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Boundary Value Problems for Second Order Impulsive Differential Equations Using Set-Valued Maps

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MARLÈNE FRIGON* AND DONAL O'REGAN**

* Département de Mathématiques et Statistique, Université de Montréal,
C.P. 6128, succ. Centre-ville,
Montréal, H3C 3J7, CANADA

** Department of Mathematics, University College Galway,
Galway, IRELAND

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ABSTRACT. In this paper, we establish some existence results for boundary value problems for second order impulsive differential equations. Our results rely on a version of the Nonlinear Alternative of A. Granas and on a version of the Schauder Fixed Point Theorem for compact, u.s.c., convex values maps.

KEY WORDS: boundary value problems, impulsive differential equations

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1. INTRODUCTION

In this paper, we give existence results for boundary value problems for second order impulsive equations, namely

$$\begin{aligned} &y''(t) = f(t, y(t), y'(t)) \quad t \in [0,1]\backslash\{t_1, \dots, t_p\}, \\ &a_0 y(0) - b_0 y'(0) = r_0, \quad a_1 y(1) + b_1 y'(1) = r_1, \\ &y(t_k^+) = I_k(y(t_k^-)), \\ &y'(t_k^+) = N_k(y(t_k^-), y'(t_k^-)), \quad k = 1, \dots, p; \end{aligned} \tag{1.1}$$

with $0 = t_0 < t_1 < \cdots < t_{p+1} = 1$, $a_0, a_1, b_0, b_1 \geq 0$ and $\max\{a_i, b_i\} > 0$, $i = 0, 1$. We define the Banach space

$$K^k = \{y : [0,1] \to \mathbb{R} : \forall j = 0, \dots, p, \exists y_j \in C^k[t_j, t_{j+1}] \text{ such that } y = y_j \text{ on } (t_j, t_{j+1}], \ \ y(0) = y_0(0)\}$$

with the norm

$$\|y\|_k = \max\{\|y\|_0, \dots, \|y^{(k)}\|_0\} \quad \text{where} \quad \|y\|_0 = \sup\{|y(t)| : t \in [0,1]\backslash\{t_1, \dots, t_p\}\}.$$

We denote $K_b^2 = \{y \in K^2 : a_0 y(0) - b_0 y'(0) = r_0, a_1 y(1) + b_1 y'(1) = r_1\}$. By a *solution* of (1.1), we mean a function $y \in K_b^2$ satisfying (1.1).

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The paper is divided into three sections. In section 2, we present two general existence principles which will be used to obtain results of the next sections.

In section 3, we establish the existence of a solution of (1.1) lying between upper and lower solutions when the function f satisfies a growth condition of Bernstein-Nagumo type (Theorem 3.2). Our result extends the theory in the literature, see [8,9]; more precisely, we do not assume a monotonicity condition on f or N_k, $k = 1, \ldots, p$, which is usually the situation in the literature. For sake of simplicity, we assume that f is continuous but the results hold if f is a Carathéodory function. We use a version of the Schauder Fixed Point Theorem for compact, u.s.c., convex values maps.

In section 4, we establish an existence result for (1.1) if the nonlinearity f grows sublinearly in the variables y, y'. To obtain the result, we use the nonlinear alternative of A. Granas for compact, u.s.c., convex values map.

For completness, we recall some definitions and we state results which will be used later. For proofs and more details, see [2].

Let X, Y be topological spaces, a multivalued mapping $F : X \to 2^Y$ with non-empty compact values is *upper semi-continuous* (u.s.c.) if $\{x \in X : F(x) \cap B \neq \emptyset\}$ is closed for all closed set $B \subset Y$. It is compact if $\overline{F(X)} = \overline{\cup_{x\in X} F(x)}$ is compact in Y, and it is completely continuous if $\overline{F(B)}$ is compact for all bounded sets $B \subset X$.

Theorem 1.1. *Let C be a convex subset of a Banach space and let $\Gamma : C \to 2^C$ be a u.s.c., compact multivalued function with each $\Gamma(c)$ a non-empty, closed, convex set. Then Γ has a fixed point ($c \in \Gamma(c)$).*

Theorem 1.2. *Let C be a convex subset of a Banach space, $U \subset C$ be bounded and relatively open, and $p^* \in U$. Suppose $H_\lambda(u) = H(u, \lambda) : \bar{U} \times [0,1] \to 2^C$ is a u.s.c., compact, multivalued function with compact, convex values such that $H(\cdot, 0) = p^*$. Then either*

(i) *there exists $u \in \bar{U}$ such that $u \in H(u, 1)$; or*
(ii) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in H(u, \lambda)$.*

2. EXISTENCE PRINCIPLES

In this section, we establish general existence principles for second order impulsive differential equations of the form:

$$\begin{aligned}
&y''(t) - \varepsilon y(t) = f(t, y(t), y'(t)) \quad t \in [0,1]\backslash\{t_1, \ldots, t_p\}, \\
&a_0 y(0) - b_0 y'(0) = r_0, \quad a_1 y(1) + b_1 y'(1) = r_1, \\
&y(t_k^+) = I_k(y(t_k^-)), \\
&y'(t_k^+) = N_k(y(t_k^-), y'(t_k^-)), \quad k = 1, \ldots, p;
\end{aligned} \tag{2.1}$$

with $0 = t_0 < t_1 < \cdots < t_{p+1} = 1$, $a_i \geq 0$, $b_i \geq 0$, $a_i + b_i > 0$, $i = 0, 1$; $\varepsilon \geq 0$ such that $\varepsilon > 0$ if $a_1 = 0$.

We assume that f, I_k, and N_k, $k = 1, \ldots, p$, satisfy the following conditions:

(H1) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous for $t \in [0,1]\backslash\{t_1, \ldots, t_p\}$, and $\lim_{t \to t_k^\pm} f(t, y, q)$ exist;
(H2) $I_k : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing;
(H3) $N_k : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

The following two theorems will be used to obtain the existence results of sections 3 and 4 respectively.

Theorem 2.1. *Assume* (H1)–(H3) *are satisfied. In addition, assume* $Im\, I_k = \mathbb{R}$, *and there exist* $c, d \geq 0$ *such that* $|f(t,y,q)| \leq c$, $|N_k(y,q)| \leq d$ *for* $t \in [0,1]$, $(y,q) \in \mathbb{R}^2$, $k = 1,\dots,p$. *Then the problem* (2.1) *has a solution.*

Proof. Take $M_p \geq 0$ such that $|y(t)| \leq M_p$ on $(t_p, 1]$ for all $y \in K_b^2$ satisfying $|y'(t_p^+)| \leq d$, and $|y''(t) - \varepsilon y(t)| \leq c$ on $(t_p, 1]$. For $k = 1,\dots,p-1$, take $M_k \geq 0$ such that

$$|y(t)| \leq M_k \quad \text{on} \quad (t_k, t_{k+1})$$

for all $y \in K_b^2$ satisfying

$$\begin{aligned} |y'(t_k^+)| &\leq d, \\ |y(t_{k+1}^-)| &\leq \max\{\,|w| : w \in I_{k+1}^{-1}(z) \quad \text{with} \quad |z| \leq M_{k+1}\,\}, \\ |y''(t) - \varepsilon y(t)| &\leq c \quad \text{on} \quad (t_k, t_{k+1}). \end{aligned}$$

Denote $\Omega = \{y \in K^1 : |y(t_k^+)| \leq M_k,\ k = 1,\dots,p\}$, and define $T : \Omega \to 2^{K \times \mathbb{R}^{2p}}$ by:

$$T(y) = (t \mapsto f(t, y(t), y'(t)), I_1^{-1}(y(t_1^+)), N_1(y(t_1^-), y'(t_1^-)), \dots, \\ I_p^{-1}(y(t_p^+)), N_p(y(t_p^-), y'(t_p^-))\,).$$

Since I_k is nondecreasing and $Im\, I_k = \mathbb{R}$, T is a multivalued mapping with non-empty, compact, convex values. Assumptions (H1)–(H3) imply that T is u.s.c. Moreover, T is bounded.

On the other hand, define $L : K_b^2 \to K \times \mathbb{R}^{2p}$ by:

$$L(y) = (y'' - \varepsilon y, y(t_1^-), y'(t_1^+), \dots, y(t_p^-), y'(t_p^+)\,).$$

It is easy to show that L is invertible. By choice of M_k, we have $L^{-1} \circ T(y) \in \Omega$ for all $y \in \Omega$. The operator $j \circ L^{-1} \circ T : \Omega \to 2^{\Omega}$ is u.s.c., compact, with non-empty, convex, compact values, where $j : K_b^2 \to K^1$ is the completely continuous inclusion. Moreover, y is a solution of (2.1) if and only if $y \in j \circ L^{-1} \circ T(y)$. Theorem 1.1 gives the conclusion. □

For $\lambda \in [0,1)$, we consider the problem:

$$\begin{aligned} &y''(t) - \varepsilon y(t) = \lambda f(t, y(t), y'(t)) \quad t \in [0,1] \backslash \{t_1, \dots, t_p\}, \\ &a_0 y(0) - b_0 y'(0) = r_0, \quad a_1 y(1) + b_1 y'(1) = r_1, \\ &y(t_k^+) = I_k(y(t_k^-)), \\ &y'(t_k^+) = \lambda N_k(y(t_k^-), y'(t_k^-)), \quad k = 1, \dots, p. \end{aligned} \tag{2.2$_\lambda$}$$

Theorem 2.2. *Assume* (H1)–(H3) *are satisfied and* $Im\, I_k = \mathbb{R}$. *In addition, suppose there exists an open set* $U \subset K^1$ *with* $y \in U \cap K^2$ *for all* y *solution of* (2.2$_0$). *Assume there is no solution* y *of* (2.2$_\lambda$) *for some* $\lambda \in (0,1)$, *such that* $y \in \partial U$. *Then the problem* (2.1) *has at least one solution* $y \in \bar{U}$.

Remark. It is easy to construct a solution y of (2.2$_0$) by first considering $[t_p, 1]$ and then continuing backwards.

Proof. Let $p^* \in U$ be a solution of (2.2$_0$), and $\Gamma : K^1 \times [0,1] \to 2^{K \times \mathbb{R}^{2p}}$ be given by:

$$\Gamma(y, \lambda) = (t \mapsto \lambda f(t, y(t), y'(t)), I_1^{-1}(y(t_1^+)), \lambda N_1(y(t_1^-), y'(t_1^-)), \dots, \\ I_p^{-1}(y(t_p^+)), \lambda N_p(y(t_p^-), y'(t_p^-))\,).$$

Γ is a multivalued mapping with nonempty, compact, convex values (since I_k is nondecreasing and $Im\, I_k = \mathbf{R}$). Also, (H1), (H2), (H3) imply that Γ is u.s.c. Thus $j \circ L^{-1} \circ \Gamma$ is u.s.c., completely continuous with nonempty, compact, convex values, where j and L are defined as in the proof of Theorem 2.1. Moreover y is a solution of (2.2_λ) if and only if $y \in j \circ L^{-1} \circ \Gamma(u, \lambda)$. Define $H : \bar{U} \times [0,1] \to 2^{K^1}$ by:

$$H(u,\lambda) = \begin{cases} j \circ L^{-1} \circ \Gamma(u, 2\lambda - 1), & \text{if } 1/2 \le \lambda \le 1, \\ 2\lambda\, j \circ L^{-1} \circ \Gamma(u,0) + (1-2\lambda)p^*, & \text{if } 0 \le \lambda < 1/2. \end{cases}$$

The result follows by applying Theorem 1.2. □

3. EXISTENCE RESULTS WITH UPPER AND LOWER SOLUTIONS

In this section, we establish the existence of a solution of (1.1) when there exist upper and lower solutions and f satisfies a Nagumo growth condition. First of all, we give the definition of upper and lower solutions of (1.1).

Definition 3.1. A function $\alpha \in K^2$ is called a *lower solution* of (1.1) if

$$\begin{aligned} &\alpha''(t) \ge f(t, \alpha(t), \alpha'(t)) \quad \text{for} \quad t \in [0,1] \backslash \{t_1, \dots, t_p\}; \\ &a_0\alpha(0) - b_0\alpha'(0) \le r_0, \;\; a_1\alpha(1) + b_1\alpha'(1) \le r_1; \\ &\alpha(t_k^+) = I_k(\alpha(t_k^-)); \\ &\alpha'(t_k^+) \ge N_k(\alpha(t_k^-), q) \quad \text{for all} \quad q \le \alpha'(t_k^-). \end{aligned}$$

Respectively, a function $\beta \in K^2$ is called an *upper solution* of (1.1) if

$$\begin{aligned} &\beta''(t) \le f(t, \beta(t), \beta'(t)) \quad \text{for} \quad t \in [0,1] \backslash \{t_1, \dots, t_p\}; \\ &a_0\beta(0) - b_0\beta'(0) \ge r_0, \;\; a_1\beta(1) + b_1\beta'(1) \ge r_1; \\ &\beta(t_k^+) = I_k(\beta(t_k^-)); \\ &\beta'(t_k^+) \le N_k(\beta(t_k^-), q) \quad \text{for all} \quad q \ge \beta'(t_k^-). \end{aligned}$$

Theorem 3.2. *Assume the following conditions are satisfied:* (H1)–(H3) *and*

(H4) *there exist $\alpha \le \beta \in K^2$ respectively lower and upper solutions of* (1.1)*;*

(H5) *for $k = 0, \dots, p$, there exists $\psi_k : [0,\infty) \to (0,\infty)$ Borel measurable such that $|f(t,y,q)| \le \psi_k(|q|)$ for $t \in (t_k, t_{k+1})$, $\alpha(t) \le y \le \beta(t)$, $q \in \mathbf{R}$; and*

$$\int_{d_k}^{\infty} \frac{x}{\psi_k(x)}\, dx > \sup\{\beta(t) - \alpha(\tau) : t, \tau \in (t_k, t_{k+1})\} = A_k,$$

where $d_k = \max\{|\alpha(t_{k+1}^-) - \beta(t_k^+)|, |\beta(t_{k+1}^-) - \alpha(t_k^+)|\}/(t_{k+1} - t_k)$.

Then the problem (1.1) *has a solution such that $\alpha(t) \le y(t) \le \beta(t)$ for all $t \in [0,1]$.*

Proof. Let $\varepsilon \ge 0$ be such that $\varepsilon > 0$ if $a_1 = 0$, and take $M > \max\{\|\beta'\|_0, \|\alpha'\|_0\}$ such that for $k = 0, \dots, p$,

$$\int_{d_k}^{M} \frac{x}{\psi_k(x)}\, dx > A_k.$$

To (t,y,q), we associate $(t,\widetilde{y},\hat{q})$ given by:

$$\widetilde{y}=\begin{cases}\beta(t), & \text{if } y>\beta(t),\\ y, & \text{if } \alpha(t)\le y\le\beta(t),\\ \alpha(t), & \text{if } y<\alpha(t),\end{cases} \quad\text{and}\quad \hat{q}=\begin{cases}M, & \text{if } q>M,\\ q, & \text{if } -M\le q\le M,\\ -M, & \text{if } q<-M.\end{cases}$$

Let $r:\mathbf{R}\to[-1,1]$ be defined by

$$r(u)=\begin{cases}u, & \text{if } |u|\le 1,\\ \frac{u}{|u|}, & \text{otherwise.}\end{cases}$$

We define $f_1:[0,1]\times\mathbf{R}^2\to\mathbf{R}$ by

$$f_1(t,y,q)=f(t,\widetilde{y},\hat{q})+r(y-\widetilde{y})-\varepsilon\widetilde{y}.$$

Consider the associated problem

$$\begin{aligned}&y''(t)-\varepsilon y(t)=f_1(t,y(t),y'(t)) \quad t\in[0,1]\backslash\{t_1,\dots,t_p\},\\ &a_0y(0)-b_0y'(0)=r_0,\quad a_1y(1)+b_1y'(1)=r_1,\\ &y(t_k^+)=I_k(\widetilde{y}(t_k^-))+y(t_k^-)-\widetilde{y}(t_k^-),\\ &y'(t_k^+)=N_k(\widetilde{y}(t_k^-),\hat{y}'(t_k^-)),\quad k=1,\dots,p;\end{aligned}\tag{3.1}$$

By Theorem 2.1, the problem (3.1) has a solution. Now, we will show that this solution y is such that $\alpha(t)\le y(t)\le\beta(t)$, and $|y'(t)|\le M$ for all $t\in[0,1]$. Then, by using the definition of f_1, we will conclude that y is a solution of (1.1).

If $\alpha(t)\le y(t)\le\beta(t)$ for all $t\in[0,1]$ does not hold, we can assume without loss of generality that $y-\beta$ attains a positive maximum on $[t_k^+,t_{k+1}^-]$ at τ_k for some $k\in\{0,\dots,p\}$. If $\tau_k=t_k^+$ or $\tau_k=t_{k+1}^-$ and $y'(\tau_k)=\beta'(\tau_k)$, or $\tau_k\in(t_k^+,t_{k+1}^-)$, then $y'(\tau_k)=\beta'(\tau_k)$. By (H4) and the definition of f_1, we get

$$\beta''(\tau_k)\ge y''(\tau_k)=f(\tau_k,\beta(\tau_k),\beta'(\tau_k))+r(y(\tau_k)-\beta(\tau_k))+\varepsilon(y(\tau_k)-\beta(\tau_k))>\beta''(\tau_k),$$

a contradiction.

Remark that $\tau_0\ne 0$, and $\tau_p\ne 1$. Indeed, if $\tau_0=0$ then $y'(0)\le\beta'(0)$, and by the boundary conditions, if $b_0=0$, then $a_0(y(0)-\beta(0))=r_0-a_0\beta(0)\le 0$, a contradiction; and if $b_0\ne 0$ then $b_0(y'(0)-\beta'(0))\ge a_0(y(0)-\beta(0))\ge 0$, so we need only to consider the case when $y'(0)=\beta'(0)$ and by the previous argument we get a contradiction. Similarly we can not have $\tau_p=1$.

It remains to discuss two cases: $\tau_k=t_{k+1}^-$ with $y'(t_{k+1}^-)>\beta'(t_{k+1}^-)$ for $k=0,\dots,p-1$, and $\tau_k=t_k^+$ with $y'(t_k^+)<\beta'(t_k^+)$ for $k=1,\dots,p$. If $\tau_k=t_{k+1}^-$ for $k\in\{0,\dots,p-1\}$ with $y'(t_{k+1}^-)>\beta'(t_{k+1}^-)$, then $\hat{y}'(t_{k+1}^-)>\beta'(t_{k+1}^-)$. On the other hand, (H4) gives

$$y(t_{k+1}^+)=I_{k+1}(\widetilde{y}(t_{k+1}^-))+y(t_{k+1}^-)-\widetilde{y}(t_{k+1}^-)>I_{k+1}(\beta(t_{k+1}^-))=\beta(t_{k+1}^+).$$

By the previous argument, we must have $\tau_{k+1}=t_{k+1}^+$ or t_{k+2}^-. Since $\tau_p\ne 1$ and by the previous reasoning, there exists $i\in\{1,\dots,p-k\}$ such that $\tau_{k+i}=t_{k+i}^+$, $\tau_{k+i-1}=t_{k+i}^-$, $y'(t_{k+i}^+)<\beta'(t_{k+i}^+)$, and $\hat{y}'(t_{k+i}^-)>\beta'(t_{k+i}^-)$. By assumption (H4), we get

$$y'(t_{k+i}^+)=N_{k+i}(\widetilde{y}(t_{k+i}^-),\hat{y}'(t_{k+i}^-))=N_{k+i}(\beta(t_{k+i}^-),\hat{y}'(t_{k+i}^-))\ge\beta'(t_{k+i}^+),$$

a contradiction.

If $\tau_k = t_k^+$ with $y'(t_k^+) < \beta'(t_k^+)$ for $k \in \{1, \ldots, p\}$, then

$$\begin{aligned}\beta(t_k^+) &< y(t_k^+)\\ &= I_k(\tilde{y}(t_k^-)) + y(t_k^-) - \tilde{y}(t_k^-)\\ &\le I_k(\beta(t_k^-)) + y(t_k^-) - \tilde{y}(t_k^-)\\ &\le \beta(t_k^+) + y(t_k^-) - \tilde{y}(t_k^-).\end{aligned}$$

Therefore, $y(t_k^-) > \tilde{y}(t_k^-)$ i.e $y(t_k^-) > \beta(t_k^-)$ by definition of $\tilde{y}$. By the previous arguments $\tau_{k-1} \notin (t_{k-1}^+, t_k^-]$, so $\tau_{k-1} = t_{k-1}^+$. Since $\tau_0 \neq 0$, we get a contradiction. Similarly, we can show that $y(t) \ge \alpha(t)$.

Now, we will show that $|y'(t)| < M$. If this is not true, assume without loss of generality that $y' \not< M$ on $[t_k^+, t_{k+1}^-]$ for some $k \in \{0, \ldots, p\}$. Since $\alpha(t) \le y(t) \le \beta(t)$, we have

$$\min\{|y'(t)| : t \in [t_k^+, t_{k+1}^-]\} \le d_k.$$

Therefore, there exist $s, \tau \in [t_k^+, t_{k+1}^-]$ such that $y'(s) = d_k$, $y'(\tau) = M$, and $d_k \le y'(t) \le M$ for t between s and τ. Consequently, $y''(t) = f(t, y(t), y'(t))$ for t between s and τ. Without loss of generality, assume $s < \tau$. Assumption (H5) gives

$$y'(t)y''(t) \le y'(t)\psi_k(y'(t)) \quad \text{for } t \in (s, \tau).$$

By dividing by ψ_k, integrating from s to τ, and using the change of variable formula, we obtain

$$\begin{aligned}\int_{d_k}^{M} \frac{x}{\psi_k(x)}\,dx = \int_{y'(s)}^{y'(\tau)} \frac{x}{\psi_k(x)}\,dx = \int_s^\tau \frac{y'(t)y''(t)}{\psi_k(y'(t))}\,dt &\le \int_s^\tau y'(t)\,dt\\ &= y(\tau) - y(s) \le A_k < \int_{d_k}^{M} \frac{x}{\psi_k(x)}\,dx,\end{aligned}$$

a contradiction. A similar argument works for the other cases, and the proof is complete. □

4. EXISTENCE RESULTS WITH SUBLINEAR GROWTH CONDITION

In this section, we obtain an existence result for (1.1) with $a_1 > 0$ when f grows sublinearly in the variables y, y'. For convenience, we assume $p = 1$.

In particular we examine the boundary value problem:

$$\begin{aligned}&y''(t) = f(t, y(t), y'(t)) \quad t \in [0,1]\backslash\{t_1\},\\ &a_0 y(0) - b_0 y'(0) = r_0, \quad a_1 y(1) + b_1 y'(1) = r_1,\\ &y(t_1^+) = I_1(y(t_1^-)),\\ &y'(t_1^+) = N_1(y(t_1^-), y'(t_1^-));\end{aligned} \tag{4.1}$$

with $0 = t_0 < t_1 < t_2 = 1$, $a_0, b_0, b_1 \ge 0$, $a_0 + b_0 > 0$, $a_1 > 0$.

Theorem 4.1. *Let* (H1),(H2),(H3) *hold and* $ImI_1 = \mathbf{R}$. *Assume*

(H6) $|f(t,u,w)| \leq A_0 + A_1|u|^\alpha + A_2|w|^\beta$ *for constants* A_0, A_1, A_2, α, β *with* $0 \leq \alpha, \beta < 1$,

(H7) *for each* $v_0, v_1 \in \mathbf{R}$ *and* $\lambda \in [0,1)$, *there exists at least one solution* (c_0, c_1) *to*

$$\begin{aligned} a_0c_0 + (a_0t_1 + b_0)c_1 &= v_0, \\ a_1I_1(c_0) + \lambda N_1(c_0, -c_1)[a_1(1-t_1) + b_1] &= v_1. \end{aligned} \tag{4.2}$$

In addition, each solution (c_0, c_1) *of* (4.2) *satisfies*

$$\begin{aligned} |c_0| &\leq B_0 + B_1|v_0|^{\sigma_1} + B_2|v_1|^{\sigma_2}; \\ |c_1| &\leq B_3 + B_4|v_0|^{\sigma_3} + B_5|v_1|^{\sigma_4}; \\ |I_1(c_0)| &\leq B_6 + B_7|v_0|^{\sigma_5} + B_8|v_1|^{\sigma_6}; \\ |N_1(-c_0, c_1)| &\leq B_9 + B_{10}|v_0|^{\sigma_7} + B_{11}|v_1|^{\sigma_8}, \end{aligned}$$

for some B_j *(independent of* λ*),* $j = 0, \ldots, 11$ *and* $\sigma_i \geq 0$ *with* $0 \leq \sigma_i\alpha, \sigma_i\beta < 1$ *for* $i = 1, \ldots, 8$,

are satisfied. Then the problem (4.1) *has at least one solution in* K_b^2.

Proof. Let y be a solution of (2.2_λ) with $p = 1$, $\varepsilon = 0$, for some $\lambda \in [0,1)$. Then

$$y(t) = \begin{cases} c_0 + c_1(t_1 - t) + \lambda \int_t^{t_1} \int_s^{t_1} f(x, y(x), y'(x))\,dx\,ds, & \text{if } 0 \leq t \leq t_1, \\ I_1(c_0) + \lambda N_1(c_0, -c_1)(t - t_1) & \\ \qquad + \lambda \int_{t_1}^t \int_{t_1}^s f(x, y(x), y'(x))\,dx\,ds, & \text{if } t_1 < t \leq 1; \end{cases} \tag{4.3}$$

and

$$y'(t) = \begin{cases} -c_1 - \lambda \int_t^{t_1} f(x, y(x), y'(x))\,dx, & \text{if } 0 \leq t \leq t_1, \\ \lambda N_1(c_0, -c_1) + \lambda \int_{t_1}^t f(x, y(x), y'(x))\,dx, & \text{if } t_1 < t \leq 1; \end{cases} \tag{4.4}$$

with c_0 and c_1 satisfying

$$\begin{aligned} &a_0c_0 + (a_0t_1 + b_0)c_1 = \\ &\qquad r_0 - a_0\lambda \int_0^{t_1} \int_s^{t_1} f(x, y(x), y'(x))\,dx\,ds - b_0\lambda \int_0^{t_1} f(x, y(x), y'(x))\,dx \\ &a_1I_1(c_0) + \lambda N_1(c_0, -c_1)[a_1(1-t_1) + b_1] = \\ &\qquad r_1 - a_1\lambda \int_{t_1}^1 \int_{t_1}^s f(x, y(x), y'(x))\,dx\,ds - b_1\lambda \int_{t_1}^1 f(x, y(x), y'(x))\,dx. \end{aligned} \tag{4.5}$$

For notational purposes, for $j = 0, 1$, let

$$\|y^{(j)}\|_{0,t_1} = \sup_{[0,t_1]} |y^{(j)}(t)| \quad \text{and} \quad \|y^{(j)}\|_{t_1,1} = \sup_{[t_1,1]} |y^{(j)}(t)|.$$

Consider first the case when $t \in [0, t_1]$. Then (4.4) implies

$$|y'(t)| \leq |c_1| + \int_0^{t_1} |f(x, y(x), y'(x))|\,dx, \quad t \in [0, t_1],$$

and this together with (H6), (H7) and (4.5) yields

$$|y'(t)| \le Q_{00} + Q_{01}\|y\|_{0,t_1}^{\sigma_3\alpha} + Q_{02}\|y'\|_{0,t_1}^{\sigma_3\beta} + Q_{03}\|y\|_{t_1,1}^{\sigma_4\alpha} + Q_{04}\|y'\|_{t_1,1}^{\sigma_4\beta} + Q_{05}\|y\|_{0,t_1}^{\alpha} + Q_{06}\|y'\|_{0,t_1}^{\beta}$$

for some constants Q_{0j}, $j = 0, \dots, 6$. Consequently, there exist constants Q_0, Q_1, Q_2, Q_3 and Q_4 with

$$\|y'\|_{0,t_1} \le Q_0 + Q_1\|y\|_{0,t_1}^{\theta_1} + Q_2\|y'\|_{0,t_1}^{\theta_2} + Q_3\|y\|_{t_1,1}^{\sigma_4\alpha} + Q_4\|y'\|_{t_1,1}^{\sigma_4\beta} \tag{4.6}$$

where $\theta_1 = \max\{\sigma_3\alpha, \alpha\}$ and $\theta_2 = \max\{\sigma_3\beta, \beta\}$. Also, (4.3) yields

$$|y(t)| \le |c_0| + |c_1| + \int_0^{t_1}\int_s^{t_1} |f(x, y(x), y'(x))|\,dx\,ds, \quad t \in [0, t_1]$$

and this together with (H6),(H7) and (4.5) yields

$$|y(t)| \le L_{01} + L_{02}\|y\|_{0,t_1}^{\sigma_1\alpha} + L_{03}\|y'\|_{0,t_1}^{\sigma_1\beta} + L_{04}\|y\|_{t_1,1}^{\sigma_2\alpha} + L_{05}\|y'\|_{t_1,1}^{\sigma_2\beta} + L_{06}\|y\|_{0,t_1}^{\sigma_3\alpha} + L_{07}\|y'\|_{0,t_1}^{\sigma_3\beta} + L_{08}\|y\|_{t_1,1}^{\sigma_4\alpha} + L_{09}\|y'\|_{t_1,1}^{\sigma_4\beta} + L_{10}\|y\|_{0,t_1}^{\alpha} + L_{11}\|y'\|_{0,t_1}^{\beta}$$

for some constants $L_{01}, \dots, L_{11}$. Consequently, there exist constants Q_5, Q_6, Q_7, Q_8 and Q_9 with

$$\|y\|_{0,t_1} \le Q_5 + Q_6\|y\|_{0,t_1}^{\theta_3} + Q_7\|y'\|_{0,t_1}^{\theta_4} + Q_8\|y\|_{t_1,1}^{\theta_5} + Q_9\|y'\|_{t_1,1}^{\theta_6} \tag{4.7}$$

where $\theta_3 = \max\{\sigma_1\alpha, \sigma_3\alpha, \alpha\}$, $\theta_4 = \max\{\sigma_1\beta, \sigma_3\beta, \beta\}$, $\theta_5 = \max\{\sigma_2\alpha, \sigma_4\alpha\}$ and $\theta_6 = \max\{\sigma_2\beta, \sigma_4\beta\}$.

Next, suppose $t \in (t_1, 1]$. Then (4.4) implies

$$|y'(t)| \le |N_1(c_0, -c_1)| + \int_{t_1}^1 |f(x, y(x), y'(x))|\,dx, \quad t \in (t_1, 1]$$

and this together with (H6), (H7) and (4.5) implies that there are constants Q_{10}, Q_{11}, Q_{12}, Q_{13} and Q_{14} with

$$\|y'\|_{t_1,1} \le Q_{10} + Q_{11}\|y\|_{0,t_1}^{\sigma_7\alpha} + Q_{12}\|y'\|_{0,t_1}^{\sigma_7\beta} + Q_{13}\|y\|_{t_1,1}^{\theta_7} + Q_{14}\|y'\|_{t_1,1}^{\theta_8} \tag{4.8}$$

where $\theta_7 = \max\{\sigma_8\alpha, \alpha\}$ and $\theta_8 = \max\{\sigma_8\beta, \beta\}$. Also (4.3) implies

$$|y(t)| \le |I_1(c_0)| + |N_1(c_0, -c_1)| + \int_{t_1}^1\int_s^{t_1} |f(x, y(x), y'(x))|\,dx\,ds, \quad t \in (t_1, 1]$$

and this together with (H6), (H7) and (4.5) implies that there are constants Q_{15}, Q_{16}, Q_{17}, Q_{18} and Q_{19} with

$$\|y\|_{t_1,1} \le Q_{15} + Q_{16}\|y\|_{0,t_1}^{\theta_9} + Q_{17}\|y'\|_{0,t_1}^{\theta_{10}} + Q_{18}\|y\|_{t_1,1}^{\theta_{11}} + Q_{19}\|y'\|_{t_1,1}^{\theta_{12}} \tag{4.9}$$

where $\theta_9 = \max\{\sigma_5\alpha, \sigma_7\alpha\}$, $\theta_{10} = \max\{\sigma_5\beta, \sigma_7\beta\}$, $\theta_{11} = \max\{\sigma_6\alpha, \sigma_8\alpha, \alpha\}$ and $\theta_{12} = \max\{\sigma_6\beta, \sigma_8\beta, \beta\}$.

Add (4.6) to (4.8) and use the fact that $\max\{\theta_2, \sigma_4\beta, \sigma_7\beta, \theta_8\} < 1$ to obtain the existence of constants Q_{20}, Q_{21} and Q_{22} with

$$\|y'\|_{0,t_1} + \|y'\|_{t_1,1} \le Q_{20} + Q_{21}\|y\|_{0,t_1}^{\theta_{13}} + Q_{22}\|y\|_{t_1,1}^{\theta_{14}} \tag{4.10}$$

where $\theta_{13} = \max\{\sigma_7\alpha, \theta_1\}$ and $\theta_{14} = \max\{\sigma_4\alpha, \theta_7\}$. Adding (4.7) and (4.9) yields the existence of constants Q_{23} and Q_{24} with

$$\|y\|_{0,t_1} + \|y\|_{t_1,1} \le Q_{23} + Q_{24}(\|y'\|_{0,t_1} + \|y'\|_{t_1,1})^{\theta_{15}} \tag{4.11}$$

where $\theta_{15} = \max\{\theta_4, \theta_6, \theta_{10}, \theta_{12}\}$. Put (4.10) into (4.11) to obtain

$$\|y\|_{0,t_1} + \|y\|_{t_1,1} \le Q_{23} + Q_{24}(Q_{20} + Q_{21}\|y\|_{0,t_1}^{\theta_{13}} + Q_{22}\|y\|_{t_1,1}^{\theta_{14}})^{\theta^{15}}$$

and since $\theta_{13}\theta_{15}, \theta_{14}\theta_{15} < 1$, there exists a constant R_0 with

$$\|y\|_{0,t_1} \le R_0 \quad \text{and} \quad \|y\|_{t_1,1} \le R_0. \tag{4.12}$$

Now (4.12) together with (4.10) implies that there exists a constant R_1 with

$$\|y'\|_{0,t_1} \le R_1 \quad \text{and} \quad \|y'\|_{t_1,1} \le R_1. \tag{4.13}$$

The result follows from Theorem 2.2 by letting $U = B(0,R)$ with $R = \max\{R_0, R_1\} + 1$. □

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