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SECOND ORDER STURM-LIOUVILLE BVP'S WITH IMPULSES AT VARIABLE MOMENTS[1]

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Abstract. In this paper, existence results are obtained for second order Sturm-Liouville BVP's with impulses at variable times. The proofs rely on a fixed point theorem for composition of acyclic maps.

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1. Introduction

In this paper, the second order problem with impulses at variable times

$$\text{(P1)}\qquad \begin{aligned} x''(t) &= f(t, x(t), x'(t)), && \text{a.e. } t \in [0,1], \\ x(t^+) &= I(x(t)), && \text{if } t = \tau(x(t)), \\ x'(t^+) &= J(x(t)), && \text{if } t = \tau(x(t)), \end{aligned}$$

with the Sturm-Liouville boundary condition

$$\text{(SL)}\qquad \begin{aligned} x(0) - ax'(0) &= \alpha, && a \geq 0, \\ x(1) + bx'(1) &= \beta, && b \geq 0, \end{aligned}$$

is studied.

The literature on second order impulsive boundary value problems is devoted totally to the case where the impulses are at fixed times (i.e. when τ is constant), see for example [3]–[5]. This is in contrast to first order initial value problems where some results were obtained with impulses at variable times, see [2], [6], [7], [10].

To our knowledge, it is the first paper to treat second order boundary value problems with impulses at variable times.

For sake of simplicity, in section 2, we begin by discussing the problem (P1), (SL) with homogenous Dirichlet boundary condition (i.e. $a = b = \alpha = \beta = 0$). To obtain the existence of a solution, we use fixed point theory for composition of acyclic maps [8], [9], and the fact that the solution set of an initial value problem has R_δ-values [1], [11]. Some of the ideas of this

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approach were used in our paper [7] on periodic first order problem with impulses at variable moments.

In section 3, we then use some of the ideas and results in section 2 to discuss the Sturm-Liouville problem (P1), (SL).

The final section examines a Sturm-Liouville problem (P2), (SL) with a different type of barrier, where

$$\text{(P2)} \qquad \begin{aligned} &x''(t) = f(t, x(t), x'(t)), && \text{a.e. } t \in [0,1],\\ &x(t^+) = I(x(t)), && \text{if } t = \tau(x'(t)),\\ &x'(t^+) = J(x(t)), && \text{if } t = \tau(x'(t)). \end{aligned}$$

For the remainder of this section, we gather together some known definitions and facts for the convenience of the reader.

We denote $\Omega^2 = \{x : [0,1] \to \mathbb{R} : \exists\, 0 < t_1 < \cdots < t_n = 1$ such that x is C^2 on $[0, t_1]$ and on $]t_{i-1}, t_i]$, $i = 2, \ldots, n$; and $\lim_{h \to 0} x^{(k)}(t_i + h)$ exists for $k = 0, 1, 2$, and $i = 1, \ldots, n\}$.

Definition 1.1. Let X, Y be two metric spaces, a multivalued mapping $\phi : X \to Y$ is *upper semi-continuous* (u.s.c.) if $\{x : \phi(x) \cap K \neq \emptyset\}$ is closed for every closed subset K of Y. It is *acyclic* if it is upper semi-continuous with compact values and for every $x \in X$, $H^m(\phi(x)) = \delta_{0m}\mathbb{Z}$, where $\{H^m\}_{m \in \mathbb{N}}$ denote the Čech cohomology functor with integer coefficients.

Definition 1.2. A nonempty compact metric space X is an *R_δ-set* if it is the intersection of a decreasing sequence of compact contractible metric spaces.

Lemma 1.3. *An upper semi-continuous multivalued mapping with compact R_δ-values is acyclic.*

The following theorem is a corollary of a fixed point result which can be found in [9].

Theorem 1.4. *Let $r < 0 < s$, $m \neq 1$, and $H : [r,s] \times [0,1] \to \mathbb{R}$ an acyclic map such that*

(i) *for every $\lambda \in [0,1[$, $r \notin H(r,\lambda)$ and $s \notin H(s,\lambda)$;*
(ii) *$H(x,0) = \{mx\}$ for every $x \in [r,s]$.*

Then $H(\cdot, 1)$ has a fixed point.

2. Homogenous Dirichlet boundary condition

In this section, we discuss in detail the second order homogenous Dirichlet boundary value problem with impulses at variable times (P1), (D), where

$$\text{(P1)} \qquad \begin{aligned} &x''(t) = f(t, x(t), x'(t)) && \text{a.e. } t \in [0,1],\\ &x(t^+) = I(x(t)), && \text{if } t = \tau(x(t)),\\ &x'(t^+) = J(x(t)), && \text{if } t = \tau(x(t)), \end{aligned}$$

$$\text{(D)} \qquad \begin{aligned} x(0) &= 0,\\ x(1) &= 0. \end{aligned}$$

We will assume that the following conditions hold:

(H1) The functions $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $I : \mathbb{R} \to \mathbb{R}$, $J : \mathbb{R} \to \mathbb{R}$ are continuous, and $\tau : \mathbb{R} \to]0,1[$ is of class C^2.
(H2) There exist $r < 0 < s$ such that $f(t,x,r) = f(t,x,s) \equiv 0$ for $t \in [0,1]$ and $x \in [-k,k]$.
(H3) $J(r) = r$, $J(s) = s$, and $J[r,s] \subset [r,s]$.
(H4) $\tau''(x) + \lambda(\tau'(x))^3 f(t,x,y) > 0$ for all $\lambda \in [0,1]$, $t \in [0,1]$, $x \in [-k,k]$, $y \in [r,s]$ such that $y\tau'(x) = 1$.
(H5) $xI(x) \geq 0$ for $x \in [r,s]$.
(H6) $\tau(I(x)) \leq \tau(x)$ for $x \in [r,s]$.

Here $k = \max|I[r,s]| + \max\{|r|,s\}$.

Theorem 2.1. *If* (H1) - (H6) *hold, then the problem* (P1), (D) *has a solution in* Ω^2.

Before we prove Theorem 2.1, we first introduce some notations and obtain some preliminary results.

For $x_0, y_0 \in \mathbb{R}$, $t_0 \in [0,1]$ and $\lambda \in [0,1]$, we consider the problem

$$\mathrm{P}(t_0,x_0,y_0,\lambda) \qquad \begin{aligned} x''(t) &= f_\lambda(t,x(t),x'(t)), \quad t \in [0,1], \\ x(t_0) &= x_0, \\ x'(t_0) &= y_0, \end{aligned}$$

where we define

$$f_\lambda(t,x,y) = \lambda f(t,\pi_1(x),\pi_2(y,\lambda)),$$

with $\pi_1 : \mathbb{R} \to [-k,k]$, and $\pi_2 : \mathbb{R} \times [0,1] \to [r,s]$ given by

$$\pi_1(x) = \begin{cases} k, & \text{if } x > k, \\ x, & \text{if } -k \leq x \leq k, \\ -k, & \text{if } x < -k; \end{cases}$$

and

$$\pi_2(x,\lambda) = \begin{cases} r, & \text{if } x < r + \frac{(1-\lambda)}{2n}r, \\ x, & \text{if } x \in [r + \frac{(1-\lambda)}{n}r, s - \frac{(1-\lambda)}{n}s], \\ s, & \text{if } x > s - \frac{(1-\lambda)}{2n}s \\ r + \frac{(1-\gamma)(1-\lambda)}{n}r, & \text{if } x = r + \frac{(1-\lambda)(2-\gamma)}{2n}r, \ \gamma \in [0,1], \\ s + \frac{(1-\gamma)(1-\lambda)}{n}s, & \text{if } x = s - \frac{(1-\lambda)(2-\gamma)}{2n}s, \ \gamma \in [0,1]; \end{cases}$$

here $n > 2$ will be fixed later (see Lemma 2.7).

Remark 2.2. Notice that f_λ is bounded.

Define the following maps:

$\mathcal{I} : [r, s] \times [0, 1] \to [0, 1] \times \mathbb{R}^2 \times [0, 1]$ by
$\mathcal{I}(y, \lambda) = (0, 0, y, \lambda)$,

$\mathcal{S} : [0, 1] \times \mathbb{R}^2 \times [0, 1] \to C^1[0, 1] \times [0, 1]$ by
$\mathcal{S}(t, x_0, y_0, \lambda) = \{(x, \lambda) : x \text{ is solution of } \mathrm{P}(t_0, x_0, y_0, \lambda)\}$,

$\mathcal{T} : Z \to [0, 1] \times C^1[0, 1] \times [0, 1]$ by
$\mathcal{T}(x, \lambda) = (\widetilde{\mathcal{T}}(x), x, \lambda)$,

$\mathcal{J} : [0, 1] \times C^1[0, 1] \times [0, 1] \to [0, 1] \times \mathbb{R}^2 \times [0, 1]$ by
$\mathcal{J}(t, x, \lambda) = (t, \lambda I(x(t)), \widetilde{J}(x'(t), \lambda), \lambda)$,

$\mathcal{E} : C^1[0, 1] \times [0, 1] \to \mathbb{R}$ by
$\mathcal{E}(x, \lambda) = x(1)$,

where

$$Z = \{(x, \lambda) \in C^2[0, 1] \times [0, 1] : x''(t) = f_\lambda(t, x(t), x'(t)), \\ x(t) \in [-k, k], x'(t) \in [r, s]\} \subset C^1[0, 1] \times [0, 1],$$

$$\widetilde{\mathcal{T}}(x) = \inf\{t > 0 : t = \tau(x(t))\},$$

and

$\widetilde{J} : \mathbb{R} \times [0, 1] \to [r, s]$ is given by $\widetilde{J}(y, \lambda) = \lambda J(\pi_3(y)) + (1 - \lambda)\pi_3(y)$,

with $\pi_3 : \mathbb{R} \to [r, s]$ defined by

$$\pi_3(y) = \begin{cases} s, & \text{if } y > s, \\ y, & \text{if } r \leq y \leq s, \\ r, & \text{if } y < r. \end{cases}$$

It is clear that $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{E}$ are continuous.

Lemma 2.3. *Suppose* (H1) *holds. Then the map* $\mathcal{S}$ *is u.s.c. with nonempty* R_δ*-values, and so, is an acyclic map.*

Proof. The problem $\mathrm{P}(t_0, x_0, y_0, \lambda)$ is equivalent to the first order system

$$\big(x'(t), y'(t)\big) = \big(y(t), f_\lambda(t, x(t), y(t))\big), \quad t \in [0, 1],$$
$$\big(x(t_0), y(t_0)\big) = \big(x_0, y_0\big).$$

Since the growth of the right hand side is at most linear, the solution set is nonempty. In fact, the conclusion follows immediately from results in [8]. □

Lemma 2.4. *Suppose that* (H1) *and* (H2) *hold. Then every solution* x *of* $\mathrm{P}(t_0, x_0, y_0, \lambda)$ *with* $y_0 \in [r, s]$ *satisfies:*

(i) $x'(t) \in [r, s]$ *for all* $t \in [t_0, 1]$;
(ii) $x(t) - x_0 \in [r, s]$ *for all* $t \in [t_0, 1]$.

Proof. Suppose that $x'[t_0, 1] \not\subset]-\infty, s]$. Then there exist $t_0 < t_1 < t_2$ such that $x'(t_1) = s$ and $x'(t) > s$ for all $t \in]t_1, t_2]$. On the other hand,

$$x''(t) = \lambda f(t, \pi(x(t)), s) = 0 \quad \text{on } [t_1, t_2],$$

and so, $x'(t) = s$ for all $t \in [t_1, t_2]$, a contradiction. A similar argument shows that $x'(t) \geq r$ for all $t \in [t_0, 1]$.

Statement (ii) is a direct consequence of (i). □

Remark 2.5. Lemma 2.4 implies that $\mathcal{S} \circ \mathcal{I}([r, s] \times [0, 1]) \subset Z$.

Lemma 2.6. *Under* (H1), $\widetilde{\mathcal{T}} : X \to]0, 1[$ *is continuous, where* $X = \{x \in C[0, 1] : x \in C^1[0, 1]$ *and* $\tau'(x(t))x'(t) < 1$ *for all* t *such that* $t = \tau(x(t))\}$.

Proof. First of all, the Intermediate Value Theorem implies that $\widetilde{\mathcal{T}}$ is well defined. Let $\varepsilon > 0$, $x \in X$, and denote $\tilde{t} = \widetilde{\mathcal{T}}(x)$. Take $0 < \gamma < \varepsilon$ such that

$$1 - \tau'(x(\tilde{t}))x'(\tilde{t}) > \gamma.$$

By continuity, there exists $0 < \eta \leq \gamma$ such that for $|t - \tilde{t}| < \eta$, we have

$$1 - \tau'(x(t))x'(t) > \gamma/2 \quad \text{and} \quad |x(t) - x(\tilde{t})| \leq 1.$$

Thus,

$$\frac{\gamma\eta}{2} \leq \int_{\tilde{t}}^{\tilde{t}+\eta} 1 - \tau'(x(t))x'(t)\, dt = \tilde{t} + \eta - \tau(x(\tilde{t} + \eta)),$$

$$-\frac{\gamma\eta}{2} \geq -\int_{\tilde{t}-\eta}^{\tilde{t}} 1 - \tau'(x(t))x'(t)\, dt = \tilde{t} - \eta - \tau(x(\tilde{t} - \eta)).$$

On the other hand, the function τ is uniformly continuous on the interval $x(\tilde{t}) + [-1 - \gamma, 1 + \gamma]$. So, there exists $0 < \delta < \gamma$ such that for all u, v in this interval with $|u - v| < \delta$, we have $|\tau(u) - \tau(v)| < \gamma\eta/4$. Thus, for $y \in X$ such that $\|x - y\| < \delta$, we have

$$\tilde{t} - \eta - \tau(y(\tilde{t} - \eta)) \leq -\frac{\gamma\eta}{4} \leq \frac{\gamma\eta}{4} \leq \tilde{t} + \eta - \tau(y(\tilde{t} + \eta)).$$

The Intermediate Value Theorem now implies that there exists $\bar{t}$ such that

$$|\bar{t} - \tilde{t}| < \eta < \varepsilon, \quad \text{and} \quad \bar{t} = \tau(y(\bar{t})).$$

We claim that $\bar{t} = \widetilde{\mathcal{T}}(y)$. Indeed, if not, there exists $\hat{t} \leq \bar{t}$ such that $\hat{t} = \tau(y(\hat{t}))$ and $\tau'(y(\hat{t}))y'(\hat{t}) \geq 1$; this contradicts the fact that $y \in X$. So, for $y \in X$ such that $\|x - y\| < \delta$, $|\widetilde{\mathcal{T}}(x) - \widetilde{\mathcal{T}}(y)| < \varepsilon$. □

Lemma 2.7. *Suppose that* (H1) *and* (H4) *hold. Then, we can choose* $n > 2$ *such that for every* $(t_0, x_0, y_0, \lambda) \in [0, 1[\times [-k, k] \times [r, s] \times [0, 1]$ *with* $\tau(x_0) \leq t_0$, *and for every solution* x *of* $\mathrm{P}(t_0, x_0, y_0, \lambda)$ *such that* $x(t) \in [-k, k]$ *and* $x'(t) \in [r, s]$ *for all* $t \in [t_0, 1]$, *we have*

$$\tau(x(t)) < t \quad \textit{for all } t \in]t_0, 1].$$

Moreover, if $\tau(x_0) = t_0$, *then* $\tau'(x(t_0))x'(t_0) < 1$.

Proof. From (H4),

$$\tau''(u) + \lambda\tau'(u)^3 f(t,u,v) > 0$$

on the compact set

$$A = \{(t,u,v,\lambda) \in [0,1] \times [-k,k] \times [r,s] \times [0,1] : v\tau'(u) = 1\}.$$

Thus, there exists $\varepsilon > 0$ such that

$$\tau''(u) + \lambda\tau'(u)^3 f(t,u,v) > \varepsilon \quad \text{for all } (t,u,v,\lambda) \in A. \tag{2.1}$$

On the other hand, by uniform continuity, there exists $\delta > 0$ such that for all $(t,u,v,\lambda) \in A$ and all $w \in [r,s]$ with $|w - v| < \delta$, we have

$$|\tau'(u)^3 (f(t,u,v) - f(t,u,w))| < \frac{\varepsilon}{2};$$

and hence, this combined with (2.1) imply that

$$\tau''(u) + \lambda\tau'(u)^3 f(t,u,w) > \frac{\varepsilon}{2}.$$

Now, choose $n > 2$ large enough so that $\max\{s, |r|\} < n\delta$. Therefore,

$$\tau''(u) + \lambda\tau'(u)^3 f(t,u,\pi_2(v,\lambda)) > 0 \quad \text{for all } (t,u,v,\lambda) \in A. \tag{2.2}$$

Now, suppose that $\tau(x(t_1)) \geq t_1$ for some $t_1 \in]t_0, 1]$. Then $h(t) = \tau(x(t)) - t$ attains a nonnegative maximum in $]t_0, 1[$ at some $\hat{t}$. Thus,

$$0 = h'(\hat{t}) = \tau'(x(\hat{t}))x'(\hat{t}) - 1,$$

and

$$0 \geq h''(\hat{t}) = \tau''(x(\hat{t}))x'(\hat{t})^2 + \tau'(x(\hat{t}))f_\lambda(\hat{t}, x(\hat{t}), x'(\hat{t}));$$

wich contradicts (2.2).

If moreover, $\tau(x_0) = t_0$, since $h(t) < 0$ for all $t > t_0$, then

$$0 \geq h'(t_0) = \tau(x(t_0))x'(t_0) - 1.$$

If $h'(t_0) = 0$, $(t_0, x(t_0), x'(t_0), \lambda) \in A$; so, equation (2.2) gives

$$0 < h''(t_0) = \tau''(x(t_0))x'(t_0)^2 + \tau'(x(t_0))x'(t_0).$$

This implies that h attains a strict local minimum at t_0, which contradicts the first part of the lemma. □

Remark 2.8. Lemmas 2.6 and 2.7 imply that $\mathcal{T}$ is continuous.

Proof of Theorem 2.1. Let n be as in Lemma 2.7. Define $H : [r,s] \times [0,1] \to \mathbb{R}$ by

$$H(y,\lambda) = y + \mathcal{E} \circ \mathcal{S} \circ \mathcal{J} \circ \mathcal{T} \circ \mathcal{S} \circ \mathcal{I}(y,\lambda).$$

From the previous lemmas, H is well defined and is a composition of acyclic maps. We claim that $H(\cdot,\lambda)$ has no fixed points on the boundary for all $\lambda \in [0,1[$. Indeed, suppose that $s \in H(s,\lambda)$ for some $\lambda \in [0,1[$. Then there exist

$$(x,\lambda) \in \mathcal{S} \circ \mathcal{J} \circ \mathcal{T} \circ \mathcal{S} \circ \mathcal{I}(s,\lambda) \quad \text{with} \quad x(1) = 0,$$

and

$$(x_1, \lambda) \in \mathcal{S} \circ \mathcal{I}(s, \lambda) \quad \text{with} \quad (x, \lambda) \in \mathcal{S} \circ \mathcal{J} \circ \mathcal{T}(x_1, \lambda).$$

That is x_1 and x are solutions of $\mathrm{P}(0, 0, s, \lambda)$ and $\mathrm{P}(\tilde{t}, \lambda I(x_1(\tilde{t})), \widetilde{J}(x_1'(\tilde{t}), \lambda), \lambda)$ respectively, where $\tilde{t} = \widetilde{\mathcal{T}}(x_1)$. Assumptions (H2) and (H3) imply that $x_1(t) = st$, $\widetilde{J}(x_1'(\tilde{t}), \lambda) = s$, and

$$x(t) = \lambda I(x_1(\tilde{t})) + s(t - \tilde{t}) \quad \text{for all } t \in [\tilde{t}, 1].$$

Finally, (H5) implies that $x(1) > 0$, a contradiction. A similar argument shows that $r \notin H(r, \lambda)$ for all $\lambda \in [0, 1[$.

Observe that

$$H(y, 0) = \{y + y(1 - t_y)\}, \quad \text{where } t_y = \tau(yt_y).$$

Define $\widetilde{H} : [r, s] \times [0, 1] \to \mathbb{R}$ by

$$\widetilde{H}(y, \lambda) = \Big\{ \big(2 - (\lambda t_y + (1 - \lambda)/2)\big) y \Big\}.$$

Notice that $\widetilde{H}$ is continuous, and $H(y, 0) = \widetilde{H}(y, 1)$. Also, $s \notin \widetilde{H}(s, \lambda)$ for $\lambda \in [0, 1]$, since $\lambda t_s + (1 - \lambda)/2 \neq 1$. Similarly, $r \notin \widetilde{H}(r, \lambda)$ for $\lambda \in [0, 1]$. Theorem 1.4 establishes the existence of $y \in [r, s]$ a fixed point of $H(\cdot, 1)$.

Consequently, there exist

$$(x_2, 1) \in \mathcal{S} \circ \mathcal{J} \circ \mathcal{T} \circ \mathcal{S} \circ \mathcal{I}(y, 1) \quad \text{with} \quad x_2(1) = 0,$$

and

$$(x_1, 1) \in \mathcal{S} \circ \mathcal{I}(y, 1) \quad \text{with} \quad (x_2, 1) \in \mathcal{S} \circ \mathcal{J} \circ \mathcal{T}(x_1, 1).$$

Now, from Lemma 2.4, we have that $x_1(t)$, $x_1'(t) \in [r, s]$, for $t \in [0.1]$, and $x_2(t) \in I(x_1(t_1)) + [r, s] \subset [-k, k]$, $x_2'(t) \in [r, s]$ for $t \in [t_1, 1]$, with $t_1 = \widetilde{\mathcal{T}}(x_1)$. Thus,

$$\begin{aligned} x_1''(t) &= f(t, x_1(t), x_1'(t)), && t \in [0, t_1], \\ x_1(0) &= 0, \\ t &\neq \tau(x_1(t)), && t \in [0, t_1[; \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} x_2''(t) &= f(t, x_2(t), x_2'(t)), && t \in [t_1, 1], \\ x_2(t_1) &= I(x_1(t_1)), \\ x_2'(t_1) &= J(x_1'(t_1)). \end{aligned} \tag{2.4}$$

Moreover, by (H6), we have

$$\tau(x_2(t_1)) = \tau(I(x_1(t_1))) \leq \tau(x_1(t_1)) = t_1.$$

So, Lemma 2.7 implies that

$$\tau(x_2(t)) < t \quad \text{for all } t \in]t_1, 1]. \tag{2.5}$$

Define

$$x(t) = \begin{cases} x_1(t), & \text{on } [0, t_1], \\ x_2(t), & \text{on }]t_1, 1]. \end{cases}$$

From (2.3), (2.4), and (2.5), we deduce that x is a solution of (P1), (D). □

Remark 2.9. We can see immediately from the above proof that (H5) can be replaced by a more general assumption, namely

(H5)* $I(st_s) + s(1-t_s) \geq 0 \geq I(rt_r) + r(1-t_r)$, where $t_s = \tau(st_s)$ and $t_r = \tau(rt_r)$.

Remark 2.10. It is possible to discuss (P1), (D) with more than one barrier if the barriers are appropriately separated (see [7]).

3. Sturm-Liouville boundary condition

In this section, we discuss the second order Sturm-Liouville boundary value problem with impulses at variable times (P1), (SL).

We fix

$$k = \max |I(\alpha + (a+1)[r,s])| + (\alpha+1)\max\{|r|, s\} + |\beta|.$$

We assume that (H1)-(H4) and the following conditions hold (with k just given):

(H7) $\tau(I(x)) \leq \tau(x)$ for $x - \alpha \in (a+1)[r,s]$.

(H8) $I(x) \geq \beta - bs$ for $\alpha + as \leq x \leq \alpha + (a+1)s$, and $I(x) \leq \beta - br$ for $\alpha + (a+1)r \leq x \leq \alpha + ar$.

Theorem 3.1. *Suppose that* (H1)-(H4), (H7) *and* (H8) *hold, then the problem* (P1), (SL) *has a solution in* Ω^2.

Proof. Define the following maps:

$$\mathcal{I}_0 : [r,s] \times [0,1] \to [0,1] \times \mathbb{R}^2 \times [0,1] \quad \text{by}$$
$$\mathcal{I}_0(y,\lambda) = (0, \alpha + ay, y, \lambda),$$
$$\mathcal{S}_0 : [0,1] \times \mathbb{R}^2 \times [0,1] \to [r,s] \times C^1[0,1] \times [0,1] \quad \text{by}$$
$$\mathcal{S}_0(t, x_0, y_0, \lambda) = \{(y_0, x, \lambda) : x \text{ is solution of } \mathrm{P}(t_0, x_0, y_0, \lambda)\},$$
$$\mathcal{T}_0 : [r,s] \times Z \to [0,1] \times [r,s] \times C^1[0,1] \times [0,1] \quad \text{by}$$
$$\mathcal{T}_0(y_0, x, \lambda) = (\widetilde{\mathcal{T}}(x), y_0, x, \lambda),$$
$$\mathcal{J}_0 : [0,1] \times [r,s] \times C^1[0,1] \times [0,1] \to [0,1] \times \mathbb{R}^2 \times [0,1] \quad \text{by}$$
$$\mathcal{J}_0(t, y_0, x, \lambda) = (t, \lambda I(x(t)) + (1-\lambda)(\beta - by_0), \widetilde{J}(x'(t), \lambda), \lambda),$$
$$\mathcal{E}_0 : [r,s] \times C^1[0,1] \times [0,1] \to \mathbb{R} \quad \text{by}$$
$$\mathcal{E}_0(y_0, x, \lambda) = x(1) + bx'(1),$$

where Z, $\widetilde{\mathcal{T}}$, $\widetilde{J}$, f_λ, and $\mathrm{P}(t_0, x_0, y_0, \lambda)$ are defined as in Section 2.

As in the previous section, we can check that $\mathcal{I}_0$, $\mathcal{J}_0$, $\mathcal{T}_0$, $\mathcal{E}_0$ are continuous, and $\mathcal{S}_0$ is u.s.c. with R_δ-values. Also, from Lemma 2.4, we have that

$$\mathcal{S}_0 \circ \mathcal{I}_0([r,s] \times [0,1]) \subset [r,s] \times \Big(Z \cap \{(x,\lambda) \in C[0,1] \times [0,1] : x(t) \in \alpha + (a+1)[r,s]\}\Big).$$

Define $H_0 : [r,s] \times [0,1] \to \mathbb{R}$ by

$$H_0(y,\lambda) = y - \beta + \mathcal{E}_0 \circ \mathcal{S}_0 \circ \mathcal{J}_0 \circ \mathcal{T}_0 \circ \mathcal{S}_0 \circ \mathcal{I}_0(y,\lambda).$$

So, H_0 is well defined and is a composition of acyclic maps. Moreover, $H_0(\cdot,\lambda)$ has no fixed points on the boundary for all $\lambda \in [0,1[$. To see this, suppose that $s \in H_0(s,\lambda)$ for some $\lambda \in [0,1[$. Then there exist

$$(s,x_1,\lambda) \in \mathcal{S}_0 \circ \mathcal{I}_0(s,\lambda),$$

and

$$(y,x,\lambda) \in \mathcal{S}_0 \circ \mathcal{J}_0 \circ \mathcal{T}_0(s,x_1,\lambda), \quad \text{with} \quad x(1) + bx'(1) = \beta.$$

Then, $x_1(t) = \alpha + as + st$, $y = \widetilde{J}(x_1'(\tilde{t}),\lambda) = s$ with $\tilde{t} = \tau(x_1(\tilde{t}))$, and

$$x(t) = \lambda I(x_1(\tilde{t})) + (1-\lambda)(\beta - bs) + s(t-\tilde{t}) \quad \text{for } t \in [\tilde{t},1].$$

Now, by (H8),

$$x(1) + bx'(1) = \lambda\big(I(x_1(\tilde{t})) + bs - \beta\big) + \beta + s(1-\tilde{t}) \geq \beta + s(1-\tilde{t}) > \beta,$$

a contradiction. A similar argument shows that $r \notin H_0(r,\lambda)$ for all $\lambda \in [0,1[$. Notice that

$$H_0(y,0) = \{y - \beta + (\beta - by + y(1-\tilde{t}_y)) + by\} = \{(2-\tilde{t}_y)y\},$$

where $\tilde{t}_y = \tau(\alpha + ay + y\tilde{t}_y)$. Define $\widetilde{H}$ as in the proof of Theorem 2.1 by replacing t_y by $\tilde{t}_y$. Essentially the same reasoning as in Section 2 establishes the result. □

4. Another type of barrier

In this section, we again discuss the second order Sturm-Liouville boundary value problem with impulses at variable times, but with a barrier different of the previous section, namely

$$\text{(P2)} \qquad \begin{aligned} &x''(t) = f(t,x(t),x'(t)), && \text{a.e. } t \in [0,1],\\ &x(t^+) = I(x(t)), && \text{if } t = \tau(x'(t)),\\ &x'(t^+) = J(x(t)), && \text{if } t = \tau(x'(t)); \end{aligned}$$

$$\text{(SL)} \qquad \begin{aligned} &x(0) - ax'(0) = \alpha, \quad a \geq 0,\\ &x(1) + bx'(1) = \beta, \quad b \geq 0, \end{aligned}$$

As in the previous section, we fix

$$k = \max|I(\alpha + (a+1)[r,s])| + (\alpha+1)\max\{|r|,s\} + |\beta|.$$

We assume that (H2), (H3), (H8) and the following conditions hold (with k just given):

- (H9) The functions $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $I : \mathbb{R} \to \mathbb{R}$, $J : \mathbb{R} \to \mathbb{R}$ are continuous, and $\tau : \mathbb{R} \to]0,1[$ is of class C^1.
- (H10) $\tau(J(y)) \leq \tau(y)$ for $y \in [r,s]$.
- (H11) $\tau'(y)f(t,x,y) < 1$ for all $t \in [0,1]$, $x \in [-k,k]$, and $y \in [r,s]$.

Theorem 4.1. *Suppose that* (H2), (H3) *and* (H8) - (H11) *hold, then the problem* (P2), (SL) *has a solution in* Ω^2.

For the proof, we need the following lemma. First of all, define

$$\mathcal{T}_1 : [r,s] \times Z \to [0,1] \times [r,s] \times C^1[0,1] \times [0,1] \quad \text{by}$$
$$\mathcal{T}_1(y_0, x, \lambda) = (\widetilde{\mathcal{T}}_1(x), y_0, x, \lambda),$$

where $\widetilde{\mathcal{T}}_1(x) = \inf\{t > 0 : t = \tau(x'(t))\}$, and Z is as before.

Lemma 4.2. *Suppose* (H9) *and* (H11) *hold. Then we can choose* $n > 2$ *such that for every* $(t_0, x_0, y_0, \lambda) \in [0,1[\times [-k,k] \times [r,s] \times [0,1]$ *with* $\tau(y_0) \leq t_0$, *and for every solution* x *of* $\mathrm{P}(t_0, x_0, y_0, \lambda)$ *such that* $x(t) \in [-k,k]$, $x'(t) \in [r,s]$ *for* $t \in [t_0, 1]$, *we have* $\tau(x'(t)) < t$ *for all* $t \in]t_0, 1]$.

Proof. Since $\tau'(y)f(t,x,y) < 1$, the same reasoning as in Lemma 2.7 implies that there exists $n > 2$ such that

$$\tau'(y)f(t,x,\pi_2(y,\lambda)) < 1 \quad \text{for all } (t,x,y) \in [0,1] \times [-k,k] \times [r,s].$$

Now, suppose the conclusion is false. Then $h(t) = \tau(x'(t)) - t$ attains a nonnegative maximum on $]t_0, 1[$ at some $\hat{t}$. Then,

$$0 = h'(\hat{t}) = \tau'(x'(\hat{t}))x''(\hat{t}) - 1 = \lambda\tau'(x'(\hat{t}))f(\hat{t}, x(\hat{t}), \pi_2(x'(\hat{t}), \lambda)) - 1 < 0,$$

a contradiction. □

Remark 4.3. Lemmas 2.6 and 4.2 imply that $\mathcal{T}_1$ is continuous.

Proof of Theorem 4.1. The proof is similar to Theorem 3.1 except that $\mathcal{T}_0$ is replaced by $\mathcal{T}_1$, and $\tilde{t}_y$ is defined by $\tau(y)$. We leave the details to the reader. □

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