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Nonlinear First-Order Initial and Periodic Problems in Banach Spaces

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Abstract—In this paper, we establish some existence results for periodic and initial value problems for first-order ordinary differential equations in Banach space, where the right member f has a decomposition $f = g + h$ with g and h satisfying, respectively, a compactness and Lipschitz assumptions. Our results extend results of [1].

Keywords—Cauchy, Periodic, Abstract equations, Nonlinear.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we are concerned with the periodic and initial value problems

$$y'(t) = f(t, y(t)) \text{ a.e. } t \in [0,T], \qquad y(0) = a \in E; \tag{1}$$

$$y'(t) + cy(t) = f(t, y(t)) \text{ a.e. } t \in [0,T], \qquad y(0) = y(T); \tag{2}$$

where E is a real Banach space and $f : [0,T] \times E \to E$ has a decomposition $f = g + h$ with g and h Carathéodory functions satisfying, respectively, a compactness and Lipschitz assumptions.

The paper is divided in three sections. In the first one, we give some preliminaries. In Section 2, we obtain existence results without assuming any growth restriction. To our knowledge, it is the first time that existence results to such problems in a Banach space (which is not necessarily Hilbert) are obtained without growth restriction on the right member. Using the semi-inner product, we impose a condition which coincides with the existence of upper and lower solutions in the scalar case. In Section 3, we show that some classical existence results for the initial value problem can be extended to problem in Banach space. The results of this paper generalize and complement results of [1].

Throughout, E is a real Banach space with norm $\|\cdot\|$. We denote by $C([0,T],E)$, the space of continuous functions $u : [0,T] \to E$. Let $u : [0,T] \to E$ be a measurable function. By $\int_0^T u(t)\,dt$, we mean the Bochner integral of u, assuming it exists. We define $W^{1,1}([0,T],E)$ as the set of

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX continuous functions u such that there exists $v \in L^1([0,T],E)$ with $u(t)-u(0)=\int_0^t v(s)\,ds$, for all t in $[0,T]$. Notice that if $u \in W^{1,1}([0,T],E)$, then u is differentiable almost everywhere on $[0,T]$, $u' \in L^1([0,T],E)$, and $u(t)-u(0)=\int_0^t u'(s)\,ds$ for t in $[0,T]$. By a *solution* to (1) or (2), we mean a function $u \in W^{1,1}([0,T],E)$ satisfying the differential equation (1) or (2), respectively. We deduce easily the following lemma from the previous definition.

LEMMA 1.1. *If $x \in W^{1,1}([0,T],E)$, then $\|x\| \in W^{1,1}([0,T],\mathbb{R})$.*

The semi-inner products on E are defined by

$$\langle x,y\rangle_+ = \|x\| \lim_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t},$$
$$\langle x,y\rangle_- = \|x\| \lim_{t\to 0^-} \frac{\|x+ty\|-\|x\|}{t}.$$

The reader is referred to [2,3] for more details.

LEMMA 1.2. *Let E be a real Banach space. The following properties are satisfied:*

(a) $|\langle x,y\rangle_\pm| \le \|x\|\,\|y\|$;
(b) $\langle \alpha x,\beta y\rangle_\pm = \alpha\beta\langle x,y\rangle_\pm$ *for all* $\alpha\beta \ge 0$;
(c) $\langle y,x+\alpha y\rangle_\pm = \langle y,x\rangle_\pm + \alpha\|y\|^2$ *for all* $\alpha \in \mathbb{R}$;
(d) $\langle x,y\rangle_- \le \langle x,y\rangle_+$;
(e) *if* $x:[0,T]\to E$ *is differentiable at* t, *then*

$$\|x(t)\|\, D^+\|x(t)\| = \langle x(t),x'(t)\rangle_+\,,$$
$$\|x(t)\|\, D^-\|x(t)\| = \langle x(t),x'(t)\rangle_-\,,$$

where D^+ and D^- are the upper right and left Dini derivatives, respectively.

A function $g:[0,T]\times E\to E$ is a *Carathéodory function* if:

(1) the map $t\to g(t,z)$ is measurable for each z in E;
(2) the map $z\to g(t,z)$ is continuous for almost all t in $[0,T]$;
(3) for each $r>0$, there exists $h_r \in L^1([0,T],\mathbb{R})$ such that $\|z\|\le r$ implies $\|g(t,z)\| \le h_r(t)$ for almost all t in $[0,T]$.

For the sake of completeness, let us recall [1, Theorem 2.1] which follows from the Krasnoselskii fixed point Theorem for contraction plus compact mappings.

THEOREM 1.3. *Assume $f:[0,T]\times E\to E$ has the decomposition $f=g+h$ with g and h Carathéodory functions such that*

(i) *for each $t\in[0,T]$, the set $\{\int_0^t g(s,u(s))\,ds : u\in C([0,T],E)\}$ is relatively compact;*
(ii) *there exists $q\in L^1([0,T],\mathbb{R})$ with $\|h(t,u)-h(t,v)\| \le q(t)\|u-v\|$ a.e. $t\in[0,T]$, and all $u,v\in E$.*

Then problem (1) has a solution.

Similarly, we obtain the following existence theorem.

THEOREM 1.4. *Let $c\neq 0$. Assume $f:[0,T]\times E\to E$ is as in the previous theorem. If*

$$\int_0^T q(t)\,dt < |c|T,$$

then problem (2) has a solution.

PROOF. We endow $C([0,T],E)$ with the norm

$$\|u\|_* = \begin{cases} \max\limits_{t\in[0,T]} \left\|e^{ct-Q(t)}u(t)\right\|, & \text{if } c>0,\\ \max\limits_{t\in[0,T]} \left\|e^{c(T-t)-\hat{Q}(t)}u(t)\right\|, & \text{if } c<0, \end{cases}$$

where $Q(t) = \int_0^t q(s)\,ds$ and $\hat{Q}(t) = \int_t^T q(t)\,dt$. Essentially, the same argument as in [1, Theorem 2.1] establishes the result. ∎

REMARK 1.5. Condition (i) is satisfied if g is a K-Carathéodory function, i.e., g is Carathéodory and is such that for each $r > 0$, there exist a nonnegative function $\eta_r \in L^1([0,T],\mathbb{R})$ and a compact set K_r in E such that $\|z\| \le r$ implies $g(t,z) \in \eta_r(t)K_r$, for almost all t in $[0,T]$, (see [4]).

In the sequel, we will say that a function $f : [0,T] \times E \to E$ satisfies *condition (A)* if

(a) f has the decomposition $f = g + h$ with g and h Carathéodory functions;
(b) for each $t \in [0,T]$ and $r > 0$, the set $\{\int_0^t g(s,u(s))\,ds : u \in C([0,T],E)$ with $\|u(s)\| \le r$, for all $s \in [0,T]\}$ is relatively compact;
(c) for each $r > 0$, there exists $q \in L^1([0,T],\mathbb{R})$ such that $\|h(t,u) - h(t,v)\| \le q(t)\|u - v\|$ a.e. $t \in [0,T]$, and all $u, v \in E$ with $\|u\|$, $\|v\| \le r$.

2. EXISTENCE RESULTS WITHOUT GROWTH CONDITION

We first present a result for the initial value problem which generalizes results in [1].

THEOREM 2.1. *Assume $f : [0,T] \times E \to E$ satisfies condition (A), and*

(i) *there exist $v \in W^{1,1}([0,T],E)$ and $M \in W^{1,1}([0,T],[0,\infty))$ such that*
$$\langle y - v(t), f(t,y) - v'(t)\rangle_- \le M(t)M'(t),$$
for a.e. $t \in [0,T]$ and all $y \in E$ with $\|y - v(t)\| = M(t)$;
(ii) *$f(t,v(t)) = v'(t)$ a.e. on $\{t \in [0,T] : M(t) = 0\}$;*
(iii) *$\|a - v(0)\| \le M(0)$.*

Then problem (1) has a solution such that $\|y(t) - v(t)\| \le M(t)$ for all $t \in [0,T]$.

PROOF. Consider the initial value problem
$$y'(t) = f\left(t, p(t,t(t))\right) \text{ a.e. } t \in [0,T], \qquad y(0) = a, \tag{3}$$
where
$$p(t,y) = \min\left\{1, \frac{M(t)}{\|y - v(t)\|}\right\} y + \left(1 - \min\left\{1, \frac{M(t)}{\|y - v(t)\|}\right\}\right) v(t).$$
It is easy to check that
$$\|p(t,y_1) - p(t,y_2)\| \le 2\,\|y_1 - y_2\|\,, \qquad \text{for all } y_1, y_2 \in E.$$
Theorem 1.3 implies that (3) has a solution y. It remains to show that $\|y(t) - v(t)\| \le M(t)$ for all $t \in [0,T]$.

Suppose there exists $t_1 \in (0,T]$ with $\|y(t_1) - v(t_1)\| > M(t_1)$. It follows from (iii) that there exists $t_0 \in [0,t_1)$ such that $\|y(t_0) - v(t_0)\| = M(t_0)$ and $\|y(t) - v(t)\| > M(t)$ for $t \in (t_0,t_1)$. We deduce from Lemmas 1.1 and 1.2 and condition (i) that a.e. on (t_0,t_1),
$$\begin{aligned}
\|y(t) - v(t)\|' &= D^+\,\|y(t) - v(t)\| = D^-\,\|y(t) - v(t)\| \\
&= \frac{\langle y(t) - v(t), y'(t) - v'(t)\rangle_-}{\|y(t) - v'(t)\|} \\
&= \frac{\langle p\left(t,y(t)\right) - v(t), y'(t) - v'(t)\rangle_-}{M(t)} \\
&\le M'(t).
\end{aligned}$$
Thus, the function $\|y - v\| - M$ is decreasing on (t_0,t_1), which is a contradiction. ∎

REMARK 2.2. In the scalar case, if $\alpha \le \beta \in W^{1,1}([0,T],\mathbb{R})$ are, respectively, lower and upper solutions of (1), then
$$v = \frac{\alpha + \beta}{2} \quad \text{and} \quad M = \frac{\beta - \alpha}{2},$$
satisfy (i), (ii), and (iii) in Theorem 2.1.

We have analogous results for the periodic problem.

THEOREM 2.3. *Let $c > 0$. Assume $f : [0,T] \times E \to E$ satisfies condition (A), and*

(i) *there exist $v \in W^{1,1}([0,T], E)$ and $M \in W^{1,1}([0,T], [0,\infty))$ such that*

$$\langle y - v(t), f(t,y) - (v'(t) + cv(t)) \rangle_- \leq M(t)\,(M'(t) + cM(t)),$$

for a.e. $t \in [0,T]$, and all $y \in E$ with $\|y - v(t)\| = M(t)$;

(ii) *$f(t, v(t)) = v'(t) + cv(t)$ a.e. on $\{t \in [0,T] : M(t) = 0\}$;*

(iii) *$\|v(0) - v(T)\| \leq M(0) - M(T)$.*

If $\int_0^T 2q(t)\,dt < cT$, where q is the function given in condition (A) for $r = \|v\|_0 + \|M\|_0$, then problem (2) has a solution such that $\|y(t) - v(t)\| \leq M(t)$ for all $t \in [0,T]$.

PROOF. The ideas in Theorem 2.1 together with Theorem 1.4 guarantees the existence of a solution y to the problem

$$y'(t) + cy(t) = f(t, p(t, y(t))) \text{ a.e. } t \in [0,T], \qquad y(0) = y(T).$$

We claim that $\|y(t) - v(t)\| \leq M(t)$ for every $t \in [0,T]$. First of all, observe that Assumptions (i), (ii), and Lemmas 1.1 and 1.2 imply that almost everywhere on $\{t : \|y(t) - v(t)\| > M(t)\}$,

$$\|y(t) - v(t)\|' \leq M'(t) + c(M(t) - \|y(t) - v(t)\|). \tag{4}$$

This inequality with the periodic condition and (iii) imply that

$$S = \{t : \|y(t) - v(t)\| \leq M(t)\} \neq \emptyset,$$

since if $S = \emptyset$, we would have

$$M(0) - M(T) < \|y(0) - v(0)\| - \|y(0) - v(T)\| \leq \|v(T) - v(0)\|.$$

Next notice that if $t_0 \in S$ and $t_0 < T$, then $[t_0, T] \subset S$. If it was not true, then $\|y(t) - v(t)\| - M(t)$ for $t \in [t_0, T]$, attains a positive maximum at $t_1 \in (t_0, T]$. Also, there exists $t_2 \in [t_0, t_1)$ with

$$0 < \|y(t) - v(t)\| - M(t) \leq \|y(t_1) - v(t)\| - M(t_1), \qquad \text{for all } t \in (t_2, t_1],$$

and $\|y(t_2) - v(t_2)\| - M(t_2) = 0$. Thus,

$$M(t_1) - M(t_2) \leq \|y(t_1) - v(t_1)\| - \|y(t_2) - v(t_2)\|.$$

On the other hand, inequality (4) implies

$$\|y(t_1) - v(t_1)\| - \|y(t_2) - v(t_2)\| < M(t_1) - M(t_2),$$

a contradiction. Thus, if $t_0 \in S$, then $[t_0, T] \subset S$. So, $T \in S$. It follows from the boundary conditions $T \in S$ and (iii) that:

$$\|y(0) - v(0)\| = \|y(T) - v(0)\| \leq M(T) + \|v(T) - v(0)\| \leq M(0),$$

so $0 \in S$. ∎

Using the change of variable $s = 1 - t$ (or again using ideas similar to those in Theorem 2.3), we get the following result.

THEOREM 2.4. *Let $c < 0$. Assume $f : [0,T] \times E \to E$ satisfies condition (A), and*

(i) *there exist $v \in W^{1,1}([0,T],E)$ and $M \in W^{1,1}([0,T],[0,\infty))$ such that*

$$\langle y - v(t), f(t,y) - (v'(t) + cv(t))\rangle_+ \geq M(t)\,(M'(t) + cM(t)),$$

for a.e. $t \in [0,T]$, and all $y \in E$ with $\|y - v(t)\| = M(t)$;

(ii) *$f(t,v(t)) = v'(t) + cv(t)$ a.e. on $\{t \in [0,T] : M(t) = 0\}$;*

(iii) *$\|v(T) - v(0)\| \leq M(T) - M(0)$.*

If $\int_0^T 2q(t)\,dt < -cT$, where q is the function given in condition (A) for $r = \|v\|_0 + \|M\|_0$, then problem (2) has a solution such that $\|y(t) - v(t)\| \leq M(t)$ for all $t \in [0,T]$.

REMARK 2.5. In the scalar case, if $\alpha \geq \beta \in W^{1,1}([0,T],\mathbb{R})$ are, respectively, lower and upper solutions of (2), then

$$v = \frac{\alpha+\beta}{2} \quad \text{and} \quad M = \frac{\alpha-\beta}{2},$$

satisfy (i), (ii), and (iii) in Theorem 2.4.

REMARK 2.6. In this paper, it is also possible to consider the more general periodic problem

$$y'(t) + c(t)\,y(t) = f(t,y(t)) \text{ a.e. } t \in [0,T], \qquad y(0) = y(T).$$

Theorem 2.3 (respectively, Theorem 2.4) holds with $c \in L^1([0,T],[0,\infty))$ (respectively, $c \in L^1([0,T],(-\infty,0]))$ and $\int_0^T 2q(t)\,dt < |\int_0^T c(t)\,dt|$.

3. RESULTS WITH GROWTH RESTRICTION

We first present an existence result with a Wintner-type growth condition.

THEOREM 3.1. *Let $f : [0,T] \times E \to E$ be a function satisfying condition (A). Assume that there are $\alpha \in L^1([0,T],[0,\infty))$ and $\varphi : [0,\infty) \to (0,\infty)$, a Borel measurable function such that for a.e. $t \in [0,T]$ and all $y \in E$,*

$$\langle y, f(t,y)\rangle_- \leq \alpha(t)\varphi(\|y\|).$$

If

$$\int_0^T \alpha(s)\,ds < \int_{\|a\|}^\infty \frac{x}{\varphi(x)}\,dx,$$

then problem (1) has a solution.

PROOF. Let $I(z) = \int_{\|a\|}^z (x/\varphi(x))\,dx$. Define $M = I^{-1}(\int_0^T \alpha(s)\,ds)$, and $\hat{p}(y)$ the radial projection onto $\{y \in E : \|y\| \leq M\}$. Consider the initial value problem

$$y'(t) = f(t,\hat{p}(y(t))) \text{ a.e. } t \in [0,T], \qquad y(0) = a. \tag{5}$$

Theorem 1.3 implies that (5) has a solution y. The assumptions, and Lemmas 1.1 and 1.2 imply that

$$\|y(t)\|' \leq \alpha(t)\frac{\varphi(\|y(t)\|)}{\|y(t)\|} \text{ a.e. on } \{t : \|y(t)\| > 0\}.$$

Lemma 3.2 in [1] applied with $R = \|a\|$, $\psi(x) = \varphi(x)/x$, and $z(t) = \|y(t)\|$ implies that $\|y(t)\| \leq M$ for all $t \in [0,T]$. ∎

THEOREM 3.2. *Let $f : [0,T] \times E \to E$ be a function satisfying condition (A). Assume there exists a Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $\|f(t,y)\| \leq g(t,\|y\|)$ for almost every $t \in [0,T]$ and all $y \in E$. In addition, assume that the problem*

$$z'(t) = g(t,z(t)) \text{ a.e. } t \in [0,T], \qquad z(0) = \|a\|,$$

has a maximal solution $r(t)$ on $[0,T]$. Then problem (1) has a solution.

PROOF. Define $\tilde{p}(t, y(t))$ the radial projection of y on $\{y \in E : \|y\| \leq r(t) + 1\}$, and consider the problem

$$y'(t) = f\left(t, \tilde{p}\left(t, y(t)\right)\right) \text{ a.e. } t \in [0, T], \qquad y(0) = a.$$

Theorem 1.3 gives the existence of y a solution to this problem. Since

$$\|y(t)\|' \leq g\left(t, \|y\|\right) \text{ a.e. } t \in [0, T],$$

using the comparison [5, Theorem 1.10.2], we deduce that $\|y(t)\| \leq r(t)$ for every $t \in [0, T]$, and the proof is complete. ∎

A special case of Theorem 3.2 is the following result.

COROLLARY 3.3. *Let $f : [0, T] \times E \to E$ be a function satisfying condition (A). Assume that there are $\alpha \in L^1([0, T], [0, \infty))$ and $\varphi : [0, \infty) \to (0, \infty)$, a continuous function such that for a.e. $t \in [0, T]$ and all $y \in E$,*

$$\|f(t, y)\| \leq \alpha(t)\varphi\left(\|y\|\right),$$

and

$$\int_0^T \alpha(s)\, ds < \int_{\|a\|}^{\infty} \frac{dx}{\varphi(x)}.$$

Then problem (1) has a solution.

REMARK 3.4. We would like to remark that Corollary 3.3 could be deduced from Theorem 3.1; this has the advantage of not having to rely on classical theory (namely, [5, Theorem 1.10.2]).

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