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MULTIPLICITY RESULTS FOR SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS

M. FRIGON AND E. MONTOKI

Abstract. Multiplicity results are obtained for systems of second order differential equations with periodic or Sturm-Liouville boundary conditions. Results rely on the notion of strict solution-tube. Different growth conditions of Wintner-Nagumo type are considered.

1. Introduction

In this paper, we establish multiplicity results for the following system of second order differential equations:

$$(1.1)\qquad \begin{aligned} &x''(t) = f(t, x(t), x'(t)), \quad \text{a.e. } t \in [0,1],\\ &x \in \mathrm{BC}. \end{aligned}$$

Here $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a Carathéodory function, and BC denotes the Strum-Liouville or the periodic boundary conditions:

$$(1.2)\qquad \begin{aligned} A_0 x(0) - \beta_0 x'(0) = r_0,\\ A_1 x(1) + \beta_1 x'(1) = r_1; \end{aligned}$$

$$(1.3)\qquad \begin{aligned} x(0) = x(1),\\ x'(0) = x'(1); \end{aligned}$$

where for $i = 0, 1$, $\beta_i \in \{0, 1\}$, and A_i is a $n \times n$ matrix such that there exists $\alpha_i \geq 0$ satisfying $\langle A_i x, x\rangle \geq \alpha_i \|x\|^2$, and $\alpha_i + \beta_i > 0$.

In the literature, there are very few multiplicity results for systems ($n > 1$) of second order differential equations. Let us mention results of Barutello, Capietto and Habets [1] where systems of two second order differential equations are considered. Multiplicity results with prescribed nodal structure were obtained for systems of second order differential equations by Capietto with Dalbono [2] and Dambrosio [4] (see also [3]). Multiplicity results for systems of superlinear second order equations were also obtained in [13] via a continuation theorem and computation of the degree associated to some scalar equations.

In the particular case of a boundary value problem for a single differential equation of second order, more results were obtained. Our results extend to systems the results of Henderson and Thompson [12] for classical upper and lower solutions. Other multiplicity results for a single equation can be found for instance in [10].

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In order to obtain multiplicity results to (1.1), we will assume the existence of solution-tubes of (1.1). This notion was introduced in [6], and generalizes the notions of upper and lower solutions of a differential equation. Also, we introduce the notion of strict solution-tube. Different growth conditions of Wintner-Nagumo type will be imposed, and for which different arguments will be needed. Indeed, in section 4, using the Schauder degree theory, we will prove a multiplicity result (which slightly generalizes a result obtained in the Ph.D. thesis of the second author [14]) when f satisfies the growth condition:

> there exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I)$ such that

$$|\langle p, f(t,x,p)\rangle| \leq \phi(\|p\|)(\gamma(t)+\|p\|) \quad \text{and} \quad \int_k^\infty \frac{s ds}{\phi(s)+s} = \infty \quad \forall k \geq 0.$$

On the other hand, in section 5, we will establish multiplicity results in the case where f satisfies the more standard growth condition:

> there exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I)$ such that

$$\|f(t,x,p)\| \leq \phi(\|p\|)(\gamma(t)+\|p\|) \quad \text{and} \quad \int_k^\infty \frac{ds}{\phi(s)} = \infty \quad \forall k \geq 0.$$

The proof of these multiplicity results will rely on degree theory for multi-valued compact upper semi-continuous maps with closed, convex values.

It is well known that for systems of second order differential equations, a Nagumo type growth condition is not sufficient to guarantee the existence of a priori bounds on the derivative of solutions. Two different types of hypothesis will be considered. The first one (see (H4)) is the well known condition introduced by Hartman [11]. The second one (see (H5) or (H8)) is a generalization of a condition introduced by the first author in [7]. It has the advantage of being trivially satisfied in the scalar case.

In what follows, we will use the following notations: $I = [0,1]$, $C^k(I,\mathbb{R}^n)$ is the space of k-times continuously differentiable functions endowed with the usual norm that we denote $\|\cdot\|_k$; $C^k_B(I,\mathbb{R}^n)$ is the subset of x in $C^k(I,\mathbb{R}^n)$ satisfying the boundary condition BC; $C_0(I,\mathbb{R}^n) = \{x \in C(I,\mathbb{R}^n) : x(0) = x(1) = 0\}$; $L^1(I,\mathbb{R}^n)$ is the space of integrable functions with the usual norm $\|\cdot\|_{L^1}$; $W^{2,1}(I,\mathbb{R}^n)$ is the Sobolev space $\{x \in C^1[0,1] : x' \text{ is absolutely continuous}\}$ endowed with the norm $\|x\|_{2,1} = \|x\|_{L^1} + \|x'\|_{L^1} + \|x''\|_{L^1}$; $W^{2,1}_B(I,\mathbb{R}^n) = W^{2,1}(I,\mathbb{R}^n) \cap C_B(I,\mathbb{R}^n)$.

We will denote single-valued maps with lower case letters while we will use capital letters for multi-valued maps. Let X and Y be topological spaces, we say that a multi-valued map $F : X \to Y$ is upper semi-continuous (u.s.c.) (resp. if $X = [0,1]$, F is measurable) if $\{x \in X : F(x) \cap B \neq \emptyset\}$ is closed (resp. measurable) for every closed set B of Y. We say that $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ a single-valued map (resp. $F : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ a multi-valued map with closed convex values) is Carathéodory if $t \mapsto f(t,x,p)$ (resp. $t \mapsto F(t,x,p)$) is measurable for all x, p; $(x,p) \mapsto f(t,x,p)$ is continuous (resp. $(x,p) \mapsto F(t,x,p)$ is u.s.c.) for a.e. $t \in I$; for every $k > 0$, there exists $d_k \in L^1(I)$ such that $f(t,B(0,k),B(0,k)) \subset B(0,d_k(t))$ (resp. $F(t,B(0,k),B(0,k)) \subset B(0,d_k(t))$) a.e. $t \in I$, where $B(0,r)$ is the closed ball of radius r centered at the origin.

2. Main Theorem

The notion of solution-tube will play a fundamental role in our main result. This notion was introduced in [6] and generalizes naturally to systems the well known notion of upper and lower solutions.

Definition 2.1. Let $v \in W^{2,1}(I, \mathbb{R}^n)$, and $\rho \in W^{2,1}(I, \mathbb{R})$. We say that (v, ρ) is a *solution-tube of* (1.1) if

(i) for a.e. $t \in I$, and every $(x, p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| = \rho(t)$ and $\langle x - v(t), p - v'(t)\rangle = \rho(t)\rho'(t)$,
$$\langle x - v(t), f(t, x, p) - v''(t)\rangle + \|p - v'(t)\|^2 \geq \rho(t)\rho''(t) + \rho'(t)^2;$$
(ii) $v''(t) = f(t, v(t), v'(t))$ a.e. on $\{t \in [0, 1] : \rho(t) = 0\}$;
(iii) if BC denotes (1.2),
$$\|A_0 v(0) - \beta_0 v'(0) - r_0\| \leq \alpha_0 \rho(0) - \beta_0 \rho'(0),$$
$$\|A_1 v(1) + \beta_1 v'(1) - r_1\| \leq \alpha_1 \rho(1) + \beta_1 \rho'(1);$$
and if BC denotes (1.3),
$$\rho(0) = \rho(1),\ v(0) = v(1),\ \|v'(0) - v'(1)\| \leq \rho'(1) - \rho'(0).$$

We denote
$$T(v, \rho) = \{x \in C(I, \mathbb{R}^n) : \|x(t) - v(t)\| \leq \rho(t)\ \forall t \in I\}.$$

Our goal is to establish multiplicity results for the system of second order differential equation (1.1). To this aim, we introduce the notion of strict solution-tube.

Definition 2.2. Let $(v, \rho) \in W^{2,1}(I, \mathbb{R}^n) \times W^{2,1}(I, \mathbb{R})$. We say that (v, ρ) is a *strict solution-tube of* (1.1) if

(i) $\rho(t) > 0$ for all $t \in I$;
(ii) for every $t \in I$, there exist $\varepsilon > 0$ and V a neighborhood of t such that
$$\langle x - v(t), f(t, x, p) - v''(t)\rangle + \|p - v'(t)\|^2 - \rho(t)\rho''(t) - \rho'(t)^2 \geq 0$$
a.e. $t \in V$ and for every $(x, p) \in S_{t,\varepsilon}$, where
$$S_{t,\varepsilon} = \{(x, p) \in \mathbb{R}^{2n} : \rho(t) - \varepsilon \leq \|x - v(t)\| \leq \rho(t),$$
$$|\langle x - v(t), p - v'(t)\rangle - \rho(t)\rho'(t)| \leq \varepsilon\};$$
(iii) if BC denotes (1.2),
$$\|A_0 v(0) - \beta_0 v'(0) - r_0\| < \alpha_0 \rho(0) - \beta_0 \rho'(0),$$
$$\|A_1 v(1) + \beta_1 v'(1) - r_1\| < \alpha_1 \rho(1) + \beta_1 \rho'(1);$$
and if BC denotes (1.3),
$$\rho(0) = \rho(1),\ v(0) = v(1),\ \|v'(0) - v'(1)\| < \rho'(1) - \rho'(0).$$

In the particular case where $n = 1$, if (v, ρ) is a strict solution-tube of (1.1) then $v - \rho$ (resp. $v + \rho$) is a strict lower (resp. upper) solution of (1.1), see [5].

We will consider the following assumptions:

(H1) $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is Carathéodory;
(H2) there exist (v_0, ρ_0) a solution-tube of (1.1) and $(v_1, \rho_1), (v_2, \rho_2)$ two strict solution-tubes of (1.1) such that $T(v_1, \rho_1) \cap T(v_2, \rho_2) = \emptyset$, and $T(v_i, \rho_i) \subset T(v_0, \rho_0)$, $i = 1, 2$;

(H3) there exist a Borel measurable function $\phi : [0,\infty[\to]0,\infty[$ and $\gamma \in L^1(I)$ such that
$$|\langle p, f(t,x,p)\rangle| \le \phi(\|p\|)(\gamma(t) + \|p\|)$$
a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \le \rho_0(t)$, and
$$\int_k^\infty \frac{sds}{\phi(s)+s} = \infty \quad \forall k \ge 0;$$
(H4) there exist $a \ge 0$ and $l \in L^1(I)$ such that
$$\|f(t,x,p)\| \le a(\langle x, f(t,x,p)\rangle + \|p\|^2) + l(t)$$
a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \le \rho_0(t)$;

(H5) there exist $R > 0$, $b > 0$, $c \ge 0$, and $h \in L^1(I)$ such that
$$(b + c\|x\|)\sigma(t,x,p) \ge \|p\| - h(t),$$
for a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \le \rho_0(t)$, $\|p\| \ge R$, where
$$\sigma(t,x,p) = \frac{\langle x, f(t,x,p)\rangle + \|p\|^2}{\|p\|} - \frac{\langle p, f(t,x,p)\rangle\langle x,p\rangle}{\|p\|^3}.$$

Observe that (H5) is trivially satisfied in the scalar case i.e. when $n = 1$.

Now, we can state our main result.

Theorem 2.3 (Main Theorem). *Assume that* (H1)–(H3), and (H4) or (H5) are satisfied then the problem (1.1) has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i,\rho_i)$, $x_i \notin T(v_j,\rho_j)$, $i = 0,1,2$, $j = 1,2$, $i \ne j$.

3. Auxiliary results

As in the method of upper and lower solutions, the assumptions on the existence of solution-tubes will permit to obtain a priori bounds on the solutions. First of all, we recall a result of [7] (see [7, Lemma 3.2]).

Lemma 3.1. *Let $\varepsilon \ge 0$, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,\mathbb{R})$ a solution-tube of* (1.1)*. If $x \in W^{2,1}_B(I,\mathbb{R}^n)$ satisfies*

$$\begin{aligned}(3.1)\quad &\frac{\langle x(t)-v(t), x''(t)-v''(t)\rangle + \|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} \\ &\quad - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3} - \varepsilon\|x(t)-v(t)\| \ge \rho''(t) - \varepsilon\rho(t)\end{aligned}$$

for a.e. $t \in \{t \in I : \|x(t)-v(t)\| > \rho(t)\}$, then $x \in T(v,\rho)$.

Next, we want to show that if x is a solution of (1.1) lying in $T(v,\rho)$ for (v,ρ) a strict solution-tube of (1.1), then x belongs to the interior of $T(v,\rho)$.

Lemma 3.2. *Let $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n)\times W^{2,1}(I,\mathbb{R})$ be a strict solution-tube of* (1.1)*. If $x \in W^{2,1}_B(I,\mathbb{R}^n) \cap T(v,\rho)$ is a solution of* (1.1)*, then $\|x(t)-v(t)\| < \rho(t)$ for all $t \in I$.*

Proof. We claim that
$$(3.2)\qquad \|x(0)-v(0)\| < \rho(0) \quad \text{and} \quad \|x(1)-v(1)\| < \rho(1).$$

Indeed, if BC denotes (1.2) and if $\|x(0) - v(0)\| = \rho(0)$, then by Definition 2.2 and since $\|x(t) - v(t)\| \le \rho(t)$ for all t,

$$\begin{aligned}
0 &\ge \frac{\beta_0}{2}\frac{d}{dt}(\|x(t) - v(t)\|^2 - \rho(t)^2)|_{t=0} \\
&= \beta_0\big(\langle x(0) - v(0), x'(0) - v'(0)\rangle - \rho(0)\rho'(0)\big) \\
&> \beta_0\langle x(0) - v(0), x'(0) - v'(0)\rangle + \rho(0)\|A_0v(0) - \beta_0v'(0) - r_0\| - \alpha_0\rho(0)^2 \\
&\ge \beta_0\langle x(0) - v(0), x'(0) - v'(0)\rangle - \langle x(0) - v(0), A_0v(0) - \beta_0v'(0) - r_0\rangle \\
&\quad - \alpha_0\|x(0) - v(0)\|^2 \\
&= \langle A_0(x(0) - v(0)), x(0) - v(0)\rangle - \alpha_0\|x(0) - v(0)\|^2 \\
&\ge 0,
\end{aligned}$$

contradiction. Similarly, we show that $\|x(1) - v(1)\| < \rho(1)$.

If BC denotes (1.3), then

$$\|x(0) - v(0)\| < \rho(0) \quad \text{if and only if} \quad \|x(1) - v(1)\| < \rho(1).$$

Therefore, if the claim is false, by Definition 2.2, $\|x(0) - v(0)\| = \rho(0) = \rho(1) = \|x(1) - v(1)\|$, and

$$\begin{aligned}
0 &\le \frac{1}{2}\Big(\frac{d}{dt}(\|x(t) - v(t)\|^2 - \rho(t)^2)|_{t=1} - \frac{d}{dt}(\|x(t) - v(t)\|^2 - \rho(t)^2)|_{t=0}\Big) \\
&= \langle x(1) - v(1), x'(1) - v'(1)\rangle - \langle x(0) - v(0), x'(0) - v'(0)\rangle \\
&\quad - \rho(1)\rho'(1) + \rho(0)\rho'(0) \\
&= \langle x(1) - v(1), v'(0) - v'(1)\rangle - \rho(1)\big(\rho'(1) - \rho'(0)\big) \\
&< \rho(1)\big(\rho'(1) - \rho'(0)\big) - \rho(1)\big(\rho'(1) - \rho'(0)\big);
\end{aligned}$$

which is a contradiction.

Now assume that $E = \{t \in]0, 1[\, : \|x(t) - v(t)\| = \rho(t)\} \ne \emptyset$. Let $t_1 = \inf E$. By (3.2), $t_1 \in]0, 1[$ and $\langle x(t_1) - v(t_1), x'(t_1) - v'(t_1)\rangle = \rho(t_1)\rho'(t_1)$. Definition 2.2 implies that there exist $\varepsilon > 0$ and V a neighborhood of t_1 such that

$$\langle y - v(t), f(t, y, p) - v''(t)\rangle + \|p - v'(t)\|^2 - \rho(t)\rho''(t) + \rho'(t)^2 \ge 0$$

a.e. $t \in V$ and for every $(y, p) \in S_{t_1,\varepsilon}$ which is defined in Definition 2.2. Since $x \in C^1$, we can find $O \subset V$ a neighborhood of t_1 such that $(x(t), x'(t)) \in S_{t_1,\varepsilon}$ for all $t \in O$. On the other hand, by definition of t_1, we can choose $t_0 \in O \cap [0, t_1[$ such that

$$\frac{d}{dt}(\|x(t) - v(t)\|^2 - \rho(t)^2)|_{t=t_0} > 0.$$

Therefore,

$$\begin{aligned}
0 &> \frac{1}{2}\int_{t_0}^{t_1} \frac{d^2}{dt^2}(\|x(t) - v(t)\|^2 - \rho(t)^2)\, dt \\
&= \int_{t_0}^{t_1} \big(\langle x(t) - v(t), f(t, x(t), x'(t)) - v''(t)\rangle + \|x'(t) - v'(t)\|^2 \\
&\qquad - \rho(t)\rho''(t) - \rho'(t)^2\big)\, dt \\
&\ge 0,
\end{aligned}$$

and the proof is complete. □

In order to obtain a priori bounds on the derivative of the solutions, we recall the following results of [7]. The first one concerns a priori bounds that can be obtained under a Wintner-Nagumo growth condition (see [7, Lemma 3.4]).

Lemma 3.3. *Let $r, k \geq 0$, $m \in L^1(I)$, and $\psi : [0,\infty[\to]0,\infty[$ a Borel measurable function such that*

$$\int_r^\infty \frac{sds}{\psi(s)} > \|m\|_{L^1} + k.$$

Then there exists $K > 0$ such that $\|x'\|_0 < K$ for every $x \in W^{2,1}(I,\mathbb{R}^n)$ satisfying

(i) $\min_{t\in I} \|x'(t)\| \leq r$;
(ii) $\|x'\|_{L^1[t_0,t_1]} \leq k$ *for every* $[t_0,t_1] \subset \{t \in I : \|x'(t)\| \geq r\}$;
(iii) $|\langle x'(t), x''(t)\rangle| \leq \psi(\|x'(t)\|)(m(t) + \|x'(t)\|)$ *a.e. on* $\{t \in I : \|x'(t)\| \geq r\}$.

The previous result shows that to obtain an a priori bound of the derivative x' with respect to norm of the uniform convergence, we need to obtain an a priori bound of x' in the L^1-norm.

The following result generalizes and simplifies [7, Lemma 3.3].

Lemma 3.4. *Let $u \in W^{2,1}(I,\mathbb{R}^n)$, $\xi > 0$, $\zeta \geq 0$, $r > 0$, $m \in L^1(I)$. Then there exists $\omega : [0,\infty[\to [0,\infty[$ an increasing function such that we have for any interval $[t_0,t_1]$ on which $\|x'(t) - u'(t)\| \geq r$,*

$$\|x' - u'\|_{L^1[t_0,t_1]} \leq \omega(\|x - u\|_0),$$

and

$$\min_{t\in I} \|x'(t) - u'(t)\| \leq \max\{r, \omega(\|x - u\|_0)\}.$$

for every $x \in W^{2,1}(I,\mathbb{R}^n)$ satisfying almost everywhere on $\{t \in I : \|x'(t) - u'(t)\| \geq r\}$,

$$(\xi + \zeta\|x(t) - u(t)\|)\sigma_u(t,x) + \frac{\zeta\langle x(t) - u(t), x'(t) - u'(t)\rangle^2}{\|x(t) - u(t)\|\,\|x'(t) - u'(t)\|} \geq \|x'(t) - u'(t)\| - m(t),$$

where

$$\sigma_u(t,x) = \frac{\langle x(t) - u(t), x''(t) - u''(t)\rangle + \|x'(t) - u'(t)\|^2}{\|x'(t) - u'(t)\|} - \frac{\langle x'(t) - u'(t), x''(t) - u''(t)\rangle\langle x(t) - u(t), x'(t) - u'(t)\rangle}{\|x'(t) - u'(t)\|^3}.$$

Proof. First of all, observe that

$$\frac{d}{dt}\frac{(\xi + \zeta\|x(t) - u(t)\|)\langle x(t) - u(t), x'(t) - u'(t)\rangle}{\|x'(t) - u'(t)\|} = (\xi + \zeta\|x(t) - u(t)\|)\sigma_u(t,x) + \frac{\zeta\langle x(t) - u(t), x'(t) - u'(t)\rangle^2}{\|x(t) - u(t)\|\,\|x'(t) - u'(t)\|}. \tag{3.3}$$

Assume that $\|x'(t) - u'(t)\| \geq r$ on $[t_0, t_1]$ then the assumptions and (3.3) yield

$$\begin{aligned}
&\int_{t_0}^{t_1} \|x'(t) - u'(t)\|\, dt \\
&\quad \leq \|m\|_{L^1[0,1]} + \int_{t_0}^{t_1} \frac{d}{dt} \frac{(\xi + \zeta\|x(t) - u(t)\|)\langle x(t) - u(t), x'(t) - u'(t)\rangle}{\|x'(t) - u'(t)\|}\, dt \\
&\quad \leq \|m\|_{L^1[0,1]} + 2(\xi + \zeta\|x - u\|_0)\|x - u\|_0 \\
&\quad = \omega(\|x - u\|_0).
\end{aligned}$$

□

An a priori bound of x' in L^1-norm can also be obtained using a condition introduced by Hartman [11].

Lemma 3.5. *Let $k \geq 0$ and $m \in L^1(I)$. Then there exists $\omega : [0, \infty[\to]0, \infty[$ an increasing function such that for every $x \in W^{2,1}(I, \mathbb{R}^n)$ satisfying*

$$\|x''(t)\| \leq k\big(\langle x(t), x''(t)\rangle + \|x'(t)\|^2\big) + m(t) \quad a.e.\ t \in I,$$

we have $\|x'\|_{L^1} \leq \omega(\|x\|_0)$.

Proof. Let $x \in W^{2,1}(I, \mathbb{R}^n)$. Observe that

$$\frac{x'(t)}{2} = x(t + \frac{1}{2}) - x(t) - \int_t^{t+\frac{1}{2}} (t + \frac{1}{2} - s)x''(s)\, ds \quad \text{for } 0 \leq t \leq \frac{1}{2}. \tag{3.4}$$

So, for $t \in [0, 1/2]$,

$$\begin{aligned}
\frac{\|x'(t)\|}{2} &\leq 2\|x\|_0 + \|m\|_{L^1} + k\int_t^{t+\frac{1}{2}} (t + \frac{1}{2} - s)\big(\langle x(s), x''(s)\rangle + \|x'(s)\|^2\big)\, ds \\
&= 2\|x\|_0 + \|m\|_{L^1} + \frac{k}{2}\int_t^{t+\frac{1}{2}} (t + \frac{1}{2} - s)\frac{d^2}{ds^2}\|x(s)\|^2\, ds.
\end{aligned}$$

Using (3.4) with $\|x\|^2$, we obtain

$$\frac{\|x'(t)\|}{2} \leq 2\|x\|_0 + \|m\|_{L^1} + k\|x\|_0^2 - \frac{k}{2}\langle x(t), x'(t)\rangle \quad \text{for } 0 \leq t \leq \frac{1}{2}. \tag{3.5}$$

Similarly, using the identity

$$\frac{x'(t)}{2} = x(t) - x(t - \frac{1}{2}) - \int_{t-\frac{1}{2}}^{t} (t - \frac{1}{2} - s)x''(s)\, ds \quad \text{for } \frac{1}{2} \leq t \leq 1, \tag{3.6}$$

we obtain

$$\frac{\|x'(t)\|}{2} \leq 2\|x\|_0 + \|m\|_{L^1} + k\|x\|_0^2 + \frac{k}{2}\langle x(t), x'(t)\rangle \quad \text{for } \frac{1}{2} \leq t \leq 1. \tag{3.7}$$

Combining (3.5) and (3.7), we deduce that

$$\begin{aligned}\|x'\|_{L^1} &= \int_0^{1/2} \|x'(t)\|\,dt + \int_{1/2}^1 \|x'(t)\|\,dt\\ &\le 4\|x\|_0 + 2\|m\|_{L^1} + 2k\|x\|_0^2 - \frac{k}{2}\int_0^{1/2}\frac{d}{dt}\|x(t)\|^2\,dt\\ &\quad + \frac{k}{2}\int_{1/2}^1 \frac{d}{dt}\|x(t)\|^2\,dt\\ &\le 4\|x\|_0 + 2\|m\|_{L^1} + 2k\|x\|_0^2 + \frac{k}{2}\big(\|x(1)\|^2 + \|x(0)\|^2 - 2\|x(1/2)\|^2\big)\\ &\le 4\|x\|_0 + 2\|m\|_{L^1} + 3k\|x\|_0^2.\end{aligned}$$

The conclusion follows if we define $\omega(r) = 3kr^2 + 4r + 2\|m\|_{L^1}$. □

4. Proof of the Main Theorem

Fix $\varepsilon \ge 0$ such that the operator $l_\varepsilon : C^1_B(I,\mathbb{R}^n) \to C_0(I,\mathbb{R}^n)$ defined by

$$l_\varepsilon(x)(t) = x'(t) - x'(0) - \varepsilon\int_0^t x(s)\,ds$$

is invertible.

For $\lambda \in [0,1]$, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,[0,\infty[)$, we define

$$f^\lambda_{v,\rho} \;:\; I \times \mathbb{R}^{2n} \to \mathbb{R}^n \quad \text{by} \quad f^\lambda_{v,\rho} = \lambda \bar f_{v,\rho} + g^\lambda_{v,\rho} - \varepsilon(1-\lambda)v,$$

where

$$\bar f_{v,\rho}(t,x,p) = \begin{cases} \frac{\rho(t)}{\|x-v(t)\|} f(t,\widetilde{x}_{v,\rho},\widehat{p}_{v,\rho}) - \varepsilon\widetilde{x}_{v,\rho}, & \text{if } \|x-v(t)\| > \rho(t),\\ f(t,x,p) - \varepsilon x, & \text{if } \|x - v(t)\| \le \rho(t)\end{cases}$$

$$g^\lambda_{v,\rho}(t,x,p) = \begin{cases} \Big(1 - \frac{\lambda\rho(t)}{\|x-v(t)\|}\Big)\Big(v''(t) + \frac{\rho''(t)}{\|x-v(t)\|}(x - v(t))\Big), & \text{if } \|x - v(t)\| > \rho(t),\\ (1-\lambda)\Big(v''(t) + \frac{\rho''(t)}{\rho(t)}(x-v(t))\Big), & \text{otherwise;}\end{cases}$$

with

$$\widetilde{x}_{v,\rho} = v(t) + \frac{\rho(t)}{\|x - v(t)\|}(x - v(t)),$$

$$\widehat{p}_{v,\rho} = p + \Big(\rho'(t) - \frac{\langle x - v(t), p - v'(t)\rangle}{\|x - v(t)\|}\Big)\Big(\frac{x - v(t)}{\|x-v(t)\|}\Big),$$

and where we mean $\rho''(t)(x - v(t))/\rho(t) = 0$ on $\{t \in [0,1] : \rho(t) = 0\}$.

Remark 4.1. For $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| > 0$,

$$\|\widetilde{x}_{v,\rho} - v(t)\| = \rho(t), \quad \langle \widetilde{x}_{v,\rho} - v(t), \widehat{p}_{v,\rho} - v'(t)\rangle = \rho(t)\rho'(t),$$

$$\|\widehat{p}_{v,\rho} - v'(t)\|^2 = \|p - v'(t)\|^2 + (\rho'(t))^2 - \frac{\langle x(t) - v(t), p - v'(t)\rangle^2}{\|x(t) - v(t)\|^2};$$

and for $x \in C^1(I,\mathbb{R}^n)$,

$$(\widetilde{x(t)}_{v,\rho}, \widehat{x'(t)}_{v,\rho}) = (x(t), x'(t)) \quad \text{a.e. on } \{t \in I : \|x(t) - v(t)\| = \rho(t) > 0\}.$$

We consider for $i = 0, 1, 2$, and $\lambda \in [0, 1]$, the problem

$$(P_{i,\lambda}) \qquad \begin{aligned} &x''(t) - \varepsilon x(t) = f^{\lambda}_{v_i,\rho_i}(t, x(t), x'(t)) \quad \text{a.e. } t \in [0, 1],\\ &x \in \mathrm{BC}. \end{aligned}$$

A priori bounds can be obtained for the solutions of $(P_{i,\lambda})$.

Proposition 4.2. *Assume that* (H1) *and* (H2) *are satisfied. Then*

(a) *every solution x of $(P_{i,\lambda})$ with $i = 0, 1, 2$, $\lambda \in [0, 1]$, is such that $x \in T(v_i, \rho_i)$;*
(b) *every solution x of $(P_{i,1})$ for $i = 1, 2$, satisfies $\|x(t) - v_i(t)\| < \rho_i(t)$ for all $t \in [0, 1]$.*

Proof. (a) For $i \in \{0, 1, 2\}$ and $\lambda \in [0, 1]$, let x be a solution of $(P_{i,\lambda})$. To simplify the notation, let v and ρ stand for v_i and ρ_i respectively, and $\tilde{x}(t)$ and $\hat{x}'(t)$ stand for $\widetilde{x(t)}_{v_i,\rho_i}$ and $\widehat{x'(t)}_{v_i,\rho_i}$. Then for almost every $t \in \{t \in I : \|x(t) - v(t)\| > \rho(t)\}$,

$$\begin{aligned}
&\frac{1}{\|x(t) - v(t)\|}\big(\langle x(t) - v(t), x''(t) - v''(t)\rangle + \|x'(t) - v'(t)\|^2\big)\\
&\quad - \frac{1}{\|x(t) - v(t)\|^3}\langle x(t) - v(t), x'(t) - v'(t)\rangle^2 - \varepsilon\|x(t) - v(t)\|\\
&= \frac{1}{\|x(t) - v(t)\|}\Big(\Big\langle x(t) - v(t), \frac{\lambda\rho(t)}{\|x(t) - v(t)\|}(f(t, \tilde{x}(t), \hat{x}'(t)) - v''(t)\Big\rangle\\
&\quad - \varepsilon\big\langle x(t) - v(t), \lambda\tilde{x}(t) + (1 - \lambda)v(t) - x(t)\big\rangle\Big)\\
&\quad + \Big(1 - \frac{\lambda\rho(t)}{\|x(t) - v(t)\|}\Big)\rho''(t)\\
&\quad + \frac{\|x'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} - \varepsilon\|x(t) - v(t)\|\\
&= \frac{\lambda}{\|x(t) - v(t)\|}\Big(\langle\tilde{x} - v(t), f(t, \tilde{x}(t), \hat{x}'(t)) - v''(t)\rangle + \|\hat{x}'(t) - v'(t)\|^2\Big)\\
&\quad + \frac{1}{\|x(t) - v(t)\|}\Big(\|x'(t) - v'(t)\|^2 - \lambda\|\hat{x}'(t) - v'(t)\|^2\Big)\\
&\quad - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} + \Big(1 - \frac{\lambda\rho(t)}{\|x(t) - v(t)\|}\Big)\rho''(t) - \varepsilon\lambda\rho(t)\\
&\geq \frac{\lambda}{\|x(t) - v(t)\|}\big(\rho(t)\rho''(t) + \rho'(t)^2\big) + \Big(1 - \frac{\lambda\rho(t)}{\|x(t) - v(t)\|}\Big)\rho''(t)\\
&\quad + \frac{1}{\|x(t) - v(t)\|}\Big((1 - \lambda)\|\hat{x}'(t) - v'(t)\|^2 - \rho'(t)^2\Big) - \varepsilon\lambda\rho(t)\\
&\geq \rho''(t) - \varepsilon\rho(t).
\end{aligned}$$

It follows from Lemma 3.1 that statement (a) holds.

(b) Observe that if x is a solution of $(P_{i,1})$ for some $i \in \{1, 2\}$, then by (a), x is a solution of (1.1). Lemma 3.2 yields to the conclusion. □

We associate to $f^{\lambda}_{v,\rho}$ an operator

$$\eta_{v,\rho} : [0, 1] \times C^1(I, \mathbb{R}^n) \to C_0(I, \mathbb{R}^n)$$

defined by

$$\eta_{v,\rho}(\lambda, x) = \int_0^t f^\lambda_{v,\rho}(s, x(s), x'(s))\, ds.$$

The following result establishes some properties of $\eta_{v,\rho}$. The proof of this result is a direct consequence of [7, Proposition 3.5].

Proposition 4.3. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function, and $(v, \rho) \in W^{2,1}(I, \mathbb{R}^n) \times W^{2,1}(I, [0, \infty[)$ a solution-tube of* (1.1)*. Then the operator $\eta_{v,\rho}$ is continuous and completely continuous.*

In order to prove our main theorem, we first establish the following general result.

Theorem 4.4. *Assume that* (H1) *and* (H2) *are satisfied. Assume also that there exists a constant $K > 0$ such that every solution x of $(P_{i,\lambda})$ with $i = 0, 1, 2$, $\lambda \in [0, 1]$, satisfies $\|x'\|_0 < K$. Then the problem* (1.1) *has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0, 1, 2$, $j = 1, 2$, $i \neq j$.*

Proof. For $i = 0, 1, 2$, let us define

$$h_i, \overline{h_i} : [0, 1] \times C^1(I, \mathbb{R}^n) \to C^1(I, \mathbb{R}^n)$$

by $h_i(\lambda, x) = l_\varepsilon \circ \eta_{v_i,\rho_i}(\lambda, x)$, and $\overline{h_i}(\lambda, x) = \lambda h_i(0, x)$ respectively. Proposition 4.3 implies that h_i, and hence $\overline{h_i}$, are continuous and completely continuous. Observe that $\overline{h_i}$ is bounded, so we can find an open bounded set $W \subset C^1(I, \mathbb{R}^n)$ such that $\overline{h_i}([0, 1] \times C^1(I, \mathbb{R}^n)) \subset W$. Therefore,

$$1 = \deg(id, W, 0) = \deg(id - \overline{h_i}(1, \cdot), W, 0) = \deg(id - h_i(0, \cdot), W, 0). \tag{4.1}$$

We denote

$$U_i = \{x \in C^1(I, \mathbb{R}^n) : \|x(t) - v_i(t)\| < \rho_i(t) + 1, \|x'(t)\| < K \ \forall t \in I\},$$
$$V_i = \{x \in C^1(I, \mathbb{R}^n) : \|x(t) - v_i(t)\| < \rho_i(t), \|x'(t)\| < K \ \forall t \in I\}.$$

Without lost of generality, we can assume that $U_i \subset W$. It follows from Proposition 4.2(a), the excision property, and the homotopic invariance of the degree, and (4.1) that

$$1 = \deg(id - h_i(1, \cdot), U_i, 0). \tag{4.2}$$

Also, for $i = 1, 2$, we have by Proposition 4.2(b), and the excision property of the degree that

$$1 = \deg(id - h_i(1, \cdot), V_i, 0). \tag{4.3}$$

On the other hand, since for $i = 1, 2$, $T(v_i, \rho_i) \subset T(v_0, \rho_0)$,

$$h_i(1, x) = h_0(1, x) \quad \text{for all } x \in \overline{V_i},$$

and hence,

$$1 = \deg(id - h_0(1, \cdot), V_i, 0) \quad \text{for } i = 1, 2. \tag{4.4}$$

Also, $V_1 \cup V_2 \subset U_0$ and $V_1 \cap V_2 = \emptyset$ since $T(v_1, \rho_1) \cap T(v_2, \rho_2) = \emptyset$, combining (4.2) and (4.4) leads to

$$\deg(id - h_0(1, \cdot), U_0 \backslash \overline{V_1 \cup V_2}, 0) = \deg(id - h_0(1, \cdot), U_0, 0)$$
$$- \deg(id - h_0(1, \cdot), V_1, 0) - \deg(id - h_0(1, \cdot), V_2, 0) = -1.$$

Therefore, there exists $x_0 \in U_0 \backslash \overline{V_1 \cup V_2}$, $x_1 \in V_1$ and $x_2 \in V_2$ solutions of $(P_{0,1})$, and hence of (1.1). □

With this general result, we can prove our main theorem.

Proof of Theorem 2.3. We have to show that there exists a constant K such that $\|x'\|_0 < K$ for every x solution of $(P_{i,\lambda})$ for $\lambda \in [0,1]$ and $i = 0,1,2$. Let x be a solution of $(P_{i,\lambda})$. From Proposition 4.2, we know that $x \in T(v_i, \rho_i)$. To simplify the notation, we write v, ρ instead of v_i, ρ_i respectively.

We have by (H3) that

$$\begin{aligned}
|\langle x'(t), x''(t)\rangle| &= |\langle x'(t), f^{\lambda}_{v,\rho}(t, x(t), x'(t)) + \varepsilon x(t)\rangle| \\
&\le \lambda |\langle x'(t), f(t, x(t), x'(t))\rangle| \\
&\qquad + (1-\lambda)\|x'(t)\|\big(\varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)|\big) \\
&\le \phi(\|x'(t)\|)\big(\gamma(t) + \|x'(t)\|\big) + \|x'(t)\|\big(\varepsilon\rho(t) + \|v''(t)\| + |\rho''(t)|\big).
\end{aligned}$$

So, for

$$\gamma_0(t) = \gamma(t) + \max_{i=0,1,2}\{\varepsilon\rho_i(t) + \|v_i''(t)\| + |\rho_i''(t)|\},$$

$$|\langle x'(t), x''(t)\rangle| \le \big(\phi(\|x'(t)\|) + \|x'(t)\|\big)\big(\gamma_0(t) + \|x'(t)\|\big). \tag{4.5}$$

Now, to verify assumptions (i) and (ii) of Lemma 3.3, we consider two cases.

1st case: (H4) is satisfied. We have

$$\begin{aligned}
\|x''(t)\| &= \|f^{\lambda}_{v,\rho}(t, x(t), x'(t)) + \varepsilon x(t)\| \\
&\le \lambda\|f(t, x(t), x'(t))\| + \varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)| \\
&\le \lambda a\Big(\langle x(t), f(t, x(t), x'(t))\rangle + \|x'(t)\|^2\Big) + l(t) \\
&\qquad + \varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)| \\
&\le a\Big(\langle x(t), f^{\lambda}_{v,\rho}(t, x(t), x'(t)) + \varepsilon x(t)\rangle + \|x'(t)\|^2\Big) \\
&\qquad - a(1-\lambda)\langle x(t), v''(t) + \Big(\frac{\rho''(t)}{\rho(t)} + \varepsilon\Big)(x(t) - v(t))\rangle \\
&\qquad + \varepsilon\|x(t) - v(t)\| + \|v''(t)\| + |\rho''(t)| + l(t) \\
&\le a\Big(\langle x(t), x''(t)\rangle + \|x'(t)\|^2\Big) + z_0(t),
\end{aligned}$$

with

$$z_0(t) = l(t) + \max_{i=0,1,2}\big\{(a\rho_i(t) + a\|v_i(t)\| + 1)(\varepsilon\rho_i(t) + \|v_i''(t)\| + \|\rho_i''(t)\|)\big\}.$$

The conclusion follows from (4.5), and Lemmas 3.3 and 3.5.

2nd case: (H5) is satisfied. Let σ_0 be the function introduced in Lemma 3.4. Observe that

$$\begin{aligned}
&\sigma_0(t,x)\\
&\quad= \frac{\langle x(t), x''(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), x''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\\
&\quad= \lambda\Big(\frac{\langle x(t), f(t,x(t),x'(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), f(t,x(t),x'(t))\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\\
&\qquad + (1-\lambda)\Big[\|x'(t)\| + \Big(\frac{\langle x(t), v''(t)\rangle}{\|x'(t)\|} - \frac{\langle x'(t), v''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\\
&\qquad + \Big(\varepsilon + \frac{\rho''(t)}{\rho(t)}\Big)\Big(\frac{\langle x(t), x(t)-v(t)\rangle}{\|x'(t)\|} - \frac{\langle x'(t), x(t)-v(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\Big]\\
&\quad\geq \lambda\Big(\frac{\langle x(t), f(t,x(t),x'(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), f(t,x(t),x'(t))\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3}\Big)\\
&\qquad + (1-\lambda)\|x'(t)\| - 2\frac{\|x(t)\|\big(\|v''(t)\| + \varepsilon\rho(t) + |\rho''(t)|\big)}{\|x'(t)\|}.
\end{aligned}$$

It follows from (H5) that on $\{t \in [0,1] : \|x'(t)\| \geq R\}$,

$$(b + c\|x(t)\|)\sigma_0(t,x) \geq \lambda(\|x'(t)\| - h(t)) + b(1-\lambda)\|x'(t)\| - \delta_0(t), \tag{4.6}$$

with

$$\delta_0(t) = \frac{2}{R}\big(b + c(\rho_0(t) + \|v_0(t)\|)\big)\big(\rho_0(t) + \|v_0(t)\|\big) \max_{i=0,1,2}\Big\{\varepsilon\rho_i(t) + |\rho_i''(t)| + \|v_i''(t)\|\Big\}.$$

The conclusion follows from (4.5), (4.6), and Lemmas 3.3 and 3.4. □

Other existence results can also be obtained. In particular, we can remove (H4) and (H5) if we impose a condition stronger than (H3) with the Sturm-Liouville boundary condition.

Theorem 4.5. *Let* BC *denote the Strum-Liouville boundary condition* (1.2) *with* $\max\{\beta_0, \beta_1\} > 0$. *Assume* (H1), (H2) *and*

(H6) *there exist* $\gamma \in L^1(I)$ *and* $\psi : [0,\infty[\to [1,\infty[$ *a Borel measurable function such that* $\|f(t,x,p)\| \leq \gamma(t)\psi(\|p\|)$ *a.e.* $t \in I$ *and for all* $(x,p) \in \mathbb{R}^{2n}$ *such that* $\|x - v_0(t)\| \leq \rho_0(t)$ *for all* $t \in [0,1]$, *and*

$$\int_k^\infty \frac{ds}{\psi(s)} = \infty \quad \forall k \geq 0.$$

Then the problem (1.1) *has at least three solutions* x_0, x_1, x_2 *such that* $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0,1,2$, $j = 1,2$, $i \neq j$.

Proof. The conclusion will follows from Theorem 4.4 if we can show that there exists a constant $K > 0$ such that $\|x'\|_0 < K$ for every x solution of $(P_{i,\lambda})$.

Let x be a solution of $(P_{i,\lambda})$. We already know that $x \in T(v_i, \rho_i)$. Set

$$\mu = \min_{i=0,1,2}\Big\{\frac{\|r_0\| + a_0(\rho_i(0) + \|v_i(0)\|)}{\beta_0}, \frac{\|r_1\| + a_1(\rho_i(1) + \|v_i(1)\|)}{\beta_1}\Big\}$$

where $\|A_j z\| \le a_j\|z\|$ for all $z \in \mathbb{R}^n$, and $j = 0, 1$. Obviously

$$\min\{\|x'(0)\|, \|x'(1)\|\} \le \mu.$$

Assume that $\|x'\|_0 > \mu$. Let $t_1 \in [0,1]$ such that $\|x'(t_1)\| = \|x'\|_0$. There exists $t_0 < t_1$ such that $\|x'(t)\| > \mu$ on (t_0, t_1) and $\|x'(t_0)\| = \mu$. We have

$$\begin{aligned}\frac{d}{dt}\|x'(t)\| &= \frac{\langle x'(t), x''(t)\rangle}{\|x'(t)\|} \\ &\le \|f(t, x(t), x'(t))\| + \Big(\varepsilon + \frac{|\rho''(t)|}{\rho(t)}\Big)\|x(t) - v(t)\| + \|v''(t)\| \\ &\le \gamma(t)\psi(\|x'(t)\|) + \varepsilon\rho(t) + |\rho''(t)| + \|v''(t)\| \\ &\le \gamma_0(t)\psi(\|x'(t)\|),\end{aligned}$$

with $\gamma_0(t) = \gamma(t) + \max_{i=0,1,2}\{\varepsilon\rho_i(t) + |\rho_i''(t)| + \|v_i''(t)\|\}$. Therefore,

$$\|\gamma_0\|_{L^1} \ge \int_{t_0}^{t_1} \frac{\|x'(t)\|'}{\psi(\|x'(t)\|)}dt = \int_{\mu}^{\|x'(t_1)\|} \frac{ds}{\psi(s)}.$$

We get the conclusion choosing $K > \mu$ such that

$$\int_{\mu}^{K} \frac{ds}{\psi(s)} > \|\gamma_0\|_{L^1}.$$

□

5. Wintner-Nagumo growth condition

In this paragraph, we present a modification of our Main Theorem in which the Wintner-Nagumo condition (H3) is replaced by a more standard one. More precisely, we establish the following result.

Theorem 5.1. *Assume that* (H1), (H2), and (H4) or (H5) are satisfied. Assume also

(H7) there exist a Borel measurable function $\psi : [0, \infty[\to]0, \infty[$ and $\gamma \in L^1(I)$ such that

$$\|f(t, x, p)\| \le \psi(\|p\|)(\gamma(t) + \|p\|)$$

a.e. $t \in I$ and for all $(x, p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \le \rho_0(t)$, and

$$\int_k^{\infty} \frac{ds}{\psi(s)} = \infty \quad \forall k \ge 0.$$

Then the problem (1.1) has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0, 1, 2$, $j = 1, 2$, $i \ne j$.

To prove this result, we consider for $i = 0, 1, 2$, and $\lambda \in [0, 1]$, the differential inclusion

$$(I_{i,\lambda}) \qquad \begin{aligned} &x''(t) - \varepsilon x(t) \in F^{\lambda}_{v_i,\rho_i}(t, x(t), x'(t)) \quad \text{a.e. } t \in [0,1], \\ &x \in \mathrm{BC}.\end{aligned}$$

where for $(v, \rho) \in W^{2,1}(I, \mathbb{R}^n) \times W^{2,1}(I, [0, \infty[)$, $F^{\lambda}_{v,\rho} : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is the multi-valued map defined by

$$F^{\lambda}_{v,\rho} = \lambda \bar{f}_{v,\rho} + G^{\lambda}_{v,\rho} - \varepsilon(1 - \lambda)v,$$

where $\bar{f}_{v,\rho}$ is defined in the previous section, and

$$G^{\lambda}_{v,\rho}(t,x,p) = \begin{cases} g^{\lambda}_{v,\rho}(t,x,p)\frac{(x-v(t))}{\|x-v(t)\|}, & \text{if } \|x-v(t)\| > \rho(t), \\ [0, g^{\lambda}_{v,\rho}(t,x,p)]\frac{(x-v(t))}{\|x-v(t)\|}, & \text{if } \|x-v(t)\| = \rho(t), \\ 0, & \text{if } \|x-v(t)\| < \rho(t); \end{cases}$$

with

$$g^{\lambda}_{v,\rho}(t,x,p) = \Bigg(\Big(1 - \frac{\lambda\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) + (1-\lambda)\Big(\frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^3} - \frac{\|p-v'(t)\|^2}{\|x-v(t)\|}\Big)\Bigg)^+.$$

Observe that $t \mapsto G^{\lambda}_{v,\rho}(t,x,p)$ is measurable for every $(x,p,\lambda) \in \mathbb{R}^n \times [0,1]$, and $(x,p,\lambda) \mapsto G^{\lambda}_{v,\rho}(t,x,p)$ is u.s.c. a.e. $t \in I$.

We associate to $F^{\lambda}_{v,\rho}$ a multi-valued operator

$$N_{v,\rho} : [0,1] \times C^1(I,\mathbb{R}^n) \to C_0(I,\mathbb{R}^n)$$

defined by

$$N_{v,\rho}(\lambda,x) = \{u \in C_0(I,\mathbb{R}^n) : \exists w \in L^1(I,\mathbb{R}^n) \text{ such that } w(t) \in F^{\lambda}_{v,\rho}(t,x(t),x'(t)) \text{ a.e. } t \in I, \text{ and } u(t) = \int_0^t w(s)\,ds\}.$$

This operator has some nice properties.

Proposition 5.2. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,[0,\infty[)$ a solution-tube of (1.1). Then the operator $N_{v,\rho}$ is completely continuous, u.s.c. with nonempty, compact, convex values.*

This result can be proved using the following lemma with [8, Lemma 2.7], and arguing as in [8, Lemmas 2.4] with $\mu : I \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$\mu(t,x,p) = \begin{cases} (\widetilde{x}_{v,\rho}, \widehat{p}_{v,\rho}) & \text{if } \|x-v(t)\| > \rho(t), \\ (x,p) & \text{otherwise}, \end{cases}$$

since we obtain a Carathéodory function if we replace in the definition of $\bar{f}_{v,\rho}$, $\widetilde{x}_{v,\rho}$ and $\widehat{p}_{v,\rho}$ respectively by x and p, and by Remark 4.1.

Lemma 5.3. *Let $f : I \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function, and $(v,\rho) \in W^{2,1}(I,\mathbb{R}^n) \times W^{2,1}(I,[0,\infty[)$ a solution-tube of (1.1). Then the operator $G^{\lambda}_{v,\rho}$ is integrably bounded on bounded sets uniformly in λ, i.e. for every $k > 0$, there exists $d_k \in L^1[0,1]$ such that $G^{\lambda}_{v,\rho}(t,x,p) \subset B(0,d_k(t))$ for a.e. $t \in [0,1]$ and for all $(x,p) \in \mathbb{R}^{2n}$ with $\|x\| \le k$, $\|p\| \le k$.*

Proof. Definition 2.1 and Remark 4.1 imply that if $G^{\lambda}_{v,\rho}(t,x,p) \neq \{0\}$, then for every $z \in G^{\lambda}_{v,\rho}(t,x,p)$,

$$\begin{aligned}
\|z\| &\leq \big|g^{\lambda}_{v,\rho}(t,x,p)\big| \\
&= \Big(1 - \frac{\lambda\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) \\
&\quad + (1-\lambda)\Big(\frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^3} - \frac{\|p-v'(t)\|^2}{\|x-v(t)\|}\Big) \\
&= \Big(1 - \frac{\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) \\
&\quad + \frac{1-\lambda}{\|x-v(t)\|}\Big[\rho(t)\rho''(t) + \frac{\rho(t)\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|} \\
&\quad + \frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^2} - \|p-v'(t)\|^2\Big] \\
&= \Big(1 - \frac{\rho(t)}{\|x-v(t)\|}\Big)\Big(\rho''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\Big) \\
&\quad + \frac{1-\lambda}{\|x-v(t)\|}\Big(\rho(t)\rho''(t) + \langle \widetilde{x}_{v,\rho} - v(t), v''(t)\rangle + \rho'(t)^2 \\
&\quad - \|\widehat{p}_{v,\rho} - v'(t)\|^2\Big) \\
&\leq |\rho''(t)| + \|v''(t)\| + \frac{1}{\|x-v(t)\|}\langle \widetilde{x}_{v,\rho} - v(t), f(t, \widetilde{x}_{v,\rho}, \widehat{p}_{v,\rho})\rangle \\
&\leq |\rho''(t)| + \|v''(t)\| + \|f(t, \widetilde{x}_{v,\rho}, \widehat{p}_{v,\rho})\|.
\end{aligned}$$

The conclusion follows from the fact that f is a Carathéodory function and

$$\|\widetilde{x}_{v,\rho}\| \leq \|v\|_0 + |\rho|_0 \quad \text{and} \quad \|\widehat{p}_{v,\rho}\| \leq 2\|p\| + \|v'\|_0 + |\rho'|_0.$$

□

In order to prove our main theorem, we can first establish a general result as in the previous section.

Theorem 5.4. *Assume that* (H1) *and* (H2) *are satisfied. Assume that there exists a constant* $K > 0$ *such that every solution* x *of* $(I_{i,\lambda})$ *with* $i = 0,1,2$, $\lambda \in [0,1]$, *satisfies* $\|x'\|_0 < K$. *Then the problem* (1.1) *has at least three distinct solutions* x_0, x_1, x_2 *such that* $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0,1,2$, $j = 1,2$, $i \neq j$.

The proof of this result is analogous to the proof of Theorem 4.4 where we replace the operators η_{v_i,ρ_i} by N_{v_i,ρ_i} and we use degree theory of compact, upper semi-continuous multi-valued maps with closed, convex values, and the following proposition.

Proposition 5.5. *Assume that* (H1) *and* (H2) *are satisfied. Then*

(a) *every solution* x *to* $(I_{i,\lambda})$ *is such that* $x \in T(v_i, \rho_i)$ *for* $i = 0,1,2$, $\lambda \in [0,1]$;
(b) *for* $i = 1,2$, *every solution* x *to* $(I_{i,1})$ *satisfies* $\|x(t) - v_i(t)\| < \rho_i(t)$ *for all* $t \in [0,1]$.

Proof. (a) Let $i \in \{0,1,2\}$, $\lambda \in [0,1]$, and let x be a solution of $(I_{i,\lambda})$. To simplify the notation, let v and ρ stand for v_i and ρ_i respectively, and $\tilde{x}(t)$ and $\hat{x}'(t)$ stand for $\widetilde{x(t)}_{v_i,\rho_i}$ and $\widehat{x'(t)}_{v_i,\rho_i}$. Then for almost every $t \in \{t \in I : \|x(t)-v(t)\| > \rho(t)\}$,

$$\begin{aligned}
&\frac{1}{\|x(t)-v(t)\|}\big(\langle x(t)-v(t), x''(t)-v''(t)\rangle + \|x'(t)-v'(t)\|^2\big) \\
&\quad - \frac{1}{\|x(t)-v(t)\|^3}\langle x(t)-v(t), x'(t)-v'(t)\rangle^2 - \varepsilon\|x(t)-v(t)\| \\
&= \frac{\lambda}{\|x(t)-v(t)\|}\Big(\langle \tilde{x}_i(t)-v(t), f(t,\tilde{x}_i(t),\hat{x}'_i(t)) - v''(t)\rangle + \|\hat{x}'_i(t)-v'(t)\|^2\Big) \\
&\quad - \varepsilon\lambda\rho(t) + \Big(\frac{\lambda\rho(t)}{\|x(t)-v(t)\|^2} - \frac{1}{\|x(t)-v(t)\|}\Big)\langle x(t)-v(t), v''(t)\rangle \\
&\quad + \frac{\|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3} - \frac{\lambda\|\hat{x}'_i(t)-v'(t)\|^2}{\|x(t)-v(t)\|} \\
&\quad + g^{\lambda}_{v,\rho}(t,x(t),x'(t)) \\
&\geq \frac{\lambda}{\|x(t)-v(t)\|}\big(\rho(t)\rho''(t) + \rho'(t)^2\big) - \varepsilon\lambda\rho(t) \\
&\quad + \Big(\frac{\lambda\rho(t)}{\|x(t)-v(t)\|^2} - \frac{1}{\|x(t)-v(t)\|}\Big)\langle x(t)-v(t), v''(t)\rangle \\
&\quad + (1-\lambda)\Big(\frac{\|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3}\Big) \\
&\quad - \frac{\lambda\rho'(t)^2}{\|x(t)-v(t)\|} + g^{\lambda}_{v,\rho}(t,x(t),x'(t)) \\
&\geq \rho''(t) - \varepsilon\rho(t).
\end{aligned}$$

The conclusion (a) follows from Lemma 3.1.

(b) We argue as in the proof of Proposition 4.2 using Lemma 3.2. □

The following lemmas will lead to the existence of a priori bounds on the derivative of the solutions of $(I_{i,\lambda})$.

Lemma 5.6. *If* (H1), (H2), (H7) *are satisfied then every solution x of $(I_{i,\lambda})$ for $i = 0, 1, 2$ and $\lambda \in [0,1]$, satisfies*

$$\|x''(t)\| \leq 2\psi(\|x'(t)\|)(\gamma(t) + \|x'(t)\|) + \varepsilon\|\rho_0\|_0 \quad \textit{a.e. } t \in [0,1].$$

Proof. Let $i \in \{0,1,2\}$, $\lambda \in [0,1]$, and x a solution to $(I_{i,\lambda})$. By Proposition 5.5, we know that $x \in T(v_i,\rho_i)$. Again, to simplify the notation, we do not write the subscripts. We have that

$$\|x''(t)\| \leq \|f(t,x(t),x'(t))\| + \varepsilon\|x(t)-v(t)\| + \begin{cases} g^{\lambda}_{v,\rho}(t,x(t),x'(t)), & \text{if } \|x(t)-v(t)\| = \rho(t), \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Let us examine the third part of the right member of this equation. Since (v,ρ) is a solution-tube of (1.1) and by Remark 4.1, we have a.e. on $\{t \in [0,1] : \|x(t)-v(t)\| =$

$\rho(t),\ g^{\lambda}_{v,\rho}(t,x(t),x'(t))>0\}$,

$$\begin{aligned}
&g^{\lambda}_{v,\rho}(t,x(t),x'(t))\\
&\quad=(1-\lambda)\left(\rho''(t)+\frac{\langle x(t)-v(t),v''(t)\rangle+\rho'(t)^2-\|\widehat{x'(t)}-v'(t)\|^2}{\|x(t)-v(t)\|}\right)\\
&\quad\le\frac{\langle x(t)-v(t),f(t,x(t),\widehat{x'(t)})\rangle}{\|x(t)-v(t)\|}\\
&\quad\le\|f(t,x(t),\widehat{x'(t)})\|=\|f(t,x(t),x'(t))\|.
\end{aligned}$$

This inequality combined with (5.1) and (H7) leads to the conclusion. □

Lemma 5.7. *Under* (H1), (H2), (H5) *there exist* $b_0>0$, $c_0\ge 0$, *and* $\delta_0\in L^1[0,1]$ *such that every solution* x *of* $(I_{i,\lambda})$ *for* $i=0,1,2$, $\lambda\in[0,1]$, *satisfies a.e. on* $\{t\in[0,1]:\|x'(t)\|\ge R\}$

$$(b_0+c_0\|x(t)\|)\sigma_0(t,x)\ge\|x'(t)\|-\delta_0(t),$$

where R *is given in* (H5) *and* σ_0 *is defined in Lemma 3.4.*

Proof. Let x be a solution to $(I_{i,\lambda})$ for some $i\in\{0,1,2\}$, and $\lambda\in[0,1]$. Again to simplify the notation, we don't write the subscripts. From Proposition 5.5, $x\in T(v,\rho)$. Also, a.e. on $\{t\in[0,1]:\|x(t)-v(t)\|=\rho(t),\ g^{\lambda}_{v,\rho}(t,x(t),x'(t))\neq 0\}$,

$$\begin{aligned}
|g^{\lambda}_{v,\rho}(t,x(t),x'(t))|&=g^{\lambda}_{v,\rho}(t,x(t),x'(t))\\
&\le z_0(t):=\max_{i=0,1,2}\{|\rho_i''(t)|+\|v_i''(t)\|\}.
\end{aligned}\tag{5.2}$$

It follows that a.e. on $\{t\in[0,1]:\|x'(t)\|\ge R\}$,

$$\begin{aligned}
&(b+c\|x(t)\|)\sigma_0(t,x)\\
&\quad=\lambda(b+c\|x(t)\|)\sigma(t,x(t),x'(t))+(1-\lambda)(b+c\|x'(t)\|)\|x'(t)\|\\
&\qquad+(b+c\|x'(t)\|)\Big(\varepsilon(1-\lambda)+\frac{\theta(t)g^{\lambda}_{v,\rho}(t,x(t),x'(t))}{\|x(t)-v(t)\|}\Big)\Big(\frac{\langle x(t),x(t)-v(t)\rangle}{\|x'(t)\|}\\
&\qquad-\frac{\langle x'(t),x(t)-v(t)\rangle\langle x(t),x'(t)\rangle}{\|x'(t)\|^3}\Big),
\end{aligned}$$

where b,c are given in (H5) and

$$\theta(t)\in\begin{cases}[0,1], & \text{if } \|x(t)-v(t)\|=\rho(t),\\ \{0\}, & \text{otherwise.}\end{cases}$$

It follows from (H5) and (5.2) that

$$(b+c\|x(t)\|)\sigma_0(t,x)\ge(\lambda+b(1-\lambda))\|x'(t)\|-h(t)-2\Big(\frac{b}{R}+c\Big)\|x(t)\|(\varepsilon\rho(t)+z_0(t)).$$

Set $\nu=\min_{\lambda\in[0,1]}\{\lambda+b(1-\lambda)\}$. The conclusion follows choosing $b_0=b/\nu$ and $c_0=c/\nu$ and

$$\nu\delta_0(t)=h(t)+\frac{2}{R}\Big(b+c(\|v_0(t)\|+\rho_0(t))\Big)\Big(\|v_0(t)\|+\rho_0(t)\Big)\Big(\varepsilon\rho_0(t)+z_0(t)\Big).$$

□

Lemma 5.8. *Under* (H1), (H2), (H4) *there exists* $m_0 \in L^1[0,1]$ *such that every solution* x *of* $(I_{i,\lambda})$ *for* $i = 0, 1, 2$, $\lambda \in [0,1]$, *satisfies*

$$\|x''(t)\| \le a\big(\langle x(t), x''(t)\rangle + \|x'(t)\|^2\big) + m_0(t) \quad a.e.\ t \in [0,1],$$

where a *is given in* (H4).

Proof. Let x be a solution of $(I_{i,\lambda})$ for some $i \in \{0,1,2\}$ and $\lambda \in [0,1]$ which is in $T(v_i, \rho_i)$ by Proposition 5.5. As always, to simplify the notation, we don't write the subscript i. We have by (H4) and (5.2)

$$\begin{aligned}\|x''(t)\| &\le \lambda\|f(t,x(t),x'(t))\| + \varepsilon(1-\lambda)\|x(t) - v(t)\| + z_0(t)\\ &\le a\lambda\big(\langle x(t), f(t,x(t),x'(t))\rangle + \|x'(t)\|^2\big) + l(t) + \varepsilon\rho_0(t) + z_0(t)\\ &\le a\big(\langle x(t), x''(t))\rangle + \|x'(t)\|^2\big) + l(t) + \varepsilon\rho_0(t) + z_0(t)\\ &\qquad + a\|x(t)\|(\varepsilon\rho_0(t) + z_0(t));\end{aligned}$$

and the proof is complete. □

Now, we can prove the main theorem of this section.

Proof of Theorem 5.1. The conclusion is a direct consequence of Theorem 5.4, Lemmas 3.3–3.5, and 5.6–5.8. □

Arguing as in the proofs of Theorems 2.3 and 5.1, we can see that we can obtain the following more general results.

Theorem 5.9. *Assume that* (H1), (H2), and (H3) or (H7) are satisfied. Assume also

(H8) there exist $R > 0$, $b > 0$, $c \ge 0$, $h \in L^1(I)$, and $u \in W^{2,1}([0,1], \mathbb{R}^n)$ such that

$$(b + c\|x - u(t)\|)\sigma^u(t,x,p) + \frac{c\langle x - u(t), p - u'(t)\rangle^2}{\|x - u(t)\|\,\|p - u'(t)\|} \ge \|p - u'(t)\| - h(t),$$

for a.e. $t \in I$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v_0(t)\| \le \rho_0(t)$, $\|p - u'(t)\| \ge R$, where

$$\sigma^u(t,x,p) = \frac{\langle x - u(t), f(t,x,p) - u''(t)\rangle + \|p - u'(t)\|^2}{\|p - u'(t)\|} - \frac{\langle p - u'(t), f(t,x,p) - u''(t)\rangle\langle x - u(t), p - u'(t)\rangle}{\|p - u'(t)\|^3}.$$

Then the problem (1.1) has at least three distinct solutions x_0, x_1, x_2 such that $x_i \in T(v_i, \rho_i)$, $x_i \notin T(v_j, \rho_j)$, $i = 0, 1, 2$, $j = 1, 2$, $i \ne j$.

Theorem 5.10. *Assume that* (H1) and

(H9) there exist (v_0, ρ_0) a solution-tube of (1.1) and for $i = 1, \ldots m$ with $m \ge 2$, there exists (v_i, ρ_i) a strict solution-tube of (1.1) such that $T(v_i, \rho_i) \cap T(v_j, \rho_j) = \emptyset$, and $T(v_i, \rho_i) \subset T(v_0, \rho_0)$, $i, j \in \{1, \ldots, m\}$, $i \ne j$.

Assume also (H3) or (H7), and (H4) or (H5) or (H8) are satisfied. Then the problem (1.1) has at least $m+1$ distinct solutions $x_1, \ldots, x_{m+1}$ such that $x_i \in T(v_i, \rho_i)$, $x_{m+1} \in T(v_0, \rho_0)$, $x_{m+1} \notin T(v_i, \rho_i)$, $i = 1, \ldots, m$.

Idea of the proof. As before, for $i = 0, \dots, m$, we define appropriate maps h_i and families of problems $(P_{i,\lambda})$ (or $(I_{i,\lambda})$) such that solutions to $(P_{i,\lambda})$ are fixed points of $h_i(\lambda, \cdot)$. A priori bounds are obtained which permit to define open sets $U_0, V_1, \dots, V_m$ with $V_i \subset U_0$, $V_i \cap V_j = \emptyset$ for $i \neq j$,

$$h_i(1, x) = h_0(1, x) \quad \text{for all } x \in \overline{V_i}, \quad i = 1, \dots, m,$$

and

$$\begin{aligned} 1 &= \deg(id - h_0(1, \cdot), U_0, 0), \\ 1 &= \deg(id - h_i(1, \cdot), V_i, 0) = \deg(id - h_0(1, \cdot), V_i, 0), \quad i = 1, \dots, m. \end{aligned}$$

Thus,

$$\begin{aligned} \deg(id - h_0(1, \cdot), U_0 \backslash \overline{V_1 \cup \cdots \cup V_m}, 0) &= \deg(id - h_0(1, \cdot), U_0, 0) \\ - \deg(id - h_0(1, \cdot), V_1, 0) - \cdots &- \deg(id - h_0(1, \cdot), V_m, 0) = 1 - m \neq 0. \end{aligned}$$

So, there exists a solution in $U_0 \backslash \overline{V_1 \cup \cdots \cup V_m}$ and hence in

$$T(v_0, \rho_0) \backslash \overline{T(v_1, \rho_1) \cup \cdots \cup T(v_m, \rho_m)},$$

and for each $i = 1, \dots m$, there exists a solution in V_i and hence in $T(v_i, \rho_i)$. □

In the scalar case, we have the following result. Let us recall that u, w are respectively strict lower and upper solutions if $\big((u + w)/2, (w - u)/2\big)$ is a strict solution-tube of (1.1). See Henderson and Thompson [12] for a result with more general upper and lower solutions.

Corollary 5.11. *Let $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function. Assume that there exist $w_1, \dots, w_m$, m strict upper solutions, and $u_1, \dots, u_m$, m strict lower solutions such that*

(i) $u_i \leq w_i$, *for* $i = 1, \dots, m$;
(ii) $u_1 \leq u_2 \leq \cdots u_m$, $w_1 \leq w_2 \leq \cdots w_m$;
(iii) $u_{i+1} \not\leq w_i$, $i = 1, \dots, m - 1$;
(iv) *there exist a Borel measurable function* $\psi : [0, \infty[\to]0, \infty[$ *and* $\gamma \in L^1(I)$ *such that*

$$\|f(t, x, p)\| \leq \psi(\|p\|)(\gamma(t) + \|p\|)$$

a.e. $t \in I$ *for all* $(x, p) \in \mathbb{R}^{2n}$ *such that* $u_1(t) \leq x \leq w_m(t)$, *and*

$$\int_k^\infty \frac{ds}{\psi(s)} = \infty \quad \forall m > 0.$$

Then the problem (1.1) *has at least* $m + 1$ *distinct solutions* $x_1, \dots, x_{m+1}$ *such that* $u_i \leq x_i \leq w_i$, $u_1 \leq x_{m+1} \leq w_m$, $x_{m+1} \notin [u_i, w_i]$, $i = 1, \dots, m$.

Finally, we present an example.

Example 5.12. Consider the following system

$$\begin{aligned} & x'' = 5x - b(x, y)y - u(t, x, y, x', y')x' + v(t, x, y, x', y') - 5 \\ (5.3) \qquad & y'' = 5y + b(x, y)x - u(t, x, y, x', y')y' + w(t, x, y, x', y') - 5 \\ & x(0) = x(1), \; y(0) = y(1), \; x'(0) = x'(1), \; y'(0) = y'(1), \end{aligned}$$

where u, v, w are bounded Carathéodory functions such that $u \geq 0$, $\|v(t,x,p,q)\| \leq 1$, $\|w(t,x,p,q)\| \leq 1$, and

$$b(x,y) = \frac{5\big((x^2+y^2-2y+3/4)^2 - (x^2+y^2-2x+3/4)^2\big)}{(x^2+y^2-2y+3/4)^2 + (x^2+y^2-2x+3/4)^2}.$$

We can find a constant $\rho_0 \geq 2$ big enough such that $\big((0,0),\rho_0\big)$ is a solution-tube of (5.3). Also, we define $\rho(t) = 1/2 + \delta(t-1/2)^2$. We can verify that for $\delta > 0$ small enough, $\big((1,0),\rho\big)$ and $\big((0,1),\rho\big)$ are strict solution-tubes of (5.3), $T\big((1,0),\rho\big) \subset T\big((0,0),\rho_0\big)$, $T\big((0,1),\rho\big) \subset T\big((0,0),\rho_0\big)$, and $T\big((1,0),\rho\big) \cap T\big((0,1),\rho\big) = \emptyset$. Finally, it is easy to check that assumptions (H3) and (H5) are satisfied.

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M. Frigon and E. Montoki
Département de mathématiques et de statistique,
Université de Montréal,
C.P. 6128, succ. Centre-Ville,
Montréal, (Québec),
H3C 3J7, CANADA.
E-mail address: frigon@dms.umontreal.ca