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ON A CRITICAL POINT THEORY FOR MULTIVALUED FUNCTIONALS AND APPLICATION TO PARTIAL DIFFERENTIAL INCLUSIONS†

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1. INTRODUCTION

The critical point theory is a domain of mathematics which has known rapid developments in the last two decades. Recently many attempts were made to develop critical point theory for nondifferential functionals, more precisely for Lipschitzian, continuous, or lower semi-continuous functionals [1–4].

The starting point of this work is a paper of Degiovanni and Marzocchi [3] in which they introduced the notion of *weak slope* of a continuous functional, and a corresponding notion of critical points. Then, they extended this notion to lower semi-continuous functional by using the weak slope of the function *graph* which is continuous, and defined on the epigraph of the functional. They observed that one difficulty with their definition is that the function *graph* can possess more critical points than the original lower semi-continuous functional, and they illustrated this fact with this example:

$$f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

In this example, (0, 0) and (0, 1) are critical points of the function *graph*. It is observed in [5], that "f exhibits a behaviour of mountain pass type and 1 is just the mountain pass level".

If we look carefully at this example, one can ask if 1 is really a mountain pass level since there is no continuous path $\phi\colon [0, 1] \to \text{graph}\, f$ from $\phi(0) = (x_0, f(x_0))$ to $\phi(1) = (x_1, f(x_1))$ for $x_0 < 0 < x_1$. From that observation, we realized that, instead of defining critical points for a lower semi-continuous functional from the function *graph*, it is also possible to do that from multivalued mappings with closed graph. To illustrate that, let us come back to the previous example. From the functional f, let us define the following two multivalued functionals:

$$F_1(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ \{0, 1\}, & \text{if } x = 0, \\ x, & \text{if } x > 0, \end{cases} \qquad \text{and} \qquad F_2(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ [0, 1], & \text{if } x = 0, \\ x, & \text{if } x > 0. \end{cases}$$

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According to our definition, 0 is not a critical point for F_1 at level 1, but it is for F_2. Observe that there is no continuous path joining $(x_0, x_0 + 1)$ to (x_1, x_1), for $x_0 < 0 < x_1$, on the graph of F_1, while there is such a path on the graph of F_2.

This example shows also that if we want to introduce a critical point theory for multivalued mapping, it is more natural to work on the graph of the functional than on its domain.

In this paper, we extend to multivalued functionals F with closed graph, the notion of weak slope introduced in [3]. From that, we define the notion of critical point and we give a Palais–Smale condition for F.

In order to obtain minimax results, we introduce the notion of invariance with respect to deformations for a family of sets. This notion contains the notion of "ambient isotopy invariance", and "homotopy stable family with boundary" introduced in [6] and [7] respectively. However, we did not include the equivariant case to this paper. Using this notion of invariance with respect to deformations, and the Deformation Theorem established in [2], we give mountain pass type results and a minimax principle.

In Section 3, we discuss possible applications of this theory to lower semi-continuous functionals.

Finally, in the last section, we give an application to a problem for partial differential inclusions. Many authors [1, 4, 8–11], starting with a problem with discontinuous nonlinearities, have considered an associated problem with multivalued nonlinearities G, where G is a subdifferential of a locally Lipschitz function. Various methods were used; however, the functional considered was always single-valued. To our knowledge, variational methods are applied for the first time to multivalued functionals to get existence result to partial differential inclusions. We should say that a part of the technique used is inspired from [12–15]; in particular the functional is defined on $L^p(\Omega)$ while, often, it is defined on $H_0^1(\Omega)$. Let us mention that we tried to give an application as simple as possible, and we are convinced that much more can be obtained from those techniques.

2. A CRITICAL POINT THEORY FOR MULTIVALUED MAPPINGS

Let (X, d) be a metric space and let $F: X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph (and nonempty values). We denote by

$$\operatorname{graph} F = \{(u, c) \in X \times \mathbb{R} : c \in F(u)\}.$$

This set is a metric space with the metric

$$d((u, c), (v, b)) = \sqrt{d(u, v)^2 + |b - c|^2}.$$

We denote π_X and $\pi_{\mathbb{R}}$ the projections of graph F on X and $\mathbb{R}$ respectively. That is $\pi_X(u, c) = u$ and $\pi_{\mathbb{R}}(u, c) = c$.

Now, we give the definition of *weak slope* of F.

Definition 2.1. Let $F: X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph, and let $(u, c) \in \operatorname{graph} F$. The *weak slope* of F at (u, c), denoted by $|dF|(u, c)$ is the supremum of $\sigma \in [0, \infty)$ such that there exist $\delta > 0$ and a continuous function

$$H = (H_1, H_2): B((u, c), \delta) \times [0, \delta] \to \operatorname{graph} F$$

(where $B((u, c), \delta)$ is the open ball in graph F centered at (u, c) of radius δ) such that

(2.1.1) $d(H((v, b), t), (v, b)) \leq t\sqrt{\ + \sigma^2}$;

(2.1.2) $H_2((v, b), t) \leq b - \sigma t$.

In the case where $F(u) = \{f(u)\}$ is a continuous single-valued function then

$$|dF|(u, f(u)) = |df|(u)$$

where $|df|(u)$ is the weak slope of f at u as defined in [3], and it coincides with the norm of the derivative when f is a function of class C^1 defined on a Finsler manifold of class C^1.

Let us remark that if F is a multivalued function as before, then the function $\mathcal{G}_F$: graph $F \to \mathbb{R}$ defined by $\mathcal{G}_F(u, c) = c$ is continuous. It is easy to check that the following equality holds:

$$|dF|(u, c) = \begin{cases} \dfrac{|d\mathcal{G}_F|(u, c)}{\sqrt{1 - |d\mathcal{G}_F|(u, c)^2}}, & \text{if } |d\mathcal{G}_F|(u, c) < 1; \\ \infty, & \text{if } |d\mathcal{G}_F|(u, c) = 1. \end{cases} \tag{2.1}$$

The lower semi-continuity of $|dF|$ is immediately deduced from (2.1) and [3, Prop. 2.6].

Proposition 2.2. Let $F: X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph. Then

$$|dF|(u, c) \leq \liminf_{n \to \infty} |dF|(u_n, c_n)$$

for every sequence $(u_n, c_n) \in$ graph F converging to (u, c).

In general, if $F = F_1 + F_2$, one can not compare $|dF|(u, c_1 + c_2)$ with $|dF_1|(u, c_1)$ and $|dF_2|(u, c_2)$. However, some information can be obtained in the particular case where F_2 is a single-valued C^1 function. For the proof, we refer to [3, Prop. 2.7].

Lemma 2.3. Let E be a Banach space, V an open subset of E, $f \in C^1(V, \mathbb{R})$, and let $G: V \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph. Denote $F(u) = G(u) + f(u)$. Then, for every $(u, c) \in$ graph F,

$$|dG|(u, c - f(u)) - \|f'(u)\| \leq |dF|(u, c) \leq |dG|(u, c - f(u)) + \|f'(u)\|.$$

In the next two definitions, we introduce the notion of critical point of F and a Palais–Smale condition.

Definition 2.4. Let $F: X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph, and let $c \in \mathbb{R}$. We say that u is a *critical point of F at level c*, if $c \in F(u)$ and $|dF|(u, c) = 0$. The set of critical points of F at level c is denoted by K_c. We say that c is a *critical value of F* if $K_c \neq \emptyset$.

Observe that since F is a multivalued mapping, $u \in X$ can be a critical point at more than one level. That means that the application associating a critical point to its critical values is a multivalued mapping. Also, it is clear from the definition that if u is a finite local minimum of F and $c = \min\{b \in F(u)\}$, then $u \in K_c$.

Definition 2.5. Let $F: X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph, and let $c \in \mathbb{R}$. The function F satisfies the *Palais-Smale condition at level c* ($(PS)_c$) if every sequence (u_n) in X for which there exists $c_n \in F(u_n)$ with $c_n \to c$ and $|dF|(u_n, c_n) \to 0$, has a convergent subsequence in X.

For sake of completeness, we state, in our context, the Noncritical Interval Theorem and the Deformation Theorem established in [2, Theorems 2.14, 2.15].

THEOREM 2.6. Let X be a complete metric space, $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph, and let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \cup \{\infty\}$ ($\alpha < \beta$). Assume that for every $c \in [\alpha, \beta]$ such that $F(X) \cap [c, \infty) \neq \emptyset$, F has no critical points at level c, and satisfies $(PS)_c$. Then there exists a continuous function $\eta = (\eta_1, \eta_2)$: graph $F \times [0, 1] \to$ graph F such that

(2.5.1) $\eta((u, b), 0) = (u, b)$;
(2.6.2) $\eta_2((u, b), t) \leq b$;
(2.6.3) if $(u, b) \in \text{graph}\, F \cap X \times (-\infty, \alpha]$, then $\eta((u, b), t) = (u, b)$;
(2.6.4) $\eta((\text{graph}\, F \cap X \times (-\infty, \beta]), 1) \subset X \times (-\infty, \alpha]$.

THEOREM 2.7. Let X be a complete metric space, $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph, and let $c \in \mathbb{R}$. Assume that F satisfies $(PS)_c$. Then, given $\varepsilon_0 > 0$, $\mathcal{O}$ a neighbourhood of $K_c \times \{c\}$ ($\mathcal{O}$ can be empty if K_c is empty), and $\sigma > 0$, there exist $0 < \varepsilon < \varepsilon_0$, and a continuous function $\eta = (\eta_1, \eta_2)$: graph $F \times [0, 1] \to$ graph F such that

(2.7.1) $d(\eta(u, b), t), (u, b)) \leq \sigma t$;
(2.7.2) $\eta_2((u, b), t) \leq b$;
(2.7.3) if $(u, b) \in \text{graph}\, F \backslash (X \times (c - \varepsilon_0, c + \varepsilon_0))$, then $\eta((u, b), t) = (u, b)$;
(2.7.4) $\eta((\text{graph}\, F \cap X \times (-\infty, c + \varepsilon]) \backslash \mathcal{O}, 1) \subset X \times (-\infty, c - \varepsilon]$.

COROLLARY 2.8. Let X be a complete metric space, $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph, and let $c \in \mathbb{R}$. Assume that F satisfies $(PS)_c$. Let $\sigma > 0$, B a closed subset of graph F, $\mathcal{O}$ and $\mathcal{V}$ neighbourhoods of $K_c \times \{c\}$ and B, respectively ($\mathcal{O}$ (resp. $\mathcal{V}$) can be empty if K_c (resp. B) is empty). Then there exist $\varepsilon > 0$, and a continuous function $\eta = (\eta_1, \eta_2)$: graph $F \times [0, 1] \to$ graph F such that

(2.8.1) $d(\eta((u, b), t), (u, b)) \leq \sigma t$;
(2.8.2) $\eta_2((u, b), t) \leq b$;
(2.8.3) $\eta = id$ on graph $F \times \{0\} \cup B \times [0, 1]$;
(2.8.4) $\eta((\text{graph}\, F \cap X \times (-\infty, c + \varepsilon]) \backslash (\mathcal{O} \cup \mathcal{V}), 1) \subset X \times (-\infty, c - \varepsilon]$.

Proof. Let η: graph $F \times [0, 1] \to$ graph F be the deformation given by the previous theorem. Let $\lambda: X \to [0, 1]$ be a Urysohn function such that $\lambda(x) = 0$ for every $x \in B$ and $\lambda(x) = 1$ for every $x \in X/\mathcal{V}$. Define $\hat{\eta}: X \times [0, 1] \to X$ be the function defined by

$$\hat{\eta}(x, t) = \eta(x, t\lambda(x)).$$

It is clear that $\hat{\eta}$ is the desired deformation. ■

Let Q be a subset of graph F. We denote

$$\Gamma(Q) = \{U \subset \text{graph}\, F : Q \subset U (U \neq \emptyset \text{ if } Q = \emptyset)\}.$$

Also, if (D, S) is a closed pair, and $\psi: S \to \text{graph}\, F$ is a continuous function, we denote

$$\Gamma(\psi(S), D) = \{f(D) : f = (f_1, f_2) \in C(D, \text{graph}\, F) \text{ and } f|_S = \psi\}.$$

Clearly $\Gamma(\psi(S), D) \subset \Gamma(\psi(S))$.

The next notions will be crucial in what follows.

Definition 2.9. Let Q be a subset of graph F, and let Γ_0 be a subset of $\Gamma(Q)$. We say that Γ_0 is *invariant with respect to* (F, Q)*-deformations* if the set $\eta(U, 1) \in \Gamma_0$ for every $U \in \Gamma_0$, and every continuous map $\eta: \text{graph}\, F \times [0, 1] \to \text{graph}\, F$ such that $\eta = id$ on $\text{graph}\, F \times \{0\} \cup Q \times [0, 1]$, and $\eta_2((u, b), t) \leq b$ for every $t \in [0, 1]$, and $(u, b) \in \text{graph}\, F$.

Remark 2.10. Instead of considering all continuous deformations, we can take a subset $\mathfrak{N}_0$ of the set of functions $\eta \in C(\text{graph}\, F \times [0, 1], \text{graph}\, F)$ satisfying $\eta = id$ on $\text{graph}\, F \times \{0\} \cup Q \times [0, 1]$, and $\eta_2((u, b), t) \leq b$ for every $t \in [0, 1]$, and $(u, b) \in \text{graph}\, F$. In that case, we can define the notion of *invariance with respect to* $\mathfrak{N}_0$.

Definition 2.11. Let A and Q be two subsets of graph F, and let Γ_0 be a nonempty subset of $\Gamma(Q)$. We say that Γ_0 *intersects* A if $U \cap A \neq \emptyset$ for every U in Γ_0.

Observe that in the previous definition, A and Q do not need to be disjoint. In particular, A could be graph F. Also, Q could be empty.

In this multivalued context, we want to present generalizations of the Mountain Pass Theorem.

THEOREM 2.12. Let X be a complete metric space, and $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph. Assume there exists a closed subset A of graph F, and there exist $Q \subset \text{graph}\, F$, and $\Gamma_0 \subset \Gamma(Q)$ nonempty and invariant with respect to (F, Q)-deformations such that Γ_0 intersects A. In addition, assume that

$$\inf_{U \in \Gamma_0} \sup \pi_{\mathbb{R}}(U \cap A) \geq \sup \pi_{\mathbb{R}}(Q),$$

with a strict inequality if $d(A, Q) = 0$. Let

$$c = \inf_{U \in \Gamma_0} \sup \pi_{\mathbb{R}}(U).$$

If $c \in \mathbb{R}$, and F satisfies $(PS)_c$ then $K_c \times \{c\} \cap \overline{\Gamma_0} \neq \emptyset$, where $\overline{\Gamma_0} = \overline{\bigcup_{U \in \Gamma_0} U}$. Moreover, if

$$c = \inf_{U \in \Gamma_0} \sup \pi_{\mathbb{R}}(U \cap A),$$

then $K_c \times \{c\} \cap A \neq \emptyset$.

The proof is analogous to the one given in [16], we include it for sake of completeness.

Proof. Let $\mathcal{V}$ be a neighbourhood of $\bar{Q}$, and $\sigma > 0$ which will be fixed later. Choose $\mathcal{O} = B(K_c \times \{c\}, \sigma)$. Let $\varepsilon > 0$ and η: graph $F \times [0, 1] \to$ graph F given by Corollary 2.8. So, $\eta_2((u, b), t) \leq b$, and $\eta = id$ on graph $F \times [0, 1] \cup \bar{Q} \times [0, 1]$. Let $U \in \Gamma_0$ be such that $\sup \pi_{\mathbb{R}}(U) \leq c + \varepsilon$. By assumption, $\eta(U, 1) \in \Gamma_0$.

Denote

$$E = \begin{cases} A, & \text{if } c = \sup_{U \in \Gamma_0} \pi_{\mathbb{R}}(U \cap A), \\ \overline{\Gamma_0}, & \text{otherwise.} \end{cases}$$

Assume $K_c \times \{c\} \cap E = \varnothing$. Observe that K_c is compact since F satisfies $(PS)_c$. Fix $\sigma > 0$ such that

$$d(K_c \times \{c\}, E) > 2\sigma,$$

and

$$d(A, Q) > 2\sigma \qquad \text{if } c = \sup \pi_{\mathbb{R}}(Q).$$

In the case where $c > \sup \pi_{\mathbb{R}}(Q)$, choose $\delta > 0$ such that

$$c > c - \delta > \sup \pi_{\mathbb{R}}(Q).$$

Fix

$$\mathcal{V} = \begin{cases} B(Q, \sigma), & \text{if } c = \sup \pi_{\mathbb{R}}(Q); \\ \text{graph } F \cap X \times (-\infty, c - \delta), & \text{otherwise.} \end{cases}$$

Let $(u, b) \in U$ be such that $\eta((u, b), 1) \in E$ and $\eta_2((u, b), 1) > c - \min\{\varepsilon, \delta\}$. Since

$$d(\eta((u, b), 1), (u, b)) \leq \sigma \qquad \text{and} \qquad \eta_2((u, b), 1) \leq b,$$

$(u, b) \notin \mathcal{O} \cup \mathcal{V}$. So, $\eta((u, b), 1) \leq c - \varepsilon$, which is a contradiction. ■

As a corollary, we get the two following theorems. The first one is obtained in taking $A = \text{graph } F$, $Q = \varnothing$ with the convention that $\sup \pi_{\mathbb{R}}(\varnothing) = -\infty$. The second one is a generalization of the well known Mountain Pass Theorem.

THEOREM 2.13 (Minimax Principle). Let X be a complete metric space, and $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph. Assume there exists $\Gamma_0 \subset \Gamma(\varnothing)$ nonempty and invariant with respect to $(F, \varnothing)$-deformations. Let

$$c = \inf_{U \in \Gamma_0} \sup \pi_{\mathbb{R}}(U).$$

If $c \in \mathbb{R}$, and F satisfies $(PS)_c$ then $K_c \times \{c\} \cap \overline{\Gamma_0} \neq \varnothing$.

THEOREM 2.14. Let X be a complete metric space, and $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph. Assume there exists a closed pair (D, S), a continuous function $\psi = (\psi_1, \psi_2): S \to \text{graph } F$, a closed subset $A \subset \text{graph } F$, and a nonempty subset $\Gamma_0 \subset \Gamma(\psi(S), D)$ invariant with respect to $(F, \psi(S))$-deformations such that Γ_0 intersects A. In addition, assume that

$$\inf_{f(D) \in \Gamma_0} \sup f_2(f^{-1}(A)) \geq \sup \psi_2(S),$$

with a strict inequality if $d(A, \psi(S)) = 0$. Let

$$c = \inf_{f(D) \in \Gamma_0} \sup f_2(D).$$

If $c \in \mathbb{R}$, and F satisfies $(PS)_c$ then $K_c \times \{c\} \cap \overline{\Gamma_0} \neq \emptyset$. Moreover, if

$$c = \inf_{f(D) \in \Gamma_0} \sup f_2(f^{-1}(A)),$$

then $K_c \times \{c\} \cap A \neq \emptyset$.

Remark 2.15. In the previous theorem, if (D, S) is a compact pair then $c \in \mathbb{R}$.

The following theorem gives the existence of a critical point which is a minimum of F on Γ_0. It is worthwhile to mention that this critical point is not necessarily a local minimum.

THEOREM 2.16. Let X be a complete metric space, and $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph. Assume there exists a closed subset A of graph F; and there exist $Q \subset$ graph F, and $\Gamma_0 \subset \Gamma(Q)$ nonempty and invariant with respect to (F, Q)-deformations such that Γ_0 intersects A, and

$$\inf_{U \in \Gamma_0} \inf \pi_{\mathbb{R}}(U \cap A) \leq \inf \pi_{\mathbb{R}}(Q).$$

with a strict inequality if $d(A, Q) = 0$. Let

$$c = \inf_{U \in \Gamma_0} \inf \pi_{\mathbb{R}}(U).$$

If $c > -\infty$ and F satisfies $(PS)_c$, then $K_c \times \{c\} \cap \overline{\Gamma_0} \neq \emptyset$.

Moreover, if

$$c = \inf_{U \in \Gamma_0} \inf \pi_{\mathbb{R}}(U \cap A),$$

then $K_c \times \{c\} \cap A \neq \emptyset$.

The proof is similar to the one of Theorem 2.12. Note that Q could be empty with the convention that $\inf \pi_{\mathbb{R}}(\emptyset) = \infty$. Taking A = graph F gives the following corollary.

COROLLARY 2.17. Let X be a complete metric space, and $F: X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph. Assume there exist a subset Q of graph F, and $\Gamma_0 \subset \Gamma(Q)$ nonempty and invariant with respect to (F, Q)-deformations. If

$$-\infty < c = \inf_{U \in \Gamma_0} \inf \pi_{\mathbb{R}}(U) < \inf \pi_{\mathbb{R}}(Q),$$

and F satisfies $(PS)_c$, then $K_c \times \{c\} \cap \overline{\Gamma_0} \neq \emptyset$.

We present some trivial examples to help to visualize the meaning of the previous results.

Example 2.18. Let $F: [-1, 1] \to \mathbb{R}$ be the multivalued function defined by $F(x) = \{|x| - 1/n : n \in \mathbb{N}\} \cup \{|x|\}$. By the previous corollary, 0 is a critical point of F at level 0 and at level $-1/n$ for every $n \in \mathbb{N}$. Observe that, in particular, 0 is not a local minimum value.

Example 2.19. Let $F: [-1, 1] \times [0, 1] \to \mathbb{R}$ be the function defined by

$$F(x, y) = \begin{cases} \{z : -2|x| \leq z \leq -3|x| + 1\}, & \text{if } y = 0; \\ \{-3|x| - 2y + 1\}, & \text{otherwise.} \end{cases}$$

Fix $S = \{-1, 1\}$, $\psi = (id, 0, -2)$. Theorem 2.14 applied with $\Gamma(\psi(S), [-1, 1])$ gives the critical point $(0, 1)$ at level -1. On the other hand, take $\Gamma_0 \subset \Gamma(\psi(S), [-1, 1])$ the set invariant with respect to $(F, \psi(S))$-deformations and generated by $\{(x, 0, -2|x|) : x \in [-1, 1]\}$. Theorem 2.14 applied with Γ_0 gives the critical point $(0, 0)$ at level 0. Observe that, in this example, it makes a difference to consider only deformations η satisfying $\eta_2((u, b), t) \leq b$. Without this restriction, it would not have been possible to obtain the critical point $(0, 0)$ from one of the previous theorems.

Example 2.20. Let $F: [-\pi^{-1}, \pi^{-1}] \to \mathbb{R}$ be the function defined by

$$F(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right) - x^2, & \text{if } x \neq 0, \\ [-1, 1], & \text{if } x = 0. \end{cases}$$

Take $(D, S) = ([-\pi^{-1}, \pi^{-1}], \{-\pi^{-1}, \pi^{-1}\})$, $\psi = (id, F|_S)$. It is clear that $\Gamma(\psi(S), D) = \emptyset$. Now, take $\Gamma_0 = \{\text{graph}\, F(D)\} \subset \Gamma(\psi(S))$, and $A = ([-(2\pi)^{-1}, (2\pi)^{-1}] \times \mathbb{R}) \cap \text{graph}\, F$. Theorem 2.12 gives the existence of the critical point 0 at level 1 while Theorem 2.14 could not be applied to get the critical point 0.

On the other hand, take $\tilde{\Gamma}_0$ the subset of $\Gamma(\emptyset)$ invariant with respect to $(F, \emptyset)$-deformations, and generated by $\text{graph}\, F \cap (D \times (-\infty, -1])$. Theorem 2.13 gives the existence of the critical point 0 at level -1.

3. LOWER SEMI-CONTINUOUS FUNCTIONALS

In this section, we want to mention how some of the previous results could be applied to lower semi-continuous functionals.

Let X be a metric space, and let $f: X \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous function. From this function, we define two multivalued mappings with closed graph:

$$\bar{f}(u) = \{c \in \mathbb{R} \cup \{\infty\} : \text{there exists } (u_n)_n \text{ such that } u_n \to u \text{ and } f(u_n) \to c\},$$

and

$$\text{conv}(f)(u) = (\text{conv}\, \bar{f}(u)).$$

Results of the previous section could be applied to $\bar{f}$ and to $\text{conv}(f)$. Let us mention that results obtained in [2, 3] for lower semi-continuous functionals concern the multivalued functional $\text{conv}(f)$.

In the particular case where $\bar{f}(u) \subset \{f(u), \infty\}$ then, as it is shown in the next lemma, using $\bar{f}$ or conv(f) makes no difference. This holds, for example, for $f = f_0 + f_1$ with f_0 continuous, and $f_1 = I_K$ where K is a closed subset of X. Moreover, in this case, if u is a critical point of $\bar{f}$, it has a unique critical value which is $f(u)$.

Next lemma establishes that u is never a critical point of conv(f) at level c if $c \in \mathrm{conv}(f)(u)\backslash\bar{f}(u)$.

LEMMA 3.1. Let $f: X \to \mathbb{R}$ be a lower semi-continuous function, and let $u \in X$ and c such that $f(u) < c < \limsup_{v \to u} f(v)$, and there is no sequence (u_n) converging to u such that $f(u_n) \to c$. Then $|d\,\mathrm{conv}(f)|(u, c) = \infty$.

Proof. Let $u \in X$ and c such that $f(u) < c < \limsup_{v \to u} f(v)$, and there is no sequence (u_n) converging to u such that $f(u_n) \to c$. Then there exists $\varepsilon > 0$ such that $B(u, 2\varepsilon) \times B(c, 2\varepsilon) \cap \mathrm{graph}\,\bar{f} = \emptyset$. Let $\alpha > 1$, and fix $\delta = \varepsilon/\alpha$. Define H: graph(conv(f)) $\cap$ $B((u, c), \delta) \times [0, \delta] \to$ graph(conv(f)) by

$$H((v, b), t) = (v, b - \alpha t).$$

The function H is well defined. Indeed, if $(v, b) \in$ graph(conv(f)) $\cap\, B((u, c), \delta)$, then $b \geq c - \varepsilon$, and $f(v) \leq c - 2\varepsilon$, and hence $b - \alpha\delta \geq f(v)$. So

$$H((v, b), t) \in \mathrm{graph}(\mathrm{conv}(f)).$$

Therefore, $|d\,\mathrm{conv}(f)\,|(u, c) \geq \alpha$. Since α is arbitrary, we get the conclusion. ∎

In general, we can not compare $|d\bar{f}|(u, c)$ and $|d\,\mathrm{conv}(f)|(u, c)$. Indeed, take

$$f(u) = \begin{cases} 0, & \text{if } u \leq 0, \\ 1 + u, & \text{if } u > 0, \end{cases}$$

then $|d\bar{f}|(0, 1) = 0$ while $|d\,\mathrm{conv}(f)|(0, 1) = 1$. For

$$g(u) = \begin{cases} 0, & \text{if } u \leq 0, \\ 1 - u, & \text{if } u > 0, \end{cases}$$

$|d\bar{g}|(0, 1) = 1$ while $|d\,\mathrm{conv}(g)|(0, 1) = 0$. Observe that, in this last example, 0 is a mountain pass point for conv(g) but not for $\bar{g}$.

It is also possible to have $|d\,\mathrm{conv}(f)|(u, f(u)) \neq |d\bar{f}|(u, f(u))$. For example, take the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x, y) = \begin{cases} x, & \text{if } x \leq 0 \text{ or } y \leq 0, \\ 2x, & \text{if } y = x(-x + 1/n),\ x \in (0, 1/n),\ n \in \mathbb{N}, \\ 3, & \text{otherwise}. \end{cases}$$

Then $|d\bar{f}|(0, 0) = 1$ while $|d\,\mathrm{conv}(f)|(0, 0) = 0$.

4. APPLICATION TO PARTIAL DIFFERENTIAL INCLUSIONS

In this section, we present an application of Theorem 2.14 to partial differential inclusions. More precisely, we give an existence result for the following problem:

$$
\begin{aligned}
-\Delta y(t) &\in G(y(t)) \quad \text{a.e. } t \in \Omega, \\
y(t) &= 0, \qquad \text{for every } t \in \partial\Omega,
\end{aligned}
\tag{4.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $G: \mathbb{R} \to \mathbb{R}$ is a multivalued mapping.

By a solution of (4.1), we mean a function $y \in H_0^1(\Omega)$ such that $\Delta y \in L^q(\Omega)$ for some q which will be determined later, and satisfying (4.1).

Now, let us state the main result of this section which is inspired of Theorem 2.15 in [17].

THEOREM, 4.1. Let $G: \mathbb{R} \to \mathbb{R}$ be an upper semi-continuous multivalued mapping with nonempty, compact, convex values. Assume

(4.1.1) there exist $0 \le \mu < (n+2)/(n-2)$, and constants a, b such that

$$|G(x)| = \max\{|z| : z \in G(x)\} \le a + b|x|^\mu \qquad \text{for every } x \in \mathbb{R};$$

(4.1.2) there exist $\beta > 2$ and $R > 0$ such that for every x such that $|x| \ge R$,

$$0 < \beta \int_0^x G(y)\,\mathrm{d}y \le \min(xG(x)) = \min\{xz : z \in G(x)\};$$

(4.1.3) the following inequality is satisfied:

$$\limsup_{x \to 0} \frac{\max(xG(x))}{x^2} \le 0,$$

where $\max(xG(x)) = \max\{xz : z \in G(x)\}$.

Then the problem (4.1) has a nontrivial solution.

Let us mention that the restriction on μ could be weakened if $n = 1$ or 2.

We denote

$$\mathcal{S}(G) = \{g: \mathbb{R} \to \mathbb{R} : g \text{ is measurable, and } g(x) \in G(x) \text{ a.e. } x \in \mathbb{R}\}.$$

It is well known that $\mathcal{S}(G)$ is nonempty. For each $g \in \mathcal{S}(G)$, we define a functional $J_g: H_0^1(\Omega) \to \mathbb{R}$ by

$$J_g(y) = \int_\Omega \frac{|\nabla y(t)|^2}{2}\,\mathrm{d}t - \int_\Omega \int_0^{y(t)} g(x)\,\mathrm{d}x\,\mathrm{d}t.$$

The assumption (4.1.1) implies that J_g is well defined.

Let us define the multivalued functional $F: L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ where $p \ge \mu + 1$ and $p \in (2, 2n/(n-2))$, by

$$F(y) = \begin{cases} \{c \in \mathbb{R} : J_g(y) \le c \text{ for some } g \in \mathcal{S}(G)\}, & \text{if } y \in H_0^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 4.2. If (4.1.1) is satisfied then for every bounded set B in $L^p(\Omega)$ and every $M \in \mathbb{R}$, the set $\mathcal{E} = \{y \in B : F(y) \cap (-\infty, M] \neq \emptyset\}$ is weakly relatively compact in $H_0^1(\Omega)$), and relatively compact in $L^r(\Omega)$ for $r \in [1, 2n(n-2)^{-1})$. Indeed, for $y \in \mathcal{E}$, let $g \in \mathcal{S}(G)$ such that $J_g(y) \leq M$. Then,

$$\int_\Omega \frac{|\nabla y(t)|^2}{2} \, dt \leq M + \int_\Omega \int_0^{y(t)} g(x) \, dx \, dt \leq M + \int_\Omega a_1 + b_1 |y(t)|^{\mu+1} \, dt \leq K.$$

So, $\mathcal{E}$ is bounded in $H_0^1(\Omega)$.

LEMMA 4.3. Assume (4.1.1) is satisfied. Let F be the functional defined above. Then F has closed graph, and nonempty, closed, convex values.

Proof. It is clear that F has nonempty, convex values. Let (y_n) and (c_n) be sequences such that $c_n \in F(y_n)$, $c_n \to c \in \mathbb{R}$, and $y_n \to y \in L^p(\Omega)$. Then there exists (g_n) a sequence in $\mathcal{S}(G)$ such that $J_{g_n}(y_n) \leq c_n$. The Remark 4.2 implies that, up to a subsequence, (y_n) is weakly convergent in $H_0^1(\Omega)$. From this, we deduce that $y \in H_0^1(\Omega)$.

Since G is upper semi-continuous with compact values, we have that $G[-k, k]$ is compact for every $k > 0$. Fix $r \in (1, \infty)$, since (g_n) is bounded in $L^r([-k, k])$ for every $k > 0$, we deduce the existence of $g \in L^r_{\mathrm{loc}}(\mathbb{R})$ such that a subsequence still denoted (g_n) converges weakly to g in $L^r[-k, k]$ for every $k > 0$, also we have

$$g(x) \in \overline{\mathrm{co}}\{g_n(x), g_{n+1}(x), \ldots\} \qquad \text{a.e. } x \in \mathbb{R}.$$

This and the fact that $g_n \in \mathcal{S}(G)$ and G is upper semi-continuous with compact, convex values imply that $g(x) \in G(x)$ a.e. $x \in \mathbb{R}$.

Without lost of generality, we assume that $y_n(t) \to y(t)$ for almost every $t \in \Omega$. Therefore,

$$\int_0^{y_n(t)} g_n(x) \, dx \to \int_0^{y(t)} g(x) \, dx \qquad \text{a.e. } t \in \Omega.$$

The Lebesgue convergence dominated theorem implies that

$$\int_\Omega \int_0^{y_n(t)} g_n(x) \, dx \, dt \to \int_\Omega \int_0^{y(t)} g(x) \, dx \, dt.$$

Thus,

$$J_g(y) \leq \liminf_{n \to \infty} J_{g_n}(y_n) \leq \lim_{n \to \infty} c_n = c,$$

and hence, $c \in F(y)$. ■

Now, we would like to show that critical points of F are solutions of (4.1).

For $y \in H_0^1(\Omega)$, we defined the possibility empty set

$$\mathcal{A}(y) = \left\{ \alpha \in L^q(\Omega) : \int_\Omega \alpha(t) w(t) \, dt \leq \int_\Omega \nabla y(t) \cdot \nabla w(t) - \min(w(t) G(y(t))) \, dt \right.$$
$$\left. \text{for every } w \in C_c^\infty(\Omega) \right\},$$

where $1/p + 1/q = 1$.

LEMMA 4.4. Assume (4.1.1) is satisfied. Let $(y, c) \in \operatorname{graph} F$ be such that $|dF|(y, c) < \infty$. Then $\Delta y \in L^q(\Omega)$. Moreover, $\mathcal{A}(y) \neq \emptyset$, and for every $\alpha \in \mathcal{A}(y)$

$$-\Delta y(t) \in \alpha(t) + G(y(t)) \qquad \text{a.e. } t \in \Omega.$$

Proof. If

$$\sup\left\{\int_\Omega \nabla y(t) \cdot \nabla w(t)\, dt : w \in C_c^\infty(\Omega) \text{ and } \|w\|_{L^p} \leq 1\right\} < \infty, \tag{4.2}$$

then

$$w \mapsto \int_\Omega \nabla y(t) \cdot \nabla w(t)\, dt$$

could be extended to a linear functional on $L^p(\Omega)$, and by the Riesz Representation Theorem, there exists $v \in L^q(\Omega)$ such that

$$\int_\Omega \nabla y(t) \cdot \nabla w(t)\, dt = -\int_\Omega v(t) w(t)\, dt$$

for every $w \in C_c^\infty(\Omega)$. In consequence, $\Delta y \in L^q(\Omega)$.

Moreover, let z be a measurable function such that $z(t) \in G(y(t))$ a.e. $t \in \Omega$. The assumption (4.1.1) implies that $z \in L^q(\Omega)$. It is clear that $\alpha = -\Delta y - z \in \mathcal{A}(y)$. Inversely, it is easy to see that if $\alpha \in \mathcal{A}(y)$, then $-\Delta y(t) \in \alpha(t) + G(y(t))$ a.e. $t \in \Omega$.

It remains to show that (4.2) is satisfied. Assume this is not true. Then for every $M > 0$, there exists $w \in C_c^\infty(\Omega)$ with $\|w\|_{L^p} \leq 1$ such that

$$\int_\Omega \nabla y(t) \cdot \nabla w(t)\, dt > M + \int_\Omega \max(w(t)G(y(t)))\, dt.$$

This inequality implies that for $\varepsilon < M/3$, there exists $\delta_1 > 0$ such that for every $(v, \xi) \in B((y, c), \delta_1) \cap \operatorname{graph} F$,

$$\int_\Omega \nabla v(t) \cdot \nabla w(t)\, dt > M - \varepsilon + \int_\Omega \max(w(t)G(y(t)))\, dt. \tag{4.3}$$

On the other hand, there exists $\delta_2 > 0$ such that for every $(v, \xi) \in B((y, c), \delta_2) \cap \operatorname{graph} F$ and every $s \in [0, \delta_2]$,

$$\int_\Omega \int_{v(t)-sw(t)}^{v(t)} g(x)\, dx\, dt - s\int_\Omega \max(w(t)G(y(t)))\, dt \leq s\varepsilon, \tag{4.4}$$

for every $g \in \mathcal{S}(G)$ such that $J_g(v) \leq \xi$. Indeed, otherwise, there would exist $(v_n, \xi_n) \to (y, c)$ in $\operatorname{graph} F \cap L^p(\Omega) \times \mathbb{R}$, and $s_n \to 0$ such that there exist $g_n \in \mathcal{S}(G)$ with $J_{g_n}(v_n) \leq \xi_n$, and

$$\int_\Omega \int_{v_n(t)-s_n w(t)}^{v_n(t)} g_n(x)\, dx\, dt - s_n \int_\Omega \max(w(t)G(y(t)))\, dt > s_n \varepsilon.$$

This is equivalent to

$$\int_\Omega \frac{1}{s_n} \int_{v_n(t)-s_n w(t)}^{v_n(t)} g_n(x)\, dx\, dt > \varepsilon + \int_\Omega \max(w(t)G(y(t)))\, dt.$$

Let $\hat{z}$ be a measurable function such that $\hat{z}(t) \in G(y(t))$ a.e. $t \in \Omega$. Denote

$$h_n(t) = \begin{cases} \dfrac{1}{s_n w(t)} \displaystyle\int_{v_n(t)-s_n w(t)}^{v_n(t)} g_n(x)\,\mathrm{d}x, & \text{if } w(t) \neq 0, \\ \hat{z}(t), & \text{if } w(t) = 0. \end{cases}$$

Since

$$\left| \frac{1}{s_n w(t)} \int_{v_n(t)-s_n w(t)}^{v_n(t)} g_n(x)\,\mathrm{d}x \right| = \left| \int_0^1 g_n(v_n(t) - \lambda s_n w(t))\,\mathrm{d}\lambda \right|$$
$$\leq \int_0^1 a + b(|v_n(t)| + |s_n w(t)|)^\mu \,\mathrm{d}\lambda,$$

The sequence (h_n) is bounded in $L^q(\Omega)$. In consequence, up to a subsequence, (h_n) converges weakly to some z in $L^q(\Omega)$, and

$$z(t) \in \overline{\mathrm{co}}\{h_n(t), h_{n+1}(t), \ldots\} \qquad \text{a.e. } t \in \Omega.$$

Moreover,

$$g_n(x) \in \bigcup_{y \in E_n(t)} G(y) \qquad \text{for every } x \in E_n(t),$$

where $E_n(t) = \mathrm{co}\{v_n(t), v_n(t) - s_n w(t)\}$. This, and the definition of h_n, imply that

$$h_n(t) \in \bigcup_{y \in E_n(t)} G(y) \qquad \text{for every } t \in \{t : w(t) \neq 0\}.$$

Without loss of generality, we can assume that $v_n(t) \to y(t)$ a.e. $t \in \Omega$.

From those facts and the assumption that G is upper semi-continuous with convex, compact values, we deduce that $z(t) \in G(y(t))$ a.e. $t \in \Omega$, and hence,

$$z(t)w(t) \leq \max(w(t)G(y(t))) \qquad \text{a.e. } t \in \Omega.$$

It follows from this inequality and the weak convergence of (h_n) to z, that

$$\varepsilon + \int_\Omega \max(w(t)G(y(t)))\,\mathrm{d}t \leq \int_\Omega \frac{1}{s_n} \int_{v_n(t)-s_n w(t)}^{v_n(t)} g_n(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_\Omega h_n(t)w(t)\,\mathrm{d}t$$
$$\to \int_\Omega z(t)w(t)\,\mathrm{d}t$$
$$\leq \int_\Omega \max(w(t)G(y(t)))\,\mathrm{d}t,$$

which is a contradiction.

Let $\delta = \min\{\delta_1, \delta_2, 2\varepsilon/\|\nabla w\|_{L^2}^2\}$. Define $H: B((y, c), \delta) \cap \operatorname{graph} F \times [0, \delta] \to \operatorname{graph} F$ by

$$H((v, \xi), s) = (v - sw, \xi - s(M - 3\varepsilon)).$$

The function H is well defined. Indeed, let $g \in \mathcal{S}(G)$ such that $J_g(v) \le \xi$. Then, from (4.3), (4.4), we get

$$\begin{aligned}
J_g(v - sw) &= J_g(v) - s\int_\Omega \nabla v(t)\cdot\nabla w(t)\,\mathrm{d}t + \frac{s^2}{2}\int_\Omega |\nabla w(t)|^2\,\mathrm{d}t \\
&\quad + \int_\Omega \int_{v(t)-sw(t)}^{v(t)} g(x)\,\mathrm{d}x\,\mathrm{d}t \\
&= J_g(v) - s\left[\int_\Omega \nabla v(t)\cdot\nabla w(t)\,\mathrm{d}t - \int_\Omega \max(w(t)G(y(t)))\,\mathrm{d}t\right] \\
&\quad + \int_\Omega \int_{v(t)-sw(t)}^{v(t)} g(x)\,\mathrm{d}x\,\mathrm{d}t - s\int_\Omega \max(w(t)G(y(t)))\,\mathrm{d}t \\
&\quad + \frac{s^2}{2}\int_\Omega |\nabla w(t)|^2\,\mathrm{d}t \\
&\le \xi - s(M-\varepsilon) + s\varepsilon + s\varepsilon \\
&= \xi - s(M - 3\varepsilon).
\end{aligned}$$

Thus, $H((v,\xi),s) \in \operatorname{graph} F$.

Therefore, $|dF|(y,c) \ge M - 3\varepsilon$. This contradicts that $|dF|(y,c) < \infty$, since M is arbitrary. ■

LEMMA 4.5. Let $y \in H_0^1(\Omega)$ such that $\Delta y \in L^q(\Omega)$. Then there exist $\alpha \in \mathcal{A}(y)$ and $w \in L^p(\Omega)$ with $\|w\|_{L^p} \le 1$ such that

$$\|\alpha\|_{L^q} = \int_\Omega \alpha(t)w(t)\,\mathrm{d}t \le \int_\Omega \tilde\alpha(t)w(t)\,\mathrm{d}t$$

for every $\tilde\alpha \in \mathcal{A}(y)$.

Proof. First of all, from the proof of Lemma 4.4, we know that if $\Delta y \in L^q(\Omega)$ then $\mathcal{A}(y) \ne \varnothing$. Moreover, it is easy to see that $\mathcal{A}(y)$ is closed, convex, and bounded.

Let us define $K\colon \mathcal{A}(y) \times \overline{B_{L^p}(0,1)} \to \mathcal{A}(y) \times \overline{B_{L^p}(0,1)}$ by

$$K(\tilde\alpha,\tilde w) = \left\{(\alpha, w) \in \mathcal{A}(y)\times\overline{B_{L^p}(0,1)} : \int_\Omega \tilde\alpha(t)w(t) - \alpha(t)\tilde w(t)\,\mathrm{d}t \ge 0\right\}.$$

For every $(\tilde\alpha,\tilde w) \in \mathcal{A}(y)\times\overline{B_{L^p}(0,1)}$, the set $K(\tilde\alpha,\tilde w)$ is closed, and convex.

The map K is KKM. Indeed, if there exist $(\tilde\alpha_1,\tilde w_1),\ldots,(\tilde\alpha_n,\tilde w_n)$ such that

$$\operatorname{conv}\{(\tilde\alpha_1,\tilde w_1),\ldots,(\tilde\alpha_n,\tilde w_n)\} \not\subset \bigcup_{i=1}^{n} K(\tilde\alpha_i,\tilde w_i),$$

this means that there exist $\lambda_1,\ldots,\lambda_n \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$, and

$$\sum_{i=1}^{N} \lambda_i(\tilde\alpha_i,\tilde w_i) \notin K(\tilde\alpha_j,\tilde w_j) \qquad \text{for } j = 1,\ldots,n.$$

That is,

$$\int_\Omega \tilde{\alpha}_j(t)\left(\sum_{i=1}^n \lambda_i \tilde{w}_i(t)\right) - \left(\sum_{i=1}^n \lambda_i \tilde{\alpha}_i(t)\right)\tilde{w}_j(t)\,dt < 0 \qquad \text{for } j = 1, \ldots, n.$$

This implies that

$$\sum_{j=1}^n \lambda_j\left(\int_\Omega \tilde{\alpha}_j(t)\left(\sum_{i=1}^n \lambda_i \tilde{w}_i(t)\right) - \left(\sum_{i=1}^n \lambda_i \tilde{\alpha}_i(t)\right)\tilde{w}_j(t)\,dt\right) < 0,$$

which is a contradiction.

The Elementary KKM Principle [18, Theorem 5.2] gives the existence of $(\alpha, w) \in K(\tilde{\alpha}, \tilde{w})$ for every $(\tilde{\alpha}, \tilde{w}) \in \mathcal{Q}(y) \times \overline{B_{L^p}(0, 1)}$. That is,

$$\int_\Omega \tilde{\alpha}(t)w(t)\,dt \geq \int_\Omega \alpha(t)\tilde{w}(t)\,dt \qquad \text{for every } (\tilde{\alpha}, \tilde{w}) \in \mathcal{Q}(y) \times \overline{B_{L^p}(0, 1)}.$$

Hence,

$$\int_\Omega \tilde{\alpha}(t)w(t)\,dt \geq \int_\Omega \alpha(t)w(t)\,dt = \|\alpha\|_{L^p}. \quad \blacksquare$$

LEMMA 4.6. Assume (4.1.1) is satisfied. If $|dF|(y, c) < \infty$ then there exists $\alpha \in \mathcal{Q}(y)$ such that $\|\alpha\|_{L^q} \leq |dF|(y, c)$.

Proof. By the previous lemmas, $\Delta y \in L^q(\Omega)$, $\mathcal{Q}(y) \neq \emptyset$, and there exist $\alpha \in \mathcal{Q}(y)$ and $\hat{w} \in L^p(\Omega)$ with $\|\hat{w}\|_{L^p} \leq 1$ such that

$$\|\alpha\|_{L^q} = \int_\Omega \alpha(t)\hat{w}(t)\,dt \leq \int_\Omega \tilde{\alpha}(t)\hat{w}(t)\,dt$$

for every $\tilde{\alpha} \in \mathcal{Q}(y)$.

If $\|\alpha\|_{L^q} = 0$, the proof is complete. If $\|\alpha\|_{L^q} > 0$. Let $\varepsilon < \|\alpha\|_{L^q}/2$. There exists $w \in C_c^\infty(\Omega)$ with $\|w\|_{L^p} \leq 1$ and $\|\hat{w} - w\|_{L^p}$ small enough such that

$$\|\alpha\|_{L^q} - \varepsilon < \int_\Omega \tilde{\alpha}(t)w(t)\,dt \tag{4.5}$$

for every $\tilde{\alpha} \in \mathcal{Q}(y)$.

Moreover, there exists $\delta_1 > 0$ such that for every $(v, \xi) \in B((y, c), \delta_1) \cap \operatorname{graph} F$, and $s \in [0, \delta_1]$, we have

$$s\int_\Omega \nabla v(t) \cdot \nabla w(t)\,dt - \int_\Omega \int_{v(t)-sw(t)}^{v(t)} g(x)\,dx\,dt \geq s(\|\alpha\|_{L^q} - \varepsilon) \tag{4.6}$$

for every $g \in \mathcal{S}(g)$ such that $J_g(v) \leq \xi$. Indeed, assume there exists $(v_n, \xi_n) \to (y, c)$ in graph $F \cap L^p(\Omega) \times \mathbb{R}$, $s_n \to 0$, and $g_n \in \mathcal{S}(G)$ such that $J_{g_n}(v_n) \leq \xi_n$ and

$$\|\alpha\|_{L^q} - \varepsilon > \int_\Omega \nabla v_n(t) \cdot \nabla w(t)\,dt - \frac{1}{s_n}\int_\Omega \int_{v_n(t)-s_n w(t)}^{v_n(t)} g_n(x)\,dx\,dt.$$

Arguing as in the proof of Lemma 4.4 gives the existence of a measurable function $z(t) \in G(y(t))$ a.e. $t \in \Omega$ such that, up to a subsequence,

$$\int_\Omega \nabla v_n(t) \cdot \nabla w(t)\,dt - \frac{1}{s_n}\int_\Omega \int_{v_n(t)-s_n w(t)}^{v_n(t)} g_n(x)\,dx\,dt$$

$$\to \int_\Omega \nabla y(t) \cdot \nabla w(t)\,dt - \int_\Omega z(t)w(t)\,dt$$

$$= \int_\Omega (-\Delta y(t) - z(t))w(t)\,dt.$$

In consequence, $\tilde{\alpha} = -\Delta y - z \in \mathcal{A}(y)$ and

$$\int_\Omega \tilde{\alpha}(t)w(t)\,dt \leq \|\alpha\|_{L^q} - \varepsilon,$$

which contradicts (4.5).

Let $\delta = \min\{\delta_1, 2\varepsilon/\|\nabla w\|_{L^2}^2\}$. Define $H\colon B((y, c), \delta) \cap \operatorname{graph} F \times [0, \delta] \to \operatorname{graph} F$ by

$$H((v, \xi), s) = (v - sw, \xi - s(\|\alpha\|_{L^q} - 2\varepsilon)).$$

The function H is well defined. Indeed, let $g \in \mathcal{S}(G)$ such that $J_g(v) \leq \xi$. Then, from (4.6), we get

$$J_g(v - sw) = J_g(v) - s\int_\Omega \nabla v(t) \cdot \nabla w(t)\,dt$$

$$+ \int_\Omega \int_{v(t)-sw(t)}^{v(t)} g(x)\,dx\,dt + \frac{s^2}{2}\int_\Omega |\nabla w(t)|^2\,dt$$

$$\leq \xi - s(\|\alpha\|_{L^q} - \varepsilon) + s\varepsilon$$

$$= \xi - s(\|\alpha\|_{L^q} - 2\varepsilon).$$

Thus, $H((v, \xi), s) \in \operatorname{graph} F$.

Therefore, $|dF|(y, c) \geq \|\alpha\|_{L^q} - 2\varepsilon$. This implies that $|dF|(y, c) \geq \|\alpha\|_{L^q}$, since ε is arbitrary. ■

In the following lemma, we show that the functional F satisfies the Palais-Smale condition. The fact that the functional is defined on $L^p(\Omega)$ simplifies the proof.

LEMMA 4.7. Assume (4.1.1) and (4.1.2) are satisfied. Then F satisfies $(PS)_c$ for every $c \in \mathbb{R}$.

Proof. Let $(y_n) \in L^p(\Omega)$ and $c_n \in F(y_n)$ such that $c_n \to c$ and $|dF|(y_n, c_n) \to 0$. Then, Lemma 4.6 gives the existence of $\alpha_n \in \mathcal{A}(y_n)$ with $\|\alpha_n\|_{L^p} \leq |dF|(y_n, c_n)$. Also, $\Delta y_n \in L^q(\Omega)$ by Lemma 4.4.

Let us denote $z_n = -\Delta y_n - \alpha_n \in G(y_n)$, and let $g_n \in \mathcal{S}(G)$ be such that $J_{g_n}(y_n) \le c_n$. So, by assumptions (4.1.1) and (4.1.2),

$$\begin{aligned}
c_n &\ge J_{g_n}(y_n)\\
&= \int_\Omega \frac{|\nabla y_n(t)|^2}{2}\,dt - \frac{1}{\beta}\int_\Omega z_n(t)y_n(t)\,dt + \frac{1}{\beta}\int_\Omega z_n(t)y_n(t)\,dt - \int_\Omega \int_0^{y_n(t)} g_n(x)\,dx\,dt\\
&= \left(\frac{1}{2} - \frac{1}{\beta}\right)\int_\Omega |\nabla y_n(t)|^2\,dt + \frac{1}{\beta}\int_\Omega \alpha_n(t)y_n(t)\,dt\\
&\quad + \int_{\{t: y_n(t) > R\}} \left(\frac{1}{\beta} z_n(t)y_n(t) - \int_0^{y_n(t)} g_n(x)\,dx\right)dt\\
&\quad + \int_{\{t: y_n(t) \le R\}} \left(\frac{1}{\beta} z_n(t)y_n(t) - \int_0^{y_n(t)} g_n(x)\,dx\right)dt\\
&\ge \left(\frac{1}{2} - \frac{1}{\beta}\right)\|y_n\|^2_{H_0^1} - k\|\alpha_n\|_{L^q}\|y_n\|_{H_0^1} - K\\
&\ge \left(\frac{1}{2} - \frac{1}{\beta}\right)\|y_n\|^2_{H_0^1} - k|dF|(y_n, c_n)\|y_n\|_{H_0^1} - K.
\end{aligned}$$

Therefore, (y_n) is bounded in $H_0^1(\Omega))$, and hence, relatively compact in $L^p(\Omega)$. ∎

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. From condition (4.1.3), for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$G(x) \le \varepsilon x \quad \text{if } x \in [0, \delta] \qquad \text{and} \qquad G(x) \ge \varepsilon x \quad \text{if } x \in [-\delta, 0].$$

The assumption (4.1.1) implies that there exists a constant $a_1 > 0$ such that

$$|G(x)| \le a_1|x|^\mu \qquad \text{if } |x| \ge \delta.$$

Thus, for every $y \in \mathbb{R}$ and every $g \in \mathcal{S}(G)$,

$$\int_0^y g(x)\,dx \le \varepsilon\frac{y^2}{2} + a_2|y|^{\mu+1}.$$

This implies that

$$J_g(y) \ge \int_\Omega \frac{|\nabla y(t)|^2}{2}\,dt - \frac{\varepsilon}{2}\|y\|^2_{L^p} - a_2\|y\|^p_{L^p} \ge \left(a_3 - \frac{\varepsilon}{2}\right)\|y\|^2_{L^p} - a_2\|y\|^p_{L^p},$$

for every $y \in H_0^1(\Omega)$ and every $g \in \mathcal{S}(g)$. Therefore, there exist $\hat{c} > 0$ and $r > 0$ such that

$$J_g(y) \ge \hat{c},$$

for every $g \in \mathcal{S}(G)$ and every $y \in H_0^1(\Omega)$ such that $\|y\|_{L^p} = r$. In other words,

$$F(y) \subset [\hat{c}, \infty],$$

for every $y \in L^p(\Omega)$ such that $\|y\|_{L^p} = r$.

On the other hand, the assumption (4.1.2) implies that there exist constants $k, l > 0$ such that for every $y \in \mathbb{R}$ and every $g \in \mathbb{S}(G)$,

$$\int_0^y g(x)\,\mathrm{d}x \geq k|y|^\beta - l. \tag{4.7}$$

Indeed, if $y > R$,

$$0 < \beta \int_0^y G(x)\,\mathrm{d}x \leq \min(yG(y)).$$

So,

$$\frac{\beta}{y} \leq \frac{\min(G(y))}{\int_0^y \min(G(x))\,\mathrm{d}x}.$$

By integrating from R to y, we get

$$\log\left(\frac{y}{R}\right)^\beta \leq \log\left(\frac{\int_0^y \min(G(x))\,\mathrm{d}x}{\int_0^R \min(G(x))\,\mathrm{d}x}\right).$$

Hence,

$$\begin{aligned} k_1 y^\beta &= \left(\frac{y}{R}\right)^\beta \int_0^R \min(G(x))\,\mathrm{d}x \\ &\leq \int_0^y \min(G(x))\,\mathrm{d}x \\ &\leq \int_0^y g(x)\,\mathrm{d}x. \end{aligned}$$

Similarly, if $y < -R$,

$$k_1|y|^\beta \leq \int_0^y g(x)\,\mathrm{d}x.$$

From (4.7), we deduce that for $y \in H_0^1(\Omega)$ and $g \in \mathbb{S}(G)$,

$$J_g(\lambda y) \leq \int_\Omega \lambda^2 \frac{|\nabla y(t)|^2}{2} - (k|\lambda y(t)|^\beta - l)\,\mathrm{d}t \to -\infty$$

as $|\lambda| \to \infty$.

From that, we can choose $(y, \xi) \in$ graph F such that $\|y\|_{L^p} > r$ and $\xi \leq 0$. Observe that $F(0) = [0, \infty)$.

Let us set $D = [0, 1]$, $S = \{0, 1\}$, $\psi(0) = (y, \xi)$, $\psi(1) = (0, 0)$, $A = \text{graph}\, F \cap \{y \in L^p(\Omega) : \|y\|_{L^p} = r\} \times \mathbb{R}$. It is clear that $\Gamma(\psi(S), D) \neq \emptyset$, $\Gamma(\psi(S), D)$ intersects A, $d(A, \psi(S)) > 0$, and that

$$\hat{c} \leq \inf_{f(D) \in \Gamma(\psi(S), D)} \sup f_2(D) = c < \infty.$$

Observe that F satisfies $(PS)_c$ by Lemma 4.7. The Theorem 2.14 gives the existence of a critical point y of F at level c. Lemmas 4.4 and 4.6 imply that y is a solution of (4.1). ■

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