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Existence Theory for Compact and Noncompact Intervals

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Abstract

Existence results are established for second order differential equations on open, half open and compact intervals.

1 Introduction

This paper presents existence results for second order differential equations on finite, semi-finite, infinite, open and half open intervals. In particular, we examine the problem

$$(1.1) \qquad y'' = f(t, y, y') \text{ a.e. on } (a, b)$$

where $-\infty \leq a < b \leq \infty$ and f is a locally Carathéodory function. We also discuss the problem (1.1) with boundary conditions:

$$(1.2) \qquad y(a) = c,$$
$$(1.3) \qquad y(b) = d.$$

The literature on second order differential equations is devoted mainly to boundary value problems on compact intervals [1,3,4,6,7,9,10,11] and to a lesser extent to problems on the semi-infinite interval [2,4,8,11,12]. This paper has two main objectives. Firstly, we are interested in obtaining new existence results for problems for differential equations on bounded and unbounded open and half open intervals. Our second objective is to give a systematic treatment of the smoothness properties of a solution of (1.1). For example, when we examine (1.1)(1.2), it is of some importance to know what conditions must one put on the nonlinearity f to have a solution in $C[a, b) \cap W^{2,1}_{loc}(a, b)$ or in $C^1[a, b) \cap W^{2,1}_{loc}(a, b)$.

Throughout this paper, we will assume the existence of lower and upper solutions of (1.1). In addition, our nonlinearity f will satisfy a growth condition of Nagumo type. Our technique uses results from the theory of boundary value problems on compact intervals together with a simple fixed point result. More precisely, the fixed point is obtained as a limit of fixed points of an appropriate sequence of operators.

2 Preliminaries

Let I be a real interval. We denote by $C^k(I)$ the space of k-times continuously differentiable functions endowed with the topology of uniform convergence on each compact subset of I. This space is a Banach space if I is compact, otherwise, it is a metrizable locally convex space. By $C_c^\infty(I)$, we mean the space of infinitely differentiable functions with compact support in I. We denote by $L^1_{loc}(I)$ the set of functions integrable on each compact subset of I. Observe that if I is compact, $L^1(I) = L^1_{loc}(I)$. Similarly, we denote by $W^{2,1}_{loc}(I)$ the set of functions $y \in W^{2,1}(J)$ for every compact subset $J \subset I$.

Definition 2.1 *We say that $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a $L^1_{loc}(I)$-Carathéodory function if*

- (i) *the map $t \mapsto f(t, y, z)$ is measurable for all $(y, z) \in \mathbb{R}^2$;*
- (ii) *the map $(y, z) \mapsto f(t, y, z)$ is continuous for almost all $t \in I$;*
- (iii) *for any $r > 0$, there exists $h_r \in L^1_{loc}(I)$ such that $|f(t, y, z)| \leq h_r(t)$ for almost all $t \in I$ and for all $|y| \leq r$, $|z| \leq r$.*

Observe that if I is a compact interval, this definition coincides with the usual definition of Carathéodory function.

Now we give the definition of lower and upper solutions of (1.1), (1.1) (1.2) and (1.1) (1.3). We take I a real interval such that $(a, b) \subset I \subset [a, b]$ where $-\infty \leq a < b \leq \infty$.

Definition 2.2 *A function $\alpha \in W^{2,1}_{loc}(I)$ is called a lower solution of (1.1) if $\alpha''(t) \geq f(t, \alpha(t), \alpha'(t))$ almost everywhere on I. When $a \in I$, α is called a lower solution of (1.1) (1.2) if it is a lower solution of (1.1) and satisfies $\alpha(a) \leq c$. Similarly, when $b \in I$, α is called a lower solution of (1.1) (1.3) if α is a lower solution of (1.1) and satisfies $\alpha(b) \leq d$.*
We define an upper solution of (1.1), (1.1) (1.2), and (1.1) (1.3) by reversing the inequalities.

Next we recall a result from the literature on second order differential equations on compact intervals. This theorem will play an important role in the proof of our existence results.

Theorem 2.3 *Let $-\infty < a < b < \infty$ and $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function. Assume there exists $\alpha, \beta \in W^{2,1}[a, b]$ respectively lower and upper solutions of (1.1) (1.2) (1.3) with $\alpha(t) \leq \beta(t)$ on $[a, b]$. In addition, assume there exist $p \in [1, \infty]$, a function $\gamma \in L^q[a, b]$ with $1/p + 1/q = 1$, $M > 0$, and a Borel measurable function $\psi : [0, \infty) \to (0, \infty)$ such that $x^{\frac{1}{p}}/\psi(x) \in L^1_{loc}[0, \infty)$,*

$$|f(t, y, z)| \leq \gamma(t)\psi(|z|) \quad a.e.\ t \in [a, b] \text{ and all } \alpha(t) \leq y \leq \beta(t),$$

and

$$[A]^{\frac{1}{p}} \|\gamma\|_{L^q[a,b]} < \int_r^M \frac{x^{\frac{1}{p}}}{\psi(x)}\, dx,$$

where $r = |d - c|/|b - a|$, and $A = \sup\{\beta(t) - \alpha(x) : t, x \in [a, b]\}$. Then the problem (1.1) (1.2) (1.3) has a solution $y \in W^{2,1}[a, b]$ such that $\alpha(t) \leq y(t) \leq \beta(t)$ and $|y'(t)| \leq M$ for all $t \in [a, b]$.

Remark. Theorem 2.3 is proved in [4,5] when $p = 1$. The case when $p > 1$ can be found in [11, chapter 5].

Now, we formulate a simple fixed point theorem. This will be used in section 3 to establish existence results for second order differential equations on semi-infinite, infinite, open and half open intervals.

Let $\{E_n\}_{n\geq 1}$ be a sequence of topological spaces such that $\ldots \subset E_2 \subset E_1$ and $\cap_{n=1}^{\infty} E_n = E$. We say that a sequence $\{x_n\}_{n\geq 1}$ such that $x_n \in E_n$ for all $n \in \mathbb{N}$ converges to $x \in E$ if the sequence $\{x_n\}_{n\geq k}$ converges to x in E_k for every $k \in \mathbb{N}$.

Theorem 2.4 (Fixed Point Theorem) *Let E be a space and $T : E \to E$ be a multivalued operator. Suppose there exist a sequence of metrizable locally convex spaces $\ldots \subset E_2 \subset E_1$, and a sequence of multivalued operators $\{T_n : E_n \to E_n\}_{n\geq 1}$ such that $\cap_{n\geq 1} E_n = E$, and for each sequences $\{x_{n_k}\}_{k\geq 1}$ and $\{v_{n_k}\}_{k\geq 1}$ with $v_{n_k} \in T_{n_k}(x_{n_k})$, converging to x and v respectively, we have $v \in T(x)$. Assume that for every $n \in \mathbb{N}$, T_n has a fixed point $x_n \in E_n$. In addition, suppose that $\{x_n\}_{n\geq k}$ is relatively compact in E_k for every $k \in \mathbb{N}$. Then T has a fixed point in E.*

Remarks. (i) The proof is immediate. The assumptions yield the existence of a subsequence S of integers and an $x \in E$ with x_n converging to x as $n \to \infty$ in S.
(ii) In the applications in section 3, E_n will be Banach spaces, the operators $T_n : E_n \to E_n$ will be single-valued and the operator $T : E \to E$ will have the following property: for any subsequence S of integers and any $x \in E$ with x_n converging to x as $n \to \infty$ in S, there exists $v \in E$ such that $T_n x_n$ converges to $v \in Tx$ as $n \to \infty$ in S.

3 Existence theory

Let I be a real interval (open, half-open, closed, finite, or infinite) and let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be a $L^1_{loc}(I)$-Carathéodory function. So, $(a, b) \subset I \subset [a, b]$ for some $-\infty \leq a < b \leq \infty$.

We denote by E a space to be specified in the applications and such that $E \subset C^1(a, b)$. Define the multivalued operator $T : E \to E$ by

$$T(u) = \{g \in E \cap W^{2,1}_{loc}(a, b) : \int_a^b g(t)\phi''(t)\,dt = \int_a^b f(t, u(t), u'(t))\phi(t)\,dt \ \ \forall \phi \in C_c^{\infty}(a, b)\}.$$

Of course, for the moment, we don't know if the operator T is well defined.

Let $\{a_k\}_{k\geq 1}$, $\{b_k\}_{k\geq 1}$, $\{c_k\}_{k\geq 1}$, $\{d_k\}_{k\geq 1}$ be sequences of real numbers such that $a \leq \ldots \leq a_2 \leq a_1 < b_1 \leq b_2 \ldots$ and $(a, b) \subset \cup_{k\geq 1}[a_k, b_k] \subset I$. For $k \in \mathbb{N}$, denote by E_k a Banach space, and by $T_k : E_k \to E_k$ an operator to be specified later and such that $E = \cap_{k\geq 1} E_k$, $E_{k-1} \subset E_k$, $E_k \subset C^1[a_k, b_k]$, and $T_k(u)(t) = F_k(u)(t)$ for every $t \in [a_k, b_k]$, where $F_k(u) \in W^{2,1}[a_k, b_k]$ is the unique function satisfying

$$\begin{aligned} &F_k(u)''(t) = f(t, u(t), u'(t)) \ \text{ a.e. on } \ (a_k, b_k), \\ &F_k(u)(a_k) = c_k, \quad F_k(u)(b_k) = d_k \end{aligned}$$

Observe that for every $u \in E_k$,

$$(3.1) \qquad \int_{a_k}^{b_k} T_k(u)(t)\phi''(t)\,dt = \int_{a_k}^{b_k} f(t, u(t), u'(t))\phi(t)\,dt \quad \text{for every } \phi \in C_c^{\infty}(a_k, b_k).$$

Our first result establishes the existence of a solution of (1.1) in $W^{2,1}_{loc}(a, b)$.

Theorem 3.1 *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be a $L^1_{loc}(a,b)$-Carathéodory function. Assume there exist $\alpha \le \beta \in W^{2,1}_{loc}(a,b)$ respectively lower and upper solution of (1.1). In addition, assume there exist $p \in [1,\infty]$, a function $\gamma \in L^q_{loc}(a,b)$ with $1/p + 1/q = 1$, and a Borel measurable function $\psi : [0,\infty) \to (0,\infty)$ such that $x^{\frac{1}{p}}/\psi(x) \in L^1_{loc}[0,\infty)$,*

$$|f(t,y,z)| \le \gamma(t)\psi(|z|) \quad \text{a.e. } t \in I \text{ and all } \alpha(t) \le y \le \beta(t),$$

and

$$\int^{\infty} \frac{x^{\frac{1}{p}}}{\psi(x)}\,dx = \infty.$$

Then (1.1) has a solution $y \in W^{2,1}(a,b)$ such that $\alpha \le y \le \beta$.

Remark. In Theorem 3.1, we can replace

$$\int^{\infty} \frac{x^{\frac{1}{p}}}{\psi(x)}\,dx = \infty \quad \text{by} \quad [A_k]^{\frac{1}{p}}\|\gamma\|_{L^q[a_k,b_k]} < \int_{r_k}^{\infty} \frac{x^{\frac{1}{p}}}{\psi(x)}\,dx.$$

where $r_k = \min\{|\beta(b_k) - \alpha(a_k)|, |\beta(a_k) - \alpha(b_k)|\}/|b_k - a_k|$, and $A_k = \sup\{\beta(t) - \alpha(x) : t, x \in [a_k, b_k]\}$.

Proof. As before, choose $\{a_k\}_{k\ge1}$, $\{b_k\}_{k\ge1}$, $\{c_k\}_{k\ge1}$, $\{d_k\}_{k\ge1}$ sequences of real numbers such that $\cup_{k\ge1}[a_k,b_k] = (a,b)$, $\alpha(a_k) \le c_k \le \beta(a_k)$, and $\alpha(b_k) \le d_k \le \beta(b_k)$. Fix $E = C^1(a,b)$, $E_k = C^1[a_k,b_k]$, and $T_k : E_k \to E_k$ the operator defined by $T_k(u) = F_k(u)$ for all $u \in E_k$ where $F_k(u)$ is defined previously.

It is clear that a fixed point of T in E is a solution of (1.1). Also, $E = \cap_{k\ge1}E_k$, and if $x_n \in E_n$ is a sequence converging to $x \in E$ and $T_n(x_n) \to v \in E$, then (3.1) and the Lebesgue dominated convergence Theorem imply $v \in T(x)$.

On the other hand, a fixed point $y_k \in E_k$ of T_k is a solution in $W^{2,1}[a_k,b_k]$ to the problem

$$\begin{aligned} &y''(t) = f(t, y(t), y'(t)) \ \text{a.e. on } (a_k,b_k), \\ &y(a_k) = c_k, \quad y(b_k) = d_k. \end{aligned} \tag{3.2}$$

By Theorem 2.3, T_k has a fixed point $y_k \in E_k$ such that $\alpha(t) \le y_k(t) \le \beta(t)$ and $|y_k'(t)| \le M_k$ for every $t \in [a_k,b_k]$ with

$$[A_k]^{\frac{1}{p}}\|\gamma\|_{L^q[a_k,b_k]} < \int_{r_k}^{M_k} \frac{x^{\frac{1}{p}}}{\psi(x)}\,dx.$$

where $r_k = \min\{|\beta(b_k) - \alpha(a_k)|, |\beta(a_k) - \alpha(b_k)|\}/|b_k - a_k|$, and $A_k = \sup\{\beta(t) - \alpha(x) : t, x \in [a_k,b_k]\}$. In fact, for every $k \in \mathbb{N}$ and for every $n \ge k$, the following inequalities are satisfied:

$$\alpha(t) \le y_n(t) \le \beta(t), \ |y_n'(t)| \le M_k \ \forall t \in [a_k,b_k], \quad \text{and} \quad |y_n''(t)| \le h_k(t) \text{ a.e. on } [a_k,b_k],$$

where $h_k \in L^1[a_k,b_k]$ is a function given in condition (iii) of definition 2.1.

The Arzela-Ascoli Theorem implies that $\{y_n\}_{n\ge k}$ is relatively compact in E_k. The conclusion follows from Theorem 2.4.

Now, we assume that $-\infty < a \in I$ and we want to point out some additional assumptions that could be imposed in order to guarantee that the solution lies in $C[a,b) \cap W^{2,1}(a,b)$.

Theorem 3.2 *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be a $L^1_{loc}(a,b)$-Carathéodory function with condition (iii) of definition 2.1 replaced by*

(*i*) *for any $r > 0$, there exists $h_r \in L^1_{loc}(a,b)$ such that $|f(t,y,z)| \le h_r(t)$ for almost all $t \in (a,b)$ and for all $|y| \le r$, $z \in \mathbb{R}$; $(t-a)h_r(t) \in L^1_{loc}[a,b)$, and*

$$\lim_{t\to a}(t-a)^2 h_r(t) = 0 \quad \text{if } h_r \notin L^1_{loc}[a,b).$$

In addition, assume there exist $\alpha \le \beta \in C[a,b) \cap W^{2,1}_{loc}(a,b)$ respectively lower and upper solutions of (1.1)(1.2). Then the problem (1.1)(1.2) has a solution $y \in C[a,b) \cap W^{2,1}_{loc}(a,b)$.

Proof. As before, choose the sequences $\{a_k\}_{k\ge1}$, $\{b_k\}_{k\ge1}$, $\{c_k\}_{k\ge1}$, $\{d_k\}_{k\ge1}$ such that $\cup_{k\ge1}[a_k,b_k] = (a,b)$, $\alpha(a_k) \le c_k \le \beta(a_k)$, $\alpha(b_k) \le d_k \le \beta(b_k)$, and $|c_k - c| = \min\{\,|x-c| : \alpha(a_k) \le x \le \beta(a_k)\,\}$. Take $E = C[a,b) \cap C^1(a,b)$, $E_k = C[a,b_k] \cap C^1[a_k,b_k]$, and define $T_k : E_k \to E_k$ by

$$T_k(u)(t) = \begin{cases} F_k(u)(t), & \text{if } t \in [a_k,b_k], \\ \frac{c_k - c}{a_k - a}(t-a) + c, & \text{if } t \in [a,a_k). \end{cases}$$

It is clear that $E = \cap_{k\ge1}E_k$, and if $x_k \to x$ and $T_k(x_k) \to v$ in E, then $v \in T(x)$.

Again, by considering the problem (3.2) and applying Theorem 2.3, we get the existence of a fixed point $y_k \in E_k$ to T_k for every $k \in \mathbb{N}$. Moreover, for every $n \ge k$, the following inequalities are satisfied:

$$\alpha(t) \le y_n(t) \le \beta(t), \quad |y_n'(t)| \le M_k \quad \text{for every } t \in [a_k,b_k],$$

$$\inf_{t\in[a,a_k]}\{\alpha(t)\} \le y_n(t) \le \sup_{t\in[a,a_k]}\{\beta(t)\} \quad \text{for every } t \in [a,a_k], \text{ and}$$

$$|y_n''(t)| \le h_k(t) \quad \text{almost everywhere on } [a_k,b_k],$$

where $M_k > 0$ is as in the proof of Theorem 3.1, and $h_k \in L^1_{loc}(a,b)$ is given in (i).

By the Arzela-Ascoli Theorem, $\{y_n\}_{n\ge k}$ is relatively compact in $C^1[a_k,b_k]$ for every $k \in \mathbb{N}$. Now, we want to show that it is relatively compact in $C[a,b_k]$. We already know that this set is bounded. All that remains to be shown is the equicontinuity at t_0 for every $t_0 \in [a,b_k]$.

First, we examine the case when $t_0 \ne a$. Choose $N \ge k$ such that $a < a_N < t_0$. Then for $n \ge N$ and $a_N \le s < t \in [a,b_k]$, we have

$$\begin{aligned} |y_n(t) - y_n(s)| &= \left|\frac{y_n(b_k) - y_n(a_N)}{b_k - a_N}(t-s) - \frac{t-s}{b_k - a_N}\int_{a_N}^{b_k}\int_{a_N}^{r} f(\tau,y_n(\tau),y_n'(\tau))\,d\tau\,dr \right. \\ &\quad \left. + \int_s^t\int_{a_N}^{r} f(\tau,y_n(\tau),y_n'(\tau))\,d\tau\,dr\right| \\ &\le K_k|t-s| + 2\|h_N\|_{L^1[a_N,b_k]}|t-s|, \end{aligned}$$

with $K_k = (1/(b_k - a_1))\sup\{\,|\beta(t)|, |\alpha(t)| : t \in [a,b_k]\,\}$. This implies the equicontinuity of $\{y_n\}_{n\ge k}$ at $t_0 \in (a,b_k]$.

On the other hand, observe that for $n \geq k$, we have

$$|y_n(t) - y_n(a)| = \begin{cases} |\frac{c_n-c}{a_n-a}(t-a)|, & \text{if } t \in [a, a_n], \\ \Big|(c_n - c) + \frac{y_n(b_k)-y_n(a_n)}{b_k-a_n}(t-a) \\ \quad - \frac{b_k-t}{b_k-a_n}\int_{a_n}^t (r-a_n) f(r, y_n(r), y_n'(r))\, dr \\ \quad - \frac{t-a_n}{b_k-a_n}\int_t^{b_k} (b_k - r) f(r, y_n(r), y_n'(r))\, dr\Big|, & \text{if } t \in (a_n, b_k]. \end{cases}$$

$$\leq \begin{cases} |\frac{c_n-c}{a_n-a}(t-a)|, & \text{if } t \in [a, a_n], \\ |c_n - c| + K_k|t-a| + \frac{b_k-t}{b_k-a_n}\int_{a_n}^t (r-a_n) h_k(r)\, dr \\ \quad + \frac{t-a_n}{b_k-a_n}\int_t^{b_k}(b_k - r) h_k(r)\, dr, & \text{if } t \in (a_n, b_k]. \end{cases}$$

$$\leq \begin{cases} |\frac{c_n-c}{a_n-a}(t-a)|, & \text{if } t \in [a, a_n], \\ |c_n - c| + K_k|t-a| + \frac{b_k-t}{b_k-a_1}\int_a^t (r-a) h_k(r)\, dr \\ \quad + \frac{t-a}{b_k-a_1}\int_t^{b_k}(b_k - r) h_k(r)\, dr, & \text{if } t \in (a_n, b_k]. \end{cases}$$

By choice of c_n, $c_n \to c$. Also, by (i) and Hospital's rule, $(r-a)h_k(r) \in L^1_{loc}[a, b)$ and

$$\lim_{t \to a}(t-a)\int_t^{b_k}(b_k - r)h_k(r)\, dr = 0.$$

Therefore, $\{y_n\}_{n \geq k}$ is equicontinuous at a, and the proof is complete.

Remark. Observe that the solution y obtained in the previous theorem is not necessarily in $C^1[a, b)$ but $y' \in L^1_{loc}[a, b)$.

Theorem 3.3 *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be a $L^1_{loc}[a, b)$-Carathéodory function. Assume there exist $\alpha \leq \beta \in W^{2,1}_{loc}[a, b)$ respectively lower and upper solutions of (1.1) (1.2). In addition, assume there exist $p \in [1, \infty]$, a function $\gamma \in L^q_{loc}[a, b)$ with $1/p + 1/q = 1$, and a Borel measurable function $\psi : [0, \infty) \to (0, \infty)$ such that $x^{\frac{1}{p}}/\psi(x) \in L^1_{loc}[0, \infty)$.*

$$|f(t, y, z)| \leq \gamma(t)\psi(|z|) \quad a.e.\ t \in I \text{ and all } \alpha(t) \leq y \leq \beta(t).$$

and

$$\int^\infty \frac{x^{\frac{1}{p}}}{\psi(x)}\, dx = \infty.$$

Then (1.1) (1.2) has a solution $y \in W^{2,1}_{loc}[a, b)$ such that $\alpha \leq y \leq \beta$.

Proof. The proof is similar to that in Theorem 3.1. We choose $a_k \equiv a$, and $c_k \equiv c$.

Remark. In Theorem 3.3, we can replace

$$\int^\infty \frac{x^{\frac{1}{p}}}{\psi(x)}\, dx = \infty \quad \text{by} \quad [A_k]^{\frac{1}{p}} \|\gamma\|_{L^q[a, b_k]} < \int_{r_k}^\infty \frac{x^{\frac{1}{p}}}{\psi(x)}\, dx.$$

where $r_k = \min\{|\beta(b_k) - \alpha(a)|, |\beta(a) - \alpha(b_k)|\}/|b_k - a|$, and $A_k = \sup\{\beta(t) - \alpha(x) : t, x \in [a, b_k]\}$.

Similar results could be obtained if we are interested in solutions in $W^{2,1}_{loc}(a,b) \cap C(a,b]$, $W^{2,1}_{loc}(a,b]$, and $W^{2,1}_{loc}(a,b) \cap C[a,b]$.

4 Examples

We give three very simple examples to illustrate the theorems of the previous section. We consider the problem

$$t^r y''(t) = f(t, y(t)) \quad \text{a.e. } t \in (0, b), \tag{4.1}$$

$$y(0) = c, \tag{4.2}$$

where $f : (0,b) \times \mathbb{R} \to \mathbb{R}$ is a $L^\infty(0,b)$–Carathéodory function.

Example 4.1 *Assume there exist $\alpha \leq \beta \in W^{2,1}_{loc}(0,b)$ such that $t^r\alpha''(t) \geq f(t, \alpha(t))$ and $t^r\beta''(t) \leq f(t, \beta(t))$ almost everywhere on $(0,b)$. Then, for any $r \in \mathbb{R}$, the problem (4.1) has a solution in $W^{2,1}_{loc}(0,b)$.*

To get more regularity on the solution, we add some restrictions.

Example 4.2 *Assume there exist $\alpha \leq \beta \in W^{2,1}_{loc}(0,b) \cap C[0,b)$ such that $t^r\alpha''(t) \geq f(t, \alpha(t))$ and $t^r\beta''(t) \leq f(t, \beta(t))$ almost everywhere on $(0,b)$, and $\alpha(0) \leq c \leq \beta(0)$. Then, for any $r < 2$, the problem (4.1) (4.2) has a solution in $W^{2,1}_{loc}(0,b) \cap C[0,b)$.*

Example 4.3 *Under the assumptions of the previous example with $\alpha \leq \beta \in W^{2,1}_{loc}[0,b)$, for any $r < 1$, the problem (4.1) (4.2) has a solution in $W^{2,1}_{loc}[0,b)$.*

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