$

and hence,

$$1 = \deg(id - h_0(1, \cdot), V_i, 0) \quad \text{for } i = 1, 2. \tag{4.4}$$

Also, $V_1 \cup V_2 \subset U_0$ and $V_1 \cap V_2 = \emptyset$ since $T(v_1, \rho_1) \cap T(v_2, \rho_2) = \emptyset$, combining (4.2) and (4.4) leads to

$$\deg(id - h_0(1, \cdot), U_0 \backslash \overline{V_1 \cup V_2}, 0) = \deg(id - h_0(1, \cdot), U_0, 0) - \deg(id - h_0(1, \cdot), V_1, 0) - \deg(id - h_0(1, \cdot), V_2, 0) = -1.$$

Therefore, there exists $x_0 \in U_0\backslash\overline{V_1 \cup V_2}$, $x_1 \in V_1$ and $x_2 \in V_2$ solutions of $(P_{0,1})$, and hence of (1.1). □

With this general result, we can prove our main theorem.

Proof of Theorem 2.3. We have to show that there exists a constant K such that $\|x'\|_0 < K$ for every x solution of $(P_{i,\lambda})$ for $\lambda \in [0,1]$ and $i = 0,1,2$. Let x be a solution of $(P_{i,\lambda})$. From Proposition 4.2, we know that $x \in T(v_i, \rho_i)$. To simplify the notation, we write v, ρ instead of v_i, ρ_i respectively.

We have by (H3) that

$$\begin{aligned}|\langle x'(t), x''(t)\rangle| &= |\langle x'(t), f^{\lambda}_{v,\rho}(t, x(t), x'(t)) + \varepsilon x(t)\rangle| \\ &\le \lambda|\langle x'(t), f(t, x(t), x'(t))\rangle| \\ &\quad + (1-\lambda)\|x'(t)\|\big(\varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)|\big) \\ &\le \phi(\|x'(t)\|)\big(\gamma(t) + \|x'(t)\|\big) + \|x'(t)\|\big(\varepsilon\rho(t) + \|v''(t)\| + |\rho''(t)|\big).\end{aligned}$$

So, for

$$\gamma_0(t) = \gamma(t) + \max_{i=0,1,2}\{\varepsilon\rho_i(t) + \|v_i''(t)\| + |\rho_i''(t)|\},$$

$$|\langle x'(t), x''(t)\rangle| \le \big(\phi(\|x'(t)\|) + \|x'(t)\|\big)\big(\gamma_0(t) + \|x'(t)\|\big). \tag{4.5}$$

Now, to verify assumptions (i) and (ii) of Lemma 3.3, we consider two cases.

1st case: (H4) is satisfied. We have

$$\begin{aligned}\|x''(t)\| &= \|f^{\lambda}_{v,\rho}(t, x(t), x'(t)) + \varepsilon x(t)\| \\ &\le \lambda\|f(t, x(t), x'(t))\| + \varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)| \\ &\le \lambda a\Big(\langle x(t), f(t, x(t), x'(t))\rangle + \|x'(t)\|^2\Big) + l(t) \\ &\quad + \varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)| \\ &\le a\Big(\langle x(t), f^{\lambda}_{v,\rho}(t, x(t), x'(t)) + \varepsilon x(t)\rangle + \|x'(t)\|^2\Big) \\ &\quad - a(1-\lambda)\langle x(t), v''(t) + \Big(\frac{\rho''(t)}{\rho(t)} + \varepsilon\Big)(x(t) - v(t))\rangle \\ &\quad + \varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)| + l(t) \\ &\le a\Big(\langle x(t), x''(t)\rangle + \|x'(t)\|^2\Big) + z_0(t),\end{aligned}$$

with

$$z_0(t) = l(t) + \max_{i=0,1,2}\big\{(a\rho_i(t) + a\|v_i(t)\| + 1)(\varepsilon\rho_i(t) + \|v_i''(t)\| + \|\rho_i''(t)\|)\big\}.$$

The conclusion follows from (4.5), and Lemmas 3.3 and 3.5.

2nd case: (H5) is satisfied. Let σ_0 be the function introduced in Lemma 3.4. Observe that

$$\begin{aligned}
&\sigma_0(t,x)\\
&\quad= \frac{\langle x(t), x''(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), x''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\\
&\quad= \lambda\Big(\frac{\langle x(t), f(t,x(t),x'(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), f(t,x(t),x'(t))\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\\
&\qquad + (1-\lambda)\Big[\|x'(t)\| + \Big(\frac{\langle x(t), v''(t)\rangle}{\|x'(t)\|} - \frac{\langle x'(t), v''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\\
&\qquad + \Big(\varepsilon + \frac{\rho''(t)}{\rho(t)}\Big)\Big(\frac{\langle x(t), x(t)-v(t)\rangle}{\|x'(t)\|} - \frac{\langle x'(t), x(t)-v(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\Big]\\
&\quad\geq \lambda\Big(\frac{\langle x(t), f(t,x(t),x'(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), f(t,x(t),x'(t))\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\\
&\qquad + (1-\lambda)\|x'(t)\| - 2\frac{\|x(t)\|\big(\|v''(t)\| + \varepsilon\rho(t) + |\rho''(t)|\big)}{\|x'(t)\|}.
\end{aligned}$$

It follows from (H5) that on $\{t \in [0,1] : \|x'(t)\| \geq R\}$,

$$(b + c\|x(t)\|)\sigma_0(t,x) \geq \lambda(\|x'(t)\| - h(t)) + b(1-\lambda)\|x'(t)\| - \delta_0(t), \tag{4.6}$$

with

$$\delta_0(t) = \frac{2}{R}\big(b + c(\rho_0(t) + \|v_0(t)\|)\big)\big(\rho_0(t) + \|v_0(t)\|\big) \max_{i=0,1,2}\Big\{\varepsilon\rho_i(t) + |\rho_i''(t)| + \|v_i''(t)\|\Big\}.$$

The conclusion follows from (4.5), (4.6), and Lemmas 3.3 and 3.4. □

Other existence results can also be obtained. In particular, we can remove (H4) and (H5) if we impose a condition stronger than (H3) with the Sturm-Liouville boundary condition.

Theorem 4.5. *Let* BC *denote the Strum-Liouville boundary condition* (1.2) *with* $\max\{\beta_0, \beta_1\} > 0$. *Assume* (H1), (H2) *and*

(H6) there exist $\gamma \in L^1(I)$ and $\psi : [0,\infty[\to [1,\infty[$ a Borel measurable function such that $\|f(t,x,p)\| \leq \gamma(t)\psi(\|p\|)$ a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \leq \rho_0(t)$ for all $t \in [0,1]$, and
$$\int_k^\infty \frac{ds}{\psi(s)} = \infty \quad \forall k \geq 0.$$

Then the problem (1.1) has at least three solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0,1,2$, $j = 1,2$, $i \neq j$.

Proof. The conclusion will follows from Theorem 4.4 if we can show that there exists a constant $K > 0$ such that $\|x'\|_0 < K$ for every x solution of $(P_{i,\lambda})$.

Let x be a solution of $(P_{i,\lambda})$. We already know that $x \in T(v_i, \rho_i)$. Set

$$\mu = \min_{i=0,1,2}\Big\{\frac{\|r_0\| + a_0(\rho_i(0) + \|v_i(0)\|)}{\beta_0}, \frac{\|r_1\| + a_1(\rho_i(1) + \|v_i(1)\|)}{\beta_1}\Big\}$$

where $\|A_j z\| \le a_j\|z\|$ for all $z \in \mathbb{R}^n$, and $j = 0, 1$. Obviously

$$\min\{\|x'(0)\|, \|x'(1)\|\} \le \mu.$$

Assume that $\|x'\|_0 > \mu$. Let $t_1 \in [0,1]$ such that $\|x'(t_1)\| = \|x'\|_0$. There exists $t_0 < t_1$ such that $\|x'(t)\| > \mu$ on (t_0, t_1) and $\|x'(t_0)\| = \mu$. We have

$$\begin{aligned}
\frac{d}{dt}\|x'(t)\| &= \frac{\langle x'(t), x''(t)\rangle}{\|x'(t)\|} \\
&\le \|f(t, x(t), x'(t))\| + \Big(\varepsilon + \frac{|\rho''(t)|}{\rho(t)}\Big)\|x(t) - v(t)\| + \|v''(t)\| \\
&\le \gamma(t)\psi(\|x'(t)\|) + \varepsilon\rho(t) + |\rho''(t)| + \|v''(t)\| \\
&\le \gamma_0(t)\psi(\|x'(t)\|),
\end{aligned}$$

with $\gamma_0(t) = \gamma(t) + \max_{i=0,1,2}\{\varepsilon\rho_i(t) + |\rho_i''(t)| + \|v_i''(t)\|\}$. Therefore,

$$\|\gamma_0\|_{L^1} \ge \int_{t_0}^{t_1} \frac{\|x'(t)\|'}{\psi(\|x'(t)\|)}dt = \int_{\mu}^{\|x'(t_1)\|} \frac{ds}{\psi(s)}.$$

We get the conclusion choosing $K > \mu$ such that

$$\int_{\mu}^{K} \frac{ds}{\psi(s)} > \|\gamma_0\|_{L^1}.$$

□

5. Wintner-Nagumo growth condition

In this paragraph, we present a modification of our Main Theorem in which the Wintner-Nagumo condition (H3) is replaced by a more standard one. More precisely, we establish the following result.

Theorem 5.1. *Assume that* (H1), (H2), and (H4) or (H5) are satisfied. Assume also

(H7) there exist a Borel measurable function $\psi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I)$ such that

$$\|f(t, x, p)\| \le \psi(\|p\|)(\gamma(t) + \|p\|)$$

a.e. $t \in I$ and for all $(x, p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \le \rho_0(t)$, and

$$\int_k^{\infty} \frac{ds}{\psi(s)} = \infty \quad \forall k \ge 0.$$

Then the problem (1.1) has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0, 1, 2$, $j = 1, 2$, $i \ne j$.

To prove this result, we consider for $i = 0, 1, 2$, and $\lambda \in [0,1]$, the differential inclusion

$$(I_{i,\lambda}) \qquad \begin{aligned} &x''(t) - \varepsilon x(t) \in F^{\lambda}_{v_i,\rho_i}(t, x(t), x'(t)) \quad \text{a.e. } t \in [0,1], \\ &x \in \mathrm{BC}. \end{aligned}$$

where for $(v, \rho) \in W^{2,1}(I, \mathbb{R}^n) \times W^{2,1}(I, [0,\infty[)$, $F^{\lambda}_{v,\rho} : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is the multi-valued map defined by

$$F^{\lambda}_{v,\rho} = \lambda \bar{f}_{v,\rho} + G^{\lambda}_{v,\rho} - \varepsilon(1 - \lambda)v,$$

where $\bar{f}_{v,\rho}$ is defined in the previous section, and

$$G^\lambda_{v,\rho}(t,x,p) = \begin{cases} g^\lambda_{v,\rho}(t,x,p)\frac{(x-v(t))}{\|x-v(t)\|}, & \text{if } \|x-v(t)\| > \rho(t),\\ [0, g^\lambda_{v,\rho}(t,x,p)]\frac{(x-v(t))}{\|x-v(t)\|}, & \text{if } \|x-v(t)\| = \rho(t),\\ 0, & \text{if } \|x-v(t)\| < \rho(t); \end{cases}$$

with

$$g^\lambda_{v,\rho}(t,x,p) = \Bigg(\Big(1 - \frac{\lambda\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) + (1-\lambda)\Big(\frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^3} - \frac{\|p-v'(t)\|^2}{\|x-v(t)\|}\Big)\Bigg)^+.$$

Observe that $t \mapsto G^\lambda_{v,\rho}(t,x,p)$ is measurable for every $(x,p,\lambda) \in \mathbb{R}^n \times [0,1]$, and $(x,p,\lambda) \mapsto G^\lambda_{v,\rho}(t,x,p)$ is u.s.c. a.e. $t \in I$.

We associate to $F^\lambda_{v,\rho}$ a multi-valued operator

$$N_{v,\rho} : [0,1] \times C^1(I,\mathbb{R}^n) \to C_0(I,\mathbb{R}^n)$$

defined by

$$N_{v,\rho}(\lambda,x) = \{u \in C_0(I,\mathbb{R}^n) : \exists w \in L^1(I,\mathbb{R}^n) \text{ such that } w(t) \in F^\lambda_{v,\rho}(t,x(t),x'(t)) \text{ a.e. } t \in I, \text{ and } u(t) = \int_0^t w(s)\,ds\}.$$

This operator has some nice properties.

Proposition 5.2. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,[0,\infty[)$ a solution-tube of (1.1). Then the operator $N_{v,\rho}$ is completely continuous, u.s.c. with nonempty, compact, convex values.*

This result can be proved using the following lemma with [8, Lemma 2.7], and arguing as in [8, Lemmas 2.4] with $\mu : I \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$\mu(t,x,p) = \begin{cases} (\widetilde{x}_{v,\rho}, \widehat{p}_{v,\rho}) & \text{if } \|x-v(t)\| > \rho(t),\\ (x,p) & \text{otherwise,} \end{cases}$$

since we obtain a Carathéodory function if we replace in the definition of $\bar{f}_{v,\rho}$, $\widetilde{x}_{v,\rho}$ and $\widehat{p}_{v,\rho}$ respectively by x and p, and by Remark 4.1.

Lemma 5.3. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,[0,\infty[)$ a solution-tube of (1.1). Then the operator $G^\lambda_{v,\rho}$ is integrably bounded on bounded sets uniformly in λ, i.e. for every $k > 0$, there exists $d_k \in L^1[0,1]$ such that $G^\lambda_{v,\rho}(t,x,p) \subset B(0,d_k(t))$ for a.e. $t \in [0,1]$ and for all $(x,p) \in \mathbb{R}^{2n}$ with $\|x\| \leq k$, $\|p\| \leq k$.*

Proof. Definition 2.1 and Remark 4.1 imply that if $G^\lambda_{v,\rho}(t,x,p) \neq \{0\}$, then for every $z \in G^\lambda_{v,\rho}(t,x,p)$,

$$\begin{aligned}
\|z\| &\le \big|g^\lambda_{v,\rho}(t,x,p)\big| \\
&= \Big(1 - \frac{\lambda\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) \\
&\quad + (1-\lambda)\Big(\frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^3} - \frac{\|p-v'(t)\|^2}{\|x-v(t)\|}\Big) \\
&= \Big(1 - \frac{\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) \\
&\quad + \frac{1-\lambda}{\|x-v(t)\|}\Big[\rho(t)\rho''(t) + \frac{\rho(t)\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|} \\
&\quad + \frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^2} - \|p-v'(t)\|^2\Big] \\
&= \Big(1 - \frac{\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) \\
&\quad + \frac{1-\lambda}{\|x-v(t)\|}\Big(\rho(t)\rho''(t) + \langle \widetilde{x}_{v,\rho} - v(t), v''(t)\rangle + \rho'(t)^2 \\
&\quad - \|\widehat{p}_{v,\rho} - v'(t)\|^2\Big) \\
&\le |\rho''(t)| + \|v''(t)\| + \frac{1}{\|x-v(t)\|}\langle \widetilde{x}_{v,\rho} - v(t), f(t,\widetilde{x}_{v,\rho},\widehat{p}_{v,\rho})\rangle \\
&\le |\rho''(t)| + \|v''(t)\| + \|f(t,\widetilde{x}_{v,\rho},\widehat{p}_{v,\rho})\|.
\end{aligned}$$

The conclusion follows from the fact that f is a Carathéodory function and

$$\|\widetilde{x}_{v,\rho}\| \le \|v\|_0 + |\rho|_0 \quad \text{and} \quad \|\widehat{p}_{v,\rho}\| \le 2\|p\| + \|v'\|_0 + |\rho'|_0.$$

□

In order to prove our main theorem, we can first establish a general result as in the previous section.

Theorem 5.4. *Assume that* (H1) *and* (H2) *are satisfied. Assume that there exists a constant $K > 0$ such that every solution x of $(I_{i,\lambda})$ with $i = 0,1,2$, $\lambda \in [0,1]$, satisfies $\|x'\|_0 < K$. Then the problem* (1.1) *has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i,\rho_i)$, $x_i \notin T(v_j,\rho_j)$, $i = 0,1,2$, $j = 1,2$, $i \neq j$.*

The proof of this result is analogous to the proof of Theorem 4.4 where we replace the operators η_{v_i,ρ_i} by N_{v_i,ρ_i} and we use degree theory of compact, upper semi-continuous multi-valued maps with closed, convex values, and the following proposition.

Proposition 5.5. *Assume that* (H1) *and* (H2) *are satisfied. Then*

(a) *every solution x to $(I_{i,\lambda})$ is such that $x \in T(v_i,\rho_i)$ for $i = 0,1,2$, $\lambda \in [0,1]$;*
(b) *for $i = 1,2$, every solution x to $(I_{i,1})$ satisfies $\|x(t) - v_i(t)\| < \rho_i(t)$ for all $t \in [0,1]$.*

Proof. (a) Let $i \in \{0,1,2\}$, $\lambda \in [0,1]$, and let x be a solution of $(I_{i,\lambda})$. To simplify the notation, let v and ρ stand for v_i and ρ_i respectively, and $\tilde{x}(t)$ and $\hat{x}'(t)$ stand for $\widetilde{x(t)}_{v_i,\rho_i}$ and $\widehat{x'(t)}_{v_i,\rho_i}$. Then for almost every $t \in \{t \in I : \|x(t)-v(t)\| > \rho(t)\}$,

$$\begin{aligned}
&\frac{1}{\|x(t)-v(t)\|}\big(\langle x(t)-v(t), x''(t)-v''(t)\rangle + \|x'(t)-v'(t)\|^2\big)\\
&\qquad - \frac{1}{\|x(t)-v(t)\|^3}\langle x(t)-v(t), x'(t)-v'(t)\rangle^2 - \varepsilon\|x(t)-v(t)\|\\
&= \frac{\lambda}{\|x(t)-v(t)\|}\Big(\langle \tilde{x}_i(t)-v(t), f(t,\tilde{x}_i(t),\hat{x}_i'(t)) - v''(t)\rangle + \|\hat{x}_i'(t)-v'(t)\|^2\Big)\\
&\qquad - \varepsilon\lambda\rho(t) + \Big(\frac{\lambda\rho(t)}{\|x(t)-v(t)\|^2} - \frac{1}{\|x(t)-v(t)\|}\Big)\langle x(t)-v(t), v''(t)\rangle\\
&\qquad + \frac{\|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3} - \frac{\lambda\|\hat{x}_i'(t)-v'(t)\|^2}{\|x(t)-v(t)\|}\\
&\qquad + g^{\lambda}_{v,\rho}(t,x(t),x'(t))\\
&\geq \frac{\lambda}{\|x(t)-v(t)\|}\big(\rho(t)\rho''(t) + \rho'(t)^2\big) - \varepsilon\lambda\rho(t)\\
&\qquad + \Big(\frac{\lambda\rho(t)}{\|x(t)-v(t)\|^2} - \frac{1}{\|x(t)-v(t)\|}\Big)\langle x(t)-v(t), v''(t)\rangle\\
&\qquad + (1-\lambda)\Big(\frac{\|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3}\Big)\\
&\qquad - \frac{\lambda\rho'(t)^2}{\|x(t)-v(t)\|} + g^{\lambda}_{v,\rho}(t,x(t),x'(t))\\
&\geq \rho''(t) - \varepsilon\rho(t).
\end{aligned}$$

The conclusion (a) follows from Lemma 3.1.

(b) We argue as in the proof of Proposition 4.2 using Lemma 3.2. □

The following lemmas will lead to the existence of a priori bounds on the derivative of the solutions of $(I_{i,\lambda})$.

Lemma 5.6. *If* (H1), (H2), (H7) *are satisfied then every solution x of $(I_{i,\lambda})$ for $i = 0,1,2$ and $\lambda \in [0,1]$, satisfies*

$$\|x''(t)\| \leq 2\psi(\|x'(t)\|)(\gamma(t) + \|x'(t)\|) + \varepsilon\|\rho_0\|_0 \quad \textit{a.e. } t \in [0,1].$$

Proof. Let $i \in \{0,1,2\}$, $\lambda \in [0,1]$, and x a solution to $(I_{i,\lambda})$. By Proposition 5.5, we know that $x \in T(v_i,\rho_i)$. Again, to simplify the notation, we do not write the subscripts. We have that

$$\begin{aligned}
\|x''(t)\| \leq \|f(t,x(t),x'(t))\| + \varepsilon\|x(t)-v(t)\|\\
+ \begin{cases} g^{\lambda}_{v,\rho}(t,x(t),x'(t)), & \text{if } \|x(t)-v(t)\| = \rho(t),\\ 0, & \text{otherwise.}\end{cases}
\end{aligned} \tag{5.1}$$

Let us examine the third part of the right member of this equation. Since (v,ρ) is a solution-tube of (1.1) and by Remark 4.1, we have a.e. on $\{t \in [0,1] : \|x(t)-v(t)\| =$

$\rho(t),\ g^{\lambda}_{v,\rho}(t,x(t),x'(t))>0\}$,

$$\begin{aligned}
&g^{\lambda}_{v,\rho}(t,x(t),x'(t))\\
&\quad=(1-\lambda)\left(\rho''(t)+\frac{\langle x(t)-v(t),v''(t)\rangle+\rho'(t)^2-\|\widehat{x'(t)}-v'(t)\|^2}{\|x(t)-v(t)\|}\right)\\
&\quad\le\frac{\langle x(t)-v(t),f(t,x(t),\widehat{x'(t)})\rangle}{\|x(t)-v(t)\|}\\
&\quad\le\|f(t,x(t),\widehat{x'(t)})\|=\|f(t,x(t),x'(t))\|.
\end{aligned}$$

This inequality combined with (5.1) and (H7) leads to the conclusion. □

Lemma 5.7. *Under* (H1), (H2), (H5) *there exist* $b_0>0$, $c_0\ge 0$, *and* $\delta_0\in L^1[0,1]$ *such that every solution* x *of* $(I_{i,\lambda})$ *for* $i=0,1,2$, $\lambda\in[0,1]$, *satisfies a.e. on* $\{t\in[0,1]:\|x'(t)\|\ge R\}$

$$(b_0+c_0\|x(t)\|)\sigma_0(t,x)\ge\|x'(t)\|-\delta_0(t),$$

where R *is given in* (H5) *and* σ_0 *is defined in Lemma 3.4.*

Proof. Let x be a solution to $(I_{i,\lambda})$ for some $i\in\{0,1,2\}$, and $\lambda\in[0,1]$. Again to simplify the notation, we don't write the subscripts. From Proposition 5.5, $x\in T(v,\rho)$. Also, a.e. on $\{t\in[0,1]:\|x(t)-v(t)\|=\rho(t),\ g^{\lambda}_{v,\rho}(t,x(t),x'(t))\ne 0\}$,

$$\begin{aligned}
|g^{\lambda}_{v,\rho}(t,x(t),x'(t))|&=g^{\lambda}_{v,\rho}(t,x(t),x'(t))\\
&\le z_0(t):=\max_{i=0,1,2}\{|\rho_i''(t)|+\|v_i''(t)\|\}.
\end{aligned}\tag{5.2}$$

It follows that a.e. on $\{t\in[0,1]:\|x'(t)\|\ge R\}$,

$$\begin{aligned}
&(b+c\|x(t)\|)\sigma_0(t,x)\\
&\quad=\lambda(b+c\|x(t)\|)\sigma(t,x(t),x'(t))+(1-\lambda)(b+c\|x'(t)\|)\|x'(t)\|\\
&\quad\quad+(b+c\|x'(t)\|)\Big(\varepsilon(1-\lambda)+\frac{\theta(t)g^{\lambda}_{v,\rho}(t,x(t),x'(t))}{\|x(t)-v(t)\|}\Big)\Big(\frac{\langle x(t),x(t)-v(t)\rangle}{\|x'(t)\|}\\
&\quad\quad-\frac{\langle x'(t),x(t)-v(t)\rangle\langle x(t),x'(t)\rangle}{\|x'(t)\|^3}\Big),
\end{aligned}$$

where b,c are given in (H5) and

$$\theta(t)\in\begin{cases}[0,1], & \text{if } \|x(t)-v(t)\|=\rho(t),\\ \{0\}, & \text{otherwise.}\end{cases}$$

It follows from (H5) and (5.2) that

$$(b+c\|x(t)\|)\sigma_0(t,x)\ge(\lambda+b(1-\lambda))\|x'(t)\|-h(t)-2\Big(\frac{b}{R}+c\Big)\|x(t)\|(\varepsilon\rho(t)+z_0(t)).$$

Set $\nu=\min_{\lambda\in[0,1]}\{\lambda+b(1-\lambda)\}$. The conclusion follows choosing $b_0=b/\nu$ and $c_0=c/\nu$ and

$$\nu\delta_0(t)=h(t)+\frac{2}{R}\Big(b+c(\|v_0(t)\|+\rho_0(t))\Big)\Big(\|v_0(t)\|+\rho_0(t)\Big)\Big(\varepsilon\rho_0(t)+z_0(t)\Big).$$

□

Lemma 5.8. *Under* (H1), (H2), (H4) *there exists* $m_0 \in L^1[0,1]$ *such that every solution* x *of* $(I_{i,\lambda})$ *for* $i = 0, 1, 2$, $\lambda \in [0,1]$, *satisfies*

$$\|x''(t)\| \le a\big(\langle x(t), x''(t)\rangle + \|x'(t)\|^2\big) + m_0(t) \quad a.e.\ t \in [0,1],$$

where a *is given in* (H4).

Proof. Let x be a solution of $(I_{i,\lambda})$ for some $i \in \{0,1,2\}$ and $\lambda \in [0,1]$ which is in $T(v_i, \rho_i)$ by Proposition 5.5. As always, to simplify the notation, we don't write the subscript i. We have by (H4) and (5.2)

$$\begin{aligned}\|x''(t)\| &\le \lambda\|f(t,x(t),x'(t))\| + \varepsilon(1-\lambda)\|x(t) - v(t)\| + z_0(t)\\ &\le a\lambda\big(\langle x(t), f(t,x(t),x'(t))\rangle + \|x'(t)\|^2\big) + l(t) + \varepsilon\rho_0(t) + z_0(t)\\ &\le a\big(\langle x(t), x''(t))\rangle + \|x'(t)\|^2\big) + l(t) + \varepsilon\rho_0(t) + z_0(t)\\ &\qquad + a\|x(t)\|(\varepsilon\rho_0(t) + z_0(t));\end{aligned}$$

and the proof is complete. □

Now, we can prove the main theorem of this section.

Proof of Theorem 5.1. The conclusion is a direct consequence of Theorem 5.4, Lemmas 3.3–3.5, and 5.6–5.8. □

Arguing as in the proofs of Theorems 2.3 and 5.1, we can see that we can obtain the following more general results.

Theorem 5.9. *Assume that* (H1), (H2), and (H3) or (H7) are satisfied. Assume also

(H8) there exist $R > 0$, $b > 0$, $c \ge 0$, $h \in L^1(I)$, and $u \in W^{2,1}([0,1], \mathbb{R}^n)$ such that

$$(b + c\|x - u(t)\|)\sigma^u(t,x,p) + \frac{c\langle x - u(t), p - u'(t)\rangle^2}{\|x - u(t)\|\,\|p - u'(t)\|} \ge \|p - u'(t)\| - h(t),$$

for a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \le \rho_0(t)$, $\|p - u'(t)\| \ge R$, where

$$\sigma^u(t,x,p) = \frac{\langle x - u(t), f(t,x,p) - u''(t)\rangle + \|p - u'(t)\|^2}{\|p - u'(t)\|} - \frac{\langle p - u'(t), f(t,x,p) - u''(t)\rangle\langle x - u(t), p - u'(t)\rangle}{\|p - u'(t)\|^3}.$$

Then the problem (1.1) has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0, 1, 2$, $j = 1, 2$, $i \ne j$.

Theorem 5.10. *Assume that* (H1) and

(H9) there exist (v_0, ρ_0) a solution-tube of (1.1) and for $i = 1, \dots m$ with $m \ge 2$, there exists (v_i, ρ_i) a strict solution-tube of (1.1) such that $T(v_i, \rho_i) \cap T(v_j, \rho_j) = \emptyset$, and $T(v_i, \rho_i) \subset T(v_0, \rho_0)$, $i, j \in \{1, \dots, m\}$, $i \ne j$.

Assume also (H3) or (H7), and (H4) or (H5) or (H8) are satisfied. Then the problem (1.1) has at least $m+1$ distinct solutions $x_1, \dots, x_{m+1}$ such that $x_i \in T(v_i, \rho_i)$, $x_{m+1} \in T(v_0, \rho_0)$, $x_{m+1} \notin T(v_i, \rho_i)$, $i = 1, \dots, m$.

Idea of the proof. As before, for $i = 0, \dots, m$, we define appropriate maps h_i and families of problems $(P_{i,\lambda})$ (or $(I_{i,\lambda})$) such that solutions to $(P_{i,\lambda})$ are fixed points of $h_i(\lambda, \cdot)$. A priori bounds are obtained which permit to define open sets $U_0, V_1, \dots, V_m$ with $V_i \subset U_0$, $V_i \cap V_j = \emptyset$ for $i \neq j$,

$$h_i(1, x) = h_0(1, x) \quad \text{for all } x \in \overline{V_i}, \quad i = 1, \dots, m,$$

and

$$\begin{aligned} 1 &= \deg(id - h_0(1, \cdot), U_0, 0), \\ 1 &= \deg(id - h_i(1, \cdot), V_i, 0) = \deg(id - h_0(1, \cdot), V_i, 0), \quad i = 1, \dots, m. \end{aligned}$$

Thus,

$$\begin{aligned} \deg(id - h_0(1, \cdot), U_0 \backslash \overline{V_1 \cup \cdots \cup V_m}, 0) &= \deg(id - h_0(1, \cdot), U_0, 0) \\ - \deg(id - h_0(1, \cdot), V_1, 0) - \cdots - \deg(id - h_0(1, \cdot), V_m, 0) &= 1 - m \neq 0. \end{aligned}$$

So, there exists a solution in $U_0 \backslash \overline{V_1 \cup \cdots \cup V_m}$ and hence in

$$T(v_0, \rho_0) \backslash \overline{T(v_1, \rho_1) \cup \cdots \cup T(v_m, \rho_m)},$$

and for each $i = 1, \dots m$, there exists a solution in V_i and hence in $T(v_i, \rho_i)$. □

In the scalar case, we have the following result. Let us recall that u, w are respectively strict lower and upper solutions if $\big((u + w)/2, (w - u)/2\big)$ is a strict solution-tube of (1.1). See Henderson and Thompson [12] for a result with more general upper and lower solutions.

Corollary 5.11. *Let $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function. Assume that there exist $w_1, \dots, w_m$, m strict upper solutions, and $u_1, \dots, u_m$, m strict lower solutions such that*

(i) *$u_i \leq w_i$, for $i = 1, \dots, m$;*
(ii) *$u_1 \leq u_2 \leq \cdots u_m$, $w_1 \leq w_2 \leq \cdots w_m$;*
(iii) *$u_{i+1} \not\leq w_i$, $i = 1, \dots, m - 1$;*
(iv) *there exist a Borel measurable function $\psi : [0, \infty[\to]0, \infty[$ and $\gamma \in L^1(I)$ such that*

$$\|f(t, x, p)\| \leq \psi(\|p\|)(\gamma(t) + \|p\|)$$

a.e. $t \in I$ for all $(x, p) \in \mathbb{R}^{2n}$ such that $u_1(t) \leq x \leq w_m(t)$, and

$$\int_k^\infty \frac{ds}{\psi(s)} = \infty \quad \forall m > 0.$$

Then the problem (1.1) *has at least $m + 1$ distinct solutions $x_1, \dots, x_{m+1}$ such that $u_i \leq x_i \leq w_i$, $u_1 \leq x_{m+1} \leq w_m$, $x_{m+1} \notin [u_i, w_i]$, $i = 1, \dots, m$.*

Finally, we present an example.

Example 5.12. Consider the following system

$$\begin{aligned} &x'' = 5x - b(x, y)y - u(t, x, y, x', y')x' + v(t, x, y, x', y') - 5 \\ &y'' = 5y + b(x, y)x - u(t, x, y, x', y')y' + w(t, x, y, x', y') - 5 \\ &x(0) = x(1),\ y(0) = y(1),\ x'(0) = x'(1),\ y'(0) = y'(1), \end{aligned} \tag{5.3}$$

where u, v, w are bounded Carathéodory functions such that $u \geq 0$, $\|v(t, x, p, q)\| \leq 1$, $\|w(t, x, p, q)\| \leq 1$, and

$$b(x, y) = \frac{5\big((x^2 + y^2 - 2y + 3/4)^2 - (x^2 + y^2 - 2x + 3/4)^2\big)}{(x^2 + y^2 - 2y + 3/4)^2 + (x^2 + y^2 - 2x + 3/4)^2}.$$

We can find a constant $\rho_0 \geq 2$ big enough such that $\big((0,0), \rho_0\big)$ is a solution-tube of (5.3). Also, we define $\rho(t) = 1/2 + \delta(t - 1/2)^2$. We can verify that for $\delta > 0$ small enough, $\big((1,0), \rho\big)$ and $\big((0,1), \rho\big)$ are strict solution-tubes of (5.3), $T\big((1,0), \rho\big) \subset T\big((0,0), \rho_0\big)$, $T\big((0,1), \rho\big) \subset T\big((0,0), \rho_0\big)$, and $T\big((1,0), \rho\big) \cap T\big((0,1), \rho\big) = \emptyset$. Finally, it is easy to check that assumptions (H3) and (H5) are satisfied.

References

[1] V. Barutello, A. Capietto and P. Habets, *Existence and multiplicity of positive solutions for a Dirichlet boundary value problem in* $\mathbb{R}^2$ Adv. Nonlinear Stud. **2** (2002), 263–278.

[2] A. Capietto and F. Dalbono, *Multiplicity results for systems of asymptotically linear second order equatins*, Adv. Nonlinear Stud. **2** (2002), 325–356.

[3] A. Capietto and W. Dambrosio, *Multiplicity results for systems of superlinear second order equations*, J. Math. Anal. Appl. **248** (2000), 532–548.

[4] A. Capietto and W. Dambrosio, *A topological degree approach to sublinear systems of second order differential equations*, Discrete Contin. Dynam Systems **6** (2000), 861–874.

[5] C. DeCoster and P. Habets, *Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results*, Non-linear analysis and boundary value problems for ordinary differential equations (Udine), CISM Courses and Lectures, **371**, Springer-Verlag, Vienna, 1996, 1–78.

[6] M. Frigon, *Boundary and periodic value problems for systems of nonlinear second order differential equations*, Topol. Methods Nonlinear Anal. **1** (1993), 259–274.

[7] M. Frigon, *Boundary and periodic value problems for systems of differential equations under Bernstein-Nagumo growth condition*, Differential Integral Equations Nonlinear Anal. **8** (1995), 1789–1849.

[8] M. Frigon, *Théorèmes d'existence de solutions d'inclusions différentielles*, "Topological methods in differential equations and inclusions", NATO ASI Series, Ser. C: Math. and Phys. Sci., Kluwer, Dordrecht, 1995, 51-87.

[9] A. Granas, and J. Dugundji, "Fixed point theory", Springer-Verlag, New York, 2003.

[10] D. Guo and V. Lakshmikantham, "Nonlinear problems in abstract cones", Academic Press, San Diego, 1988.

[11] P. Hartman, *On boundary value problems for systems of ordinary, nonlinear, second order differential equations*, Trans. Amer. Math. Soc., **96** (1960), 493–509.

[12] J. Henderson and H.B. Thompson, *Existence of multiple solutions for second order boundary value problems*, J. Differential Equations **166** (2000), 443–454.

[13] M. Henrard, *Infinitely many solutions of weakly coupled superlinear systems*, Adv. Differential Equations **2** (1997), 753–778.

[14] E. Montoki, "Existence et multiplicité de solutions de systèmes d'équations et de systèmes d'inclusions différentielles avec opérateurs maximaux monotones", Ph.D. Thesis, Université de Montréal, Montréal, 2004.

M. Frigon and E. Montoki
Département de mathématiques et de statistique,
Université de Montréal,
C.P. 6128, succ. Centre-Ville,
Montréal, (Québec),
H3C 3J7, CANADA.
E-mail address: frigon@dms.umontreal.ca