Bull. London Math. Soc. 35 (2003) 672–680 © 2003 London Mathematical Society
DOI: 10.1112/S0024609303002212

FIXED POINTS OF CONE-COMPRESSING AND CONE-EXTENDING OPERATORS IN FRÉCHET SPACES

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Abstract

A generalization of norm type cone-compression and -expansion results due to Krasnosel'skiĭ (see *Dokl. Akad. Nauk SSSR NS* 135 (1960) 527–530) is presented here for single-valued completely continuous maps defined on a Fréchet space. Applications to second-order differential equations on the half line are presented, and the existence of nontrivial solutions is established.

1. *Introduction*

We present a generalization – for single-valued completely continuous maps defined on a Fréchet space – of norm type cone-compression and -expansion results due to Krasnosel'skiĭ [**5**], establishing the existence of a fixed point in the intersection of an annulus and a cone. Even though our result holds for k-set contraction, we choose for the sake of simplicity to present it for completely continuous maps. Here, we consider admissible maps in the sense of Frigon [**3**], and we therefore use the theory of upper semi-continuous multivalued maps. This type of result was also obtained by Agarwal and O'Regan [**1**] for appropriate sequences of upper semi-continuous multivalued maps and decreasing sequences of Banach spaces.

Applications to second-order differential equations on the half line are presented, and the existence of nontrivial solutions is also established. For other results on these types of problem, the interested reader can consult [**4**] or [**8**].

Our main result will rely on the following particular case of a result due to Petryshyn [**7**, Theorem 3] for upper semi-continuous multivalued maps.

Theorem 1.1. *Let E be a Banach space and let $C \subset E$ be a closed cone. Let U and V be bounded open sets in E such that $0 \in U \subset \overline{U} \subset V$, and let $F : V \cap C \longrightarrow C$ be a compact, upper semi-continuous multivalued map with nonempty compact convex values. Assume that:*

(1) $\|y\| \geqslant \|x\|$, *for all $y \in F(x)$ and $x \in \partial U \cap C$;*
(2) $\|y\| \leqslant \|x\|$, *for all $y \in F(x)$ and $x \in \partial V \cap C$;*

or

(1′) $\|y\| \leqslant \|x\|$, *for all $y \in F(x)$ and $x \in \partial U \cap C$;*
(2′) $\|y\| \geqslant \|x\|$, *for all $y \in F(x)$ and $x \in \partial V \cap C$.*

Then F has a fixed point in $C \cap \overline{V} \setminus \overline{U}$.

For the sake of completeness, we recall some notations and definitions – given in [**3**] – that will be needed in what follows.

Received 25 July 2002; revised 18 November 2002.

2000 *Mathematics Subject Classification* 47H10, 34B40.

This work was partially supported by CRSNG Canada.

Let E be a Fréchet space with the topology generated by a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. For the sake of simplicity, it is assumed that the following condition is satisfied:

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \dots, \quad \text{for every } x \in E. \tag{$*$}$$

For $r > 0$ and $x \in E$, we denote $B(x,r) = \{y \in E : \|x-y\|_n \leqslant r,\ \forall n \in \mathbb{N}\}$. A subset X of E is *bounded* if, for every $n \in \mathbb{N}$, there exists $r_n > 0$ such that $\|x\|_n \leqslant r_n$ for every $x \in X$.

To E we associate, for every $n \in \mathbb{N}$, a Banach space $\mathbb{E}_n$ as follows. For each $n \in \mathbb{N}$, we write

$$x \sim_n y, \quad \text{if and only if } \|x-y\|_n = 0.$$

This defines an equivalence relation on E. We denote by $E_n = E/\sim_n$ the quotient space, and by $\mathbb{E}_n$ the completion of E_n with respect to $\|\cdot\|_n$. (The norm on E_n induced by $\|\cdot\|_n$ and its extension to $\mathbb{E}_n$ are still denoted by $\|\cdot\|_n$.) This construction defines a continuous map $\mu_n : E \longrightarrow \mathbb{E}_n$. For $r > 0$ and $x \in \mathbb{E}_n$, we denote $B_n(x,r) = \{y \in \mathbb{E}_n : \|x-y\|_n \leqslant r\}$.

For each subset $X \subset E$, and each $n \in \mathbb{N}$, we set $X_n = \mu_n(X)$, and we denote by $\overline{X_n}$, and ∂X_n, respectively, the closure and the boundary of X_n with respect to $\|\cdot\|_n$ in $\mathbb{E}_n$. Similarly, for every $m \geqslant n$, we can define an equivalence relation on $\mathbb{E}_m$, still denoted $\sim_n$, which defines a continuous map $\mu_{n,m} : \mathbb{E}_m \longrightarrow \mathbb{E}_n$, since $\mathbb{E}_m/\sim_n$ can be regarded as a subset of $\mathbb{E}_n$. In fact, E is the projective limit of $(\mathbb{E}_n)_{n\in\mathbb{N}}$.

LEMMA 1.2. *Assume that the condition* $(*)$ *is satisfied, and let* X *be a closed subset of* $\mathbb{E}$. *Then, for every sequence* $(z_n)_{n\in\mathbb{N}}$ *with* $z_n \in \overline{X_n}$, *such that* $(\mu_{n,m}(z_m))_{m\geqslant n}$ *is a Cauchy sequence in* $\overline{X_n}$ *for every* $n \in \mathbb{N}$, *there exists an* $x \in X$ *such that* $(\mu_{n,m}(z_m))_{m\geqslant n}$ *converges to* $\mu_n(x) \in X_n$ *for every* $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, let $D(n) \subset \mathbb{E}_n$. We define

$$\begin{aligned} D(\infty) = \{x \in E :\ & \exists N_0 \subset \mathbb{N} \text{ infinite and } z_n \in D(n) \text{ for } n \in N_0 \text{ such that} \\ & \forall n \in \mathbb{N},\ \mu_{n,m}(z_m) \to \mu_n(x) \text{ as } m \to \infty \text{ with } m \in N_0 \text{ and } m \geqslant n\}. \end{aligned} \tag{1.1}$$

Let us recall the notion of a *pseudo-interior*, introduced in [**3**].

DEFINITION 1.3. Let X be a subset of E. The *pseudo-interior* of X is defined as follows:

$$\text{pseudo-int}(X) = \{x \in X : \mu_n(x) \in \overline{X_n} \setminus \partial X_n, \text{ for every } n \in \mathbb{N}\}.$$

The set X is *pseudo-open* if $X = \text{pseudo-int}(X)$.

We define, for every $n \in \mathbb{N}$, the multivalued map $S_n : X \longrightarrow X$ by

$$S_n(x) = \{y \in X : \|x-y\|_n = 0\}.$$

DEFINITION 1.4. Let X be a closed subset of E. A compact map $f : X \longrightarrow E$ is called *admissible* if, for every $n \in \mathbb{N}$, the following conditions hold.

(1) The multivalued map $F_n : X_n \longrightarrow \mathbb{E}_n$ defined by

$$F_n(\mu_n(x)) = \overline{\text{co}}(\mu_n \circ f \circ S_n(x))$$

admits an upper semi-continuous extension $\mathbb{F}_n : \overline{X_n} \longrightarrow \mathbb{E}_n$ with convex, compact values.

(2) For every $\varepsilon > 0$, there exists $m \geqslant n$ such that for every $x \in X$,

$$\operatorname{diam}_n(f(S_m(x))) < \varepsilon.$$

REMARK 1.5. Notice that if Y is a closed convex subset of E and $f : X \longrightarrow E$ is an admissible map such that $f(X) \subset Y$, then, for every $n \in \mathbb{N}$, the extension $\mathbb{F}_n$ can be chosen such that $\mathbb{F}_n(\overline{X_n}) \subset \overline{Y_n}$, since $F_n(X_n) \subset \overline{Y_n}$. Indeed, otherwise, its intersection with $\overline{Y_n}$ is also an extension of F_n.

2. *Fixed point results*

THEOREM 2.1. *Let $(E, \{\|\cdot\|_n\}_{n\in\mathbb{N}})$ be a Fréchet space, let C be a closed cone in E, and let $f : C \longrightarrow C$ be admissible and completely continuous. Assume that there exist U and V, two bounded pseudo-open subsets of E such that $0 \in U \subset \overline{U} \subset V$, and for every $n \in \mathbb{N}$,*

(1) $\|y\|_n \geqslant \|x\|_n$, *for all $y \in \mathbb{F}_n(x)$ and $x \in \partial U_n \cap \overline{C_n}$;*
(2) $\|y\|_n \leqslant \|x\|_n$, *for all $y \in \mathbb{F}_n(x)$ and $x \in \partial V_n \cap \overline{C_n}$;*

or

(1′) $\|y\|_n \leqslant \|x\|_n$, *for all $y \in \mathbb{F}_n(x)$ and $x \in \partial U_n \cap \overline{C_n}$;*
(2′) $\|y\|_n \geqslant \|x\|_n$, *for all $y \in \mathbb{F}_n(x)$ and $x \in \partial V_n \cap \overline{C_n}$;*

(*where $\mathbb{F}_n$ is as given in the definition of an admissible map*). *Then there exists $x \in C \cap \overline{V} \cap D(\infty)$, with $x = f(x)$; here,*

$$D(n) = \overline{C_n} \cap \overline{V_n \setminus U_n}$$

and $D(\infty)$ is as defined in (1.1).

Proof. Since f is admissible, for every $n \in \mathbb{N}$ we have $\mathbb{F}_n : \overline{C_n} \longrightarrow \mathbb{E}_n$ an upper semi-continuous extension of F_n, defined by

$$F_n(\mu_n(x)) = \overline{\mathrm{co}}(\mu_n \circ f \circ S_n(x)) \subset \overline{C_n}.$$

Note that for every $n \in \mathbb{N}$, $\overline{C_n}$ is a cone. Indeed, let $\hat{x}, \hat{y} \in C_n$, and let $\lambda \in [0,1]$. For every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$, we have $\lambda x + (1-\lambda)y \in C$, and hence

$$\mu_n(\lambda x + (1-\lambda)y) = \lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \lambda\hat{x} + (1-\lambda)\hat{y} \in C_n.$$

Similarly, it is easy to show that $t\hat{x} \in C_n$ for every $t \geqslant 0$. So C_n (and hence $\overline{C_n}$) is a cone.

Observe that for every $n \in \mathbb{N}$, we know that U_n and V_n are open, and that $0 \in U_n \subset \overline{U_n} \subset V_n$. Also, from Remark 1.5, we can consider that $\mathbb{F}_n : \overline{V_n} \cap \overline{C_n} \longrightarrow \overline{C_n}$ for every $n \in \mathbb{N}$.

Theorem 1.1 implies that $\mathbb{F}_n$ has a fixed point $x_n \in D(n)$ for every $n \in \mathbb{N}$. Obviously, $\mu_{n,m}(x_m) \in \mathbb{F}_n(\mu_{n,m}(x_m))$ for every $m \geqslant n$. The compactness and the upper semi-continuity of $\mathbb{F}_1$ permit us to deduce the existence of a subsequence $(\mu_{1,m}(x_m))_{m\in N_1}$, converging to $z_1 \in \overline{V_1}$, such that $z_1 \in \mathbb{F}_1(z_1)$. Now take $N_1' = \{m \in N_1 : m \geqslant 2\}$. The same argument, applied to $(\mu_{2,m}(x_m))_{m\in N_1'}$, implies the existence of a subsequence $(\mu_{2,m}(x_m))_{m\in N_2}$, converging to $z_2 \in \overline{V_2}$, such that $z_2 \in \mathbb{F}_2(z_2)$. Moreover, $\mu_{1,2}(z_2) = z_1$. By repeating the argument, we obtain

$$\ldots \subset N_2 \subset N_1' \subset N_1 \subset \mathbb{N},$$

and, for every $n \in \mathbb{N}$,

$$z_n \in \mathbb{F}_n(z_n) \cap \overline{V_n} \text{ such that } (\mu_{n,m}(x_m))_{m\in N_n} \text{ converges to } z_n.$$

By a diagonalization process, we deduce the existence of a subsequence $(x_m)_{m\in N_0}$ such that $(\mu_{n,m}(x_m))_{m\in N_0}$ converges to z_n for every $n \in \mathbb{N}$. Lemma 1.2 and the definition given in (1.1) imply the existence of $x \in C \cap \overline{V} \cap D(\infty)$ such that $\mu_n(x) \in \mathbb{F}_n(\mu_n(x))$ for every $n \in \mathbb{N}$.

To conclude, we have to show that $x = f(x)$. If this is not true, there exists $n \in \mathbb{N}$ such that $\|x - f(x)\|_n = d > 0$. Since f is admissible, there exists $m \geqslant n$ such that

$$d/2 > \operatorname{diam}_n(f(S_m(x))) = \operatorname{diam}_n(\operatorname{co}(f(S_m(x)))).$$

The fact that $\mu_m(x) \in \mathbb{F}_m(\mu_m(x))$ implies that there exists $y \in \operatorname{co}(f(S_m(x)))$ such that $\|x - y\|_m < d/2$. Thus

$$\begin{aligned} d &= \|x - f(x)\|_n \\ &\leqslant \|x - y\|_n + \|y - f(x)\|_n \\ &< \|x - y\|_m + d/2 \\ &< d, \end{aligned}$$

a contradiction. □

Remark 2.2. The last theorem is true for f defined on $C \cap \overline{V}$ if

$$\overline{C_n} \cap \overline{V_n} \subset \overline{\mu_n(C \cap \overline{V})}$$

for every $n \in \mathbb{N}$. In that case, it is also true if Condition (1) in Definition 1.4 is replaced by the following condition.

(1′) For every $n \in \mathbb{N}$, the multivalued map $F_n : C_n \cap V_n \longrightarrow \mathbb{E}_n$ defined by

$$F_n(\mu_n(x)) = \overline{\operatorname{co}}(\mu_n \circ f \circ S_n(x))$$

admits an upper semi-continuous extension $\mathbb{F}_n : \overline{C_n} \cap \overline{V_n} \longrightarrow \mathbb{E}_n$ with convex, compact values.

Remark 2.3. Theorem 2.1 can be generalized to k-set contraction.

Corollary 2.4. *Let $(E, \{\|\cdot\|_n\}_{n\in\mathbb{N}})$ be a Fréchet space, let C be a closed cone in E, and let $f : C \longrightarrow C$ be admissible and completely continuous. Assume that there exist $R > r > 0$ such that, for every $n \in \mathbb{N}$,*

(1) *$\|y\|_n \geqslant r$, for all $y \in \mathbb{F}_n(x)$ and $x \in \partial B_n(0,r) \cap \overline{C_n}$;*
(2) *$\|y\|_n \leqslant R$, for all $y \in \mathbb{F}_n(x)$ and $x \in \partial B_n(0,R) \cap \overline{C_n}$;*

or

(1′) *$\|y\|_n \leqslant r$, for all $y \in \mathbb{F}_n(x)$ and $x \in \partial B_n(0,r) \cap \overline{C_n}$;*
(2′) *$\|y\|_n \geqslant R$, for all $y \in \mathbb{F}_n(x)$ and $x \in \partial B_n(0,R) \cap \overline{C_n}$.*

Then there exists $x \in C \cap B(0,R) \cap D(\infty)$, with $x = f(x)$; here

$$D(n) = \overline{C_n} \cap \overline{B_n(0,R) \setminus B_n(0,r)},$$

and $D(\infty)$ is as defined in (1.1).

3. *Application*

To illustrate how easily the fixed point theory of the previous section can be applied in practice, we consider the boundary value problem

$$\begin{aligned} y''(t) - m^2 y(t) + q(t)g(y(t)) &= 0, \qquad t \in [0,\infty), \\ y(0) &= 0, \\ \lim_{t\to\infty} y(t) &= 0, \end{aligned} \tag{3.1}$$

where $m > 0$ is a constant.

To establish the existence of a solution, we will need some lower- and upper-type inequalities for the Green's function k of the boundary value problem

$$\begin{aligned} y''(t) - m^2 y(t) &= 0, \qquad t \in [0,\infty), \\ y(0) &= 0, \\ \lim_{t\to\infty} y(t) &= 0. \end{aligned} \tag{3.2}$$

It is easy to see that

$$k(t,s) = \begin{cases} \dfrac{e^{-mt}}{2m}(e^{ms} - e^{-ms}), & \text{if } s \leqslant t, \\ \dfrac{e^{-ms}}{2m}(e^{mt} - e^{-mt}), & \text{if } s > t. \end{cases}$$

The following inequalities will be needed:

$$k(t,s)e^{-mt} \leqslant k(s,s)e^{-ms}, \quad \text{for all } t,s \in [0,\infty). \tag{3.3}$$

Also, for any $a, b \in (0,\infty)$, $a < b$, we have

$$k(t,s) \geqslant k_0 e^{-ms} k(s,s), \quad \text{for all } t \in [a,b], s \in [0,\infty); \tag{3.4}$$

here, $k_0 = \min\{e^{-mb}, e^{ma} - e^{-ma}\}$. Now, for $t, s \in [0,\infty)$, we have

$$\begin{aligned} \frac{k(t,s)e^{-mt}}{k(s,s)e^{-ms}} &= \begin{cases} \dfrac{e^{-2mt}}{e^{-2ms}}, & \text{if } s \leqslant t, \\ \dfrac{1 - e^{-2mt}}{1 - e^{-2ms}}, & \text{if } s > t, \end{cases} \\ &\leqslant \begin{cases} \dfrac{e^{-2ms}}{e^{-2ms}}, & \text{if } s \leqslant t, \\ \dfrac{1 - e^{-2ms}}{1 - e^{-2ms}}, & \text{if } s > t, \end{cases} \\ &= 1, \end{aligned}$$

whereas for $t \in [a,b]$ and $s \in [0,\infty)$, we have

$$\begin{aligned} \frac{k(t,s)}{k(s,s)e^{-ms}} &= \begin{cases} e^{2ms-mt}, & \text{if } s \leqslant t, \\ \dfrac{e^{mt} - e^{-mt}}{1 - e^{-2ms}}, & \text{if } s > t, \end{cases} \\ &\geqslant \begin{cases} e^{-mb}, & \text{if } s \leqslant t, \\ e^{ma} - e^{-ma}, & \text{if } s > t, \end{cases} \\ &\geqslant k_0. \end{aligned}$$

We are now in position to prove our main existence result.

THEOREM 3.1. *Let* $q : [0,\infty) \longrightarrow [0,\infty)$ *be measurable, and let* $g : [0,\infty) \longrightarrow [0,\infty)$ *be continuous and nondecreasing. Suppose that the following conditions are satisfied:*

(i) $\lim_{t\to\infty} e^{-mt} \int_0^t e^{ms} q(s)\,ds = 0$;

(ii) $\lim_{t\to\infty} e^{mt} \int_t^\infty e^{-ms} q(s)\,ds = 0$;

(iii) *there exists* $R > 0$ *such that*

$$R \geqslant \sup_{t\in[0,\infty)} e^{-mt} \int_0^\infty k(t,s)q(s)g(e^{ms}R)\,ds;$$

(iv) *for fixed* $a, b \in (0,\infty)$ *with* $a < b$, *there exists* $r \in (0,R)$ *such that*

$$r \leqslant g(k_0 r) \sup_{t\in[a,b]} e^{-mt} \int_a^b k(t,s)q(s)\,ds,$$

where k_0 *is as defined in inequality* (3.4).

Then (3.1) *has a solution* y *with*

$$\sup_{t\in[0,\infty)} |e^{-mt}y(t)| \leqslant R \quad \text{and} \quad y(t) \geqslant k_0 r, \quad \text{for } t \in [a,b].$$

Proof. Choose $\{b_n\}$ an increasing sequence such that $b_1 = b$ and $b_n \to \infty$. We endow $C([0,\infty))$ with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$, defined as follows:

$$\|y\|_n = \sup_{t\in[0,b_n]} |e^{-mt}y(t)|.$$

We denote $E = (C([0,\infty)), \{\|\cdot\|_n\}_{n\in\mathbb{N}})$ and

$$C = \{y \in E : y(t) \geqslant 0 \text{ on } [0,\infty) \text{ and } y(t) \geqslant k_0\|y\|_n, \forall t \in [a,b], \forall n \in \mathbb{N}\}.$$

For $n \in \mathbb{N}$,

$$\mathbb{E}_n = (C[0,b_n], \|\cdot\|_n),$$
$$C_n = \overline{C_n} = \{y \in \mathbb{E}_n : y(t) \geqslant 0 \text{ on } [0,b_n] \text{ and } y(t) \geqslant k_0\|y\|_n, \forall t \in [a,b]\},$$

and

$$D(n) = C_n \cap \{z \in \mathbb{E}_n : r \leqslant \|z\|_n \leqslant R\}.$$

Notice that

$$D(\infty) \subset \{y \in B(0,R) : y(t) \geqslant 0 \text{ on } [0,\infty) \text{ and } y(t) \geqslant k_0 r, \forall t \in [a,b]\}.$$

Let $f : C \cap B(0,R) \longrightarrow E$ be given by

$$f(y)(t) = \int_0^\infty k(t,s)q(s)g(y(s))\,ds, \quad \text{for } t \in [0,\infty).$$

We will establish the existence of a solution to (3.1) by applying Corollary 2.4. First, notice that $C_n \cap B_n(0,R) \subset \mu_n(C \cap B(0,R))$ for every $n \in \mathbb{N}$. To see this, fix $n \in \mathbb{N}$, and let $y \in C_n \cap B_n(0,R)$. If we let z be defined by

$$z(t) = \begin{cases} y(t), & \text{if } t \in [0,b_n], \\ y(b_n), & \text{if } t > b_n, \end{cases}$$

then $z \in C \cap B(0,R)$, and $\|y - z\|_n = 0$. Thus $y \in \mu_n(C \cap B(0,R))$.

Next, we show that $f : C \cap B(0,R) \longrightarrow C$. To see this, let $y \in C \cap B(0,R)$. Clearly, $f(y)(t) \geqslant 0$ for $t \in [0,\infty)$. Now fix $n \in \mathbb{N}$; then inequalities (3.3) and (3.4) imply that for $t \in [a,b]$,

$$\begin{aligned} f(y)(t) &= \int_0^\infty k(t,s)q(s)g(y(s))\,ds \\ &\geqslant k_0 \int_0^\infty e^{-ms}k(s,s)q(s)g(y(s))\,ds \\ &\geqslant k_0 \int_0^\infty e^{-mx}k(x,s)q(s)g(y(s))\,ds \\ &= k_0 e^{-mx} f(y(x)), \end{aligned}$$

for any $x \in [0,\infty)$. As a result, we have

$$f(y)(t) \geqslant k_0 \|f(y)\|_n, \quad \text{for all } t \in [a,b].$$

Thus $f(y) \in C$.

The argument in [**3**, Lemma 5.4 and Proposition 5.5] guarantees that the function $f : C \cap B(0,R) \longrightarrow C$ is admissible and compact. It remains for us to check Conditions (1) and (2) in Corollary 2.4.

First, however, we must describe the extension $\mathbb{F}_n$ that we are considering. Fix $n \in \mathbb{N}$ and $u \in C_n \cap B_n(0,R)$. Let

$$S_n^*(u) = \{v \in C \cap B(0,R) : v \text{ is a continuous extension of } u\},$$

and observe that

$$\begin{aligned} F_n(u) &= \overline{\text{co}}\big(\mu_n \circ f \circ S_n\big(\mu_n^{-1}(u)\big)\big) \\ &= \overline{f \circ S_n^*}(u), \end{aligned}$$

and

$$\overline{f \circ S_n^*}(u)(t) = \int_0^\infty k_n(t,s,u(s))\,ds,$$

where

$$k_n(t,s,x) = \begin{cases} k(t,s)q(s)g(x), & \text{if } s \leqslant b_n, \\ \{k(t,s)q(s)g(e^{ms}y) : |y| \leqslant R\}, & \text{if } s > b_n. \end{cases}$$

We know from [**3**, Proposition 5.5] that $\overline{f \circ S_n^*}$ is continuous and has compact, convex values. We therefore take $\mathbb{F}_n = F_n = \overline{f \circ S_n^*}$.

Fix $n \in \mathbb{N}$, let $x \in \partial B_n(0,R) \cap C_n$, and let $y \in \mathbb{F}_n(x)$. We must show that

$$\|y\|_n \leqslant R. \tag{3.5}$$

Since g is nondecreasing and $\|x\|_n = R$, for $t \in [0,b_n]$ we have

$$\begin{aligned} |y(t)| &\leqslant \int_0^{b_n} k(t,s)q(s)g(x(s))\,ds + \int_{b_n}^\infty k(t,s)q(s)g(e^{ms}R)\,ds \\ &\leqslant \int_0^{b_n} k(t,s)q(s)g(e^{ms}R)\,ds + \int_{b_n}^\infty k(t,s)q(s)g(e^{ms}R)\,ds \\ &= \int_0^\infty k(t,s)q(s)g(e^{ms}R)\,ds. \end{aligned}$$

Taken together with assumption (iii), this yields

$$\|y\|_n = \sup_{t\in[0,b_n]} |e^{-mt}y(t)| \leqslant \sup_{t\in[0,b_n]} \left(e^{-mt}\int_0^\infty k(t,s)q(s)g(e^{ms}R)\,ds\right) \leqslant R;$$

so (3.5) holds.

Next, fix $n \in \mathbb{N}$, let $x \in \partial B_n(0,r) \cap C_n$, and let $y \in \mathbb{F}_n(x)$. We must show that

$$\|y\|_n \geqslant r. \tag{3.6}$$

Again, since g is nondecreasing, $\|x\|_n = r$ and $x(t) \geqslant k_0 r$ for $t \in [a,b]$, we have, for $t \in [0, b_n]$,

$$\begin{aligned} e^{-mt} y(t) &\geqslant e^{-mt} \int_0^{b_n} k(t,s) q(s) g(x(s))\, ds \\ &\geqslant e^{-mt} \int_a^b k(t,s) q(s) g(x(s))\, ds \\ &\geqslant e^{-mt} \int_a^b k(t,s) q(s) g(k_0 r)\, ds. \end{aligned}$$

Consequently,

$$\|y\|_n \geqslant g(k_0 r) e^{-mt} \int_a^b k(t,s) q(s)\, ds, \quad \text{for all } t \in [0, b_n],$$

which, together with assumption (iv), yields

$$\|y\|_n \geqslant g(k_0 r) \sup_{t \in [a,b]} e^{-mt} \int_a^b k(t,s) q(s)\, ds \geqslant r.$$

Thus, (3.6) holds.

We apply Corollary 2.4 and Remark 2.2 to deduce the result. □

REMARK 3.2.

(a) If there exists $r < R$ with

$$r \leqslant k_0 g(k_0 r) \sup_{t \in [a,b]} \int_a^b e^{-m(t+s)} k(s,s) q(s)\, ds,$$

then (3.4) guarantees that assumption (iv) is satisfied.

(b) If there exists $R > 0$ with

$$R \geqslant \int_0^\infty e^{-ms} k(s,s) q(s) g(e^{ms} R)\, ds,$$

then (3.3) guarantees that assumption (iii) is satisfied.

EXAMPLE. Let

$$q(s) = e^{-\lambda s} \quad \text{and} \quad g(x) = Ax^\alpha + Bx^\beta + C,$$

with $\lambda > 0$, $\alpha, \beta \in (0,1)$, $A, B > 0$ and $C \geqslant 0$. It is clear that g is continuous and nondecreasing. Also, assumptions (iii) and (iv) of the previous theorem are satisfied. To see this, notice that $\max\{\alpha, \beta\} < 1$ guarantees that

$$\lim_{r \to \infty} \frac{r}{A_0 r^\alpha + A_1 r^\beta + A_2} = \infty,$$

for any constants $A_0, A_1 > 0$, $A_2 \geqslant 0$. Now, let $a, b \in (0,1)$ with $a < b$, and notice that (iv) is verified since

$$\lim_{r \to 0} \frac{r}{g(k_0 r)} = \lim_{r \to 0} \frac{r}{A k_0^\alpha r^\alpha + B k_0^\beta r^\beta + C} = 0.$$

Therefore, the problem (3.1) has a non-trivial solution. Notice that if $C = 0$, then $y \equiv 0$ is a solution of (3.1).

The ideas in this section can easily be extended to the more general boundary value problem

$$\begin{aligned} y''(t) - m^2 y(t) + g(t, y(t)) = 0, \qquad & t \in [0, \infty), \\ y(0) = 0, \qquad & \\ \lim_{t \to \infty} y(t) = 0. \qquad & \end{aligned} \tag{3.7}$$

Here, $g : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is such that:

(1) $s \mapsto g(s, y)$ is measurable for any $y \in [0, \infty)$;

(2) $y \mapsto g(s, y)$ is continuous and nondecreasing for almost every $s \in [0, \infty)$;

(3) for any $r > 0$, there exists $h_r \in L^1([0, \infty))$ such that $g(s, y) \leqslant h_r(s)$ for all $y \in [0, r]$ and almost all $s \in [0, \infty)$, and

$$\lim_{t \to \infty} e^{-mt} \int_0^t e^{ms} h_r(s)\, ds = 0 \quad \text{and} \quad \lim_{t \to \infty} e^{mt} \int_t^\infty e^{-ms} h_r(s)\, ds = 0;$$

(4) there exists $R > 0$ such that

$$R \geqslant e^{-mt} \int_0^\infty k(t, s) g(s, e^{ms} R)\, ds;$$

(5) for fixed $a, b \in (0, \infty)$ with $a < b$, there exists $r \in (0, R)$ such that

$$r \leqslant \sup_{t \in [a,b]} e^{-mt} \int_a^b k(t, s) g(s, k_0 r)\, ds.$$

Then (3.7) has a solution y with

$$\sup_{t \in [0, \infty)} |e^{-mt} y(t)| \leqslant R \quad \text{and} \quad y(t) \geqslant k_0 r, \quad \text{for } t \in [a, b].$$

The details are essentially the same, so they are left to the reader.

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