Differential Equations and Dynamical Systems, Volume 2, Number 1, January 1994, pp. 41 - 48.

Existence Results for Initial Value Problems in Banach Spaces

Marlène Frigon* † and Donal O'Regan‡

(Submitted by: V. Lakshmikantham)

Abstract

In this paper, we establish some existence results for the initial value problem $y' = f(t, y)$, $y(0) = a \in E$ where E is a real Banach space and f has a decomposition $f = g + h$ with g and h satisfying respectively, a compactness and Lipschitz assumptions. Our results rely on Krasnoselskii fixed point theorem.

1 Introduction and Preliminaries

In this paper, we are concerned with the initial value problem:

$$y'(t) = f(t, y(t)), \quad t \in [0, T], \qquad y(0) = a \in E; \tag{1.1}$$

where E is a real Banach space and $f : [0, T] \times E \to E$ has a decomposition $f = g + h$ with g and h Carathéodory functions satisfying respectively, a compactness and Lipschitz assumptions. Our results rely on Krasnoselskii's fixed point theorem for contraction plus compact mappings and donot use homotopy arguments. The paper is divided in to three sections. In the first one, we give some preliminaries. Our main

Received on November 22, 1993.

AMS (MOS) 1991 Subject classifications: 34G20, 34A12.

*Work supported in part by grants from NSERC-Canada and FCAR-Québec.

†Dèpartment de Mathèmatiques et Statistique, Universitè de Montrèal, C.P.6128, SUCC. A, Montrèal, H3C 3J7, Canada.

‡Department of Mathematics, University College Galway, Galway, Ireland.

existence result is obtained in Section 2. This result will be used in Section 3 to obtain more applicable existence results. More precisely, in Section 3.1, we give existence theorems of Wintner type, which generalize some results of ([4],[8]) where $h \equiv 0$. In ([4],[8]), proofs rely on topological transversality theory. In Section 3.2, existence theorems are obtained under an assumption which is equivalent to an assumption of existence of upper and lower solutions to (1.1) in the scalar case. No growth condition is assumed.

Throughout, E is a real Banach space with norm $\|\cdot\|$. In case $E = H$, a Hilbert space, we denote the inner product by $\langle \, , \, \rangle$ so that $\|u\|^2 = \langle u, u\rangle$ for $u \in H$. We denote by $C([0,T], E)$ the space of continuous functions $u : [0,T] \to E$. We set $C_a([0,T],E) = \{u \in C([0,T],E) : u(0) = a\}$.

Let $u : [0,T] \to E$ be a measurable function. By $\int_0^T u(t)\,dt$, we mean the Bochner integral of u, assuming it exists (see [3] for properties of the Bochner integral). We define the Sobolev class $W^{1,1}([0,T],E)$ by the space of continuous functions u such that there exists $\nu \in L^1([0,T],E)$ with $u(t) - u(0) = \int_0^t \nu(s)\,ds$, for all t in $[0,T]$. Notice that if $u \in W^{1,1}([0,T],E)$ then u is differentiable almost everywhere on $[0,T]$, $u' \in L^1([0,T],E)$, and $u(t) - u(0) = \int_0^t u'(s)\,ds$ for t in $[0,T]$. Also, if E is a reflexive Banach space, $u \in W^{1,1}([0,T],E)$ if and only if u is absolutely continuous. By a *solution* to (1.1), we mean a function $u \in W^{1,1}([0,T],E)$ satisfying the differential equation (1.1) a.e. on $[0,T]$ and $y(0) = a$.

A function $g : [0,T] \times E \to E$ is a *Carathéodory function* if: (1) the map $t \to g(t,z)$ is measurable for each z in E; (2) the map $z \to g(t,z)$ is continuous for almost all t in $[0,T]$; (3) for each $r > 0$, there exists $h_r \in L^1([0,T],R)$ such that $\|z\| \leq r$ implies $\|g(t,z)\| \leq h_r(t)$ for almost all t in $[0,T]$.

For the sake of completness, we state the Krasnoselskii's fixed point Theorem (see for proof [6],[11]).

Theorem 1.1 *Let C be a nonempty closed convex subset of a Banach space E. Suppose T_1, T_2 map C into E and*

(i) $T_1(C) + T_2(C) \subset C$;

(ii) $T_1 : C \to E$ is continuous and compact;

(iii) $T_2 : C \to E$ is a contraction mapping.

Then there exists $y \in C$ with $T_1(y) + T_2(y) = y$.

2 Main Existence Result

In this section, we establish an existence result for the initial value problem (1.1), which will be used to prove the forthcoming theorems.

Theorem 2.1 *Assume $f : [0,T] \times E \to E$ has the decomposition $f = g^* + h^*$ with g^* and h^* Carathéodory functions such that*

(2.1) for each $t \in [0,T]$, the set $\{\int_0^t g^*(s,u(s))\,ds : u \in C_a([0,T],E)\}$ is relatively compact;

(2.2) there exists $q^* \in L^1([0,T],R)$ *with* $\|h^*(t,u) - h^*(t,v)\| \le q^*(t)\|u - v\|$ *a.e.* $t \in [0,T]$ *and all* $u, v \in E$.

Then (1.1) has a solution.

Proof. Let us endow $C([0,T],E)$ with the norm

$$\|u\|_Q = \max_{t\in[0,T]} \|e^{-Q(t)}u(t)\| \quad \text{where} \quad Q(t) = \int_0^t q^*(s)\,ds.$$

A solution to (1.1) is a fixed point of the operator $T : C_a([0,T],E) \to C_a([0,T],E)$ defined by:

$$\begin{aligned}(Ty)(t) &= a + \int_0^t f(s,y(s))\,ds = a + \int_0^t h^*(s,y(s))\,ds + \int_0^t g^*(s,y(s))\,ds \\ &\equiv (T_2y)(t) + (T_1y)(t),\end{aligned}$$

where $(T_1y)(t) = \int_0^t g^*(s,y(s))\,ds$ and $(T_2y)(t) = a + \int_0^t h^*(s,y(s))\,ds$. Since g^* and h^* are Carathéodory, T_1 and T_2 are well defined and continuous. Condition (2.1) together with the Arzela-Ascoli Theorem imply that T_1 is compact. Also, T_2 is a contraction mapping since

$$\begin{aligned}\|T_2u - T_2v\|_Q &= \max_{t\in[0,T]} \left\| e^{-Q(t)} \int_0^t [h^*(s,u(s)) - h^*(s,v(s))]\,ds \right\| \\ &\le \max_{t\in[0,T]} \left[e^{-Q(t)} \int_0^t q^*(s) e^{Q(s)} e^{-Q(s)} \|u(s) - v(s)\|\,ds \right] \\ &\le \|u - v\|_Q \max_{t\in[0,T]} \left[e^{-Q(t)}[e^{Q(t)} - 1] \right] \\ &= (1 - e^{-Q(T)})\,\|u - v\|_Q \equiv k_0\|u - v\|_Q\end{aligned}$$

with $k_0 < 1$. Then, Krasnoselskii's fixed point Theorem (Theorem 1.1) gives the existence of a fixed point to T i.e. a solution to (1.1).

Remark 2.2.

(a) Condition (2.1) is satisfied if g^* is a K-Carathéodory function i.e. g^* is Carathéodory and satisfies property-K (see [4]):

for each $r > 0$, there exist a nonnegative function $\eta_r \in L^1([0,T],R)$ and a compact set K_r in E such that $\|z\| \le r$ implies $g^*(t,z) \in \eta_r(t)K_r$ for almost all t in $[0,T]$.

(b) A more general version of this theorem could be given in terms of measure of non-compactness and condensing mappings.

3 Other Existence Results

3.1 Wintner type existence results

We first present an existence result if $E = H$, a real Hilbert space.

Theorem 3.1. *Assume* $f : [0,T] \times H \to H$ *has the decomposition* $f = g + h$ *with* g *and* h *Carathéodory functions such that*

(3.1) *there are* $r \in L^1([0,T],[0,\infty))$ *and* $\varphi : [0,\infty) \to (0,\infty)$ *a Borel measurable function such that* $\langle y, f(t,y)\rangle \le r(t)\varphi(\|y\|)$ *for a.e.* $t \in [0,T]$ *and all* $y \in H$, *and* $\int_0^T r(s)\,ds < \int_{\|a\|}^{\infty} \frac{x}{\varphi(x)}\,dx$.

Let $I(z) = \int_{\|a\|}^{z} \frac{x}{\varphi(x)}\,dx$ *and notice* $I : [\,\|a\|,\infty) \to [0,\infty)$ *is strictly increasing. Define* $M = I^{-1}(\int_0^T r(s)\,ds)$ *and assume*

(3.2) *for each* $t \in [0,T]$, *the set* $\{\int_0^t g(s,u(s))\,ds : u \in C_a([0,T],H)$ *with* $\|u(s)\| \le M$ *for all* $s \in [0,T]\,\}$ *is relatively compact;*

(3.3) *there exists* $q \in L^1([0,T],R)$ *with* $\|h(t,u)-h(t,v)\| \le q(t)\|u-v\|$ *a.e.* $t \in [0,T]$ *and all* $u,v \in H$ *with* $\|u\|$, $\|v\| \le M$.

Then (1.1) has a solution.

Proof. Consider the initial value problem:

$$y'(t) = g_1(t,y(t)) + h_1(t,y(t)) \quad t \in [0,T], \qquad y(0) = a; \tag{3.4}$$

with $g_1(t,y) = g(t,p(y))$, $h_1(t,y) = h(t,p(y))$ and

$$p(y) = \begin{cases} y, & \text{if } \|y\| \le M \\ M\frac{y}{\|y\|}, & \text{if } \|y\| > M \end{cases}$$

is the radial retraction of H onto $\overline{B(0,M)} = \{y : \|y\| \le M\}$. It is easy to check that p is nonexpansive i.e.

$$\|p(y_1) - p(y_2)\| \le \|y_1 - y_2\| \quad \text{for all } \; y_1, y_2 \in H.$$

This together with (3.3) implies that

$$\begin{aligned} \|h_1(t,u) - h_1(t,v)\| &= \|h(t,p(u)) - h(t,p(v))\| \le q(t)\|p(u) - p(v)\| \\ &\le q(t)\|u-v\|; \end{aligned}$$

so (2.2) is satisfied with $h^* = h_1$ and $q^* = q$. Notice as well that $g^* = g_1$ satisfies (2.1). Consequently, Theorem 2.1 implies that (3.4) has a solution y. The following Lemma 3.2 applied with $R = \|a\|$, $\psi(x) = \varphi(x)/x$, and $z(t) = \|y(t)\|$ implies that $\|y(t)\| \le M$ for all $t \in [0,T]$. Indeed, if $\|y(t)\| > M$ for some t, then there exists $[t_0,t_1] \subset [0,T]$ such that $\|y\| \in W^{1,1}([t_0,t_1],R)$, $\|y(t_0)\| = \|a\|$, $\|y(t_1)\| = M$ and $\|a\| \le \|y(t)\| \le M$ for $t \in [t_0,t_1]$. If $\int_{t_0}^{t_1} r(s)\,ds = \int_0^T r(s)\,ds$, $r(t) = 0$ a.e. on $[0,T]\backslash[t_0,t_1]$. Thus $\|y(t)\| = M$ for $t > t_1$ and $\|y(t)\| = \|a\|$ for $t < t_0$, a contradiction. Otherwise, by Lemma 3.2, $\|y(t)\| < M$ for $t \in [t_0,t_1]$, a contradiction. Consequently, y is a solution to the original problem (1.1).

Lemma 3.2. Let $R \ge 0$, $r \in L^1([0,T],[0,\infty))$ and ψ: $[0,\infty) \to (0,\infty)$ be a Borel function such that

$$\int_0^T r(s)\,ds < \int_R^{\infty} \frac{1}{\psi(x)}\,dx.$$

Let M_0 be such that $\int_0^T r(s)\,ds = \int_R^{M_0} \frac{1}{\psi(x)}\,dx$. Then for any $[t_0, t_1] \subset [0, T]$ and any z in $W^{1,1}([t_0, t_1], [0, \infty))$ with $z'(t) \le r(t)\psi(z(t))$ for a.e. $t \in [t_0, t_1]$ and $z(t_0) \le R$, we have $z(t) \le M_0$ for all $t \in [t_0, t_1]$. Moreover, if $\int_{t_0}^{t_1} r(s)\,ds < \int_0^T r(s)\,ds$, $z(t) < M_0$ for all $t \in [t_0, t_1]$.

Proof. Since $z'(t) \le r(t)\psi(z(t))$ for a.e. $t \in [t_0, t_1]$ and $z(t_0) \le R$, for any $t \in [t_0, t_1]$ such that $z(t) > R$, there exists $\bar{t}_0 \in [t_0, t]$ with $z(\bar{t}_0) = R$. Dividing by $\psi(z(t))$, integrating from $\bar{t}_0$ to t, and using the change of variable formula we get

$$\int_R^{z(t)} \frac{1}{\psi(x)}\,dx \le \int_{\bar{t}_0}^t r(s)\,ds \le \int_0^T r(s)\,ds = \int_R^{M_0} \frac{1}{\psi(x)}\,dx.$$

Therefore $z(t) \le M_0$, and $z(t) < M_0$ if $\int_{\bar{t}_0}^t r(s)\,ds < \int_0^T r(s)\,ds$. This completes the proof.

We next derive an existence result for a real Banach space E.

Theorem 3.3. *Assume $f : [0, T] \times E \to E$ has the decomposition $f = g + h$ with g and h Carathéodory functions such that*

(3.5) there exist a nondecreasing Borel measurable function $\varphi : [0, \infty) \to (0, \infty)$, and $r \in L^1([0, T], [0, \infty))$ such that $\|f(t, y)\| \le r(t)\varphi(\|y\|)$ for a.e. $t \in [0, T]$ and all $y \in E$, and $\int_0^T r(s)\,ds < \int_{\|a\|}^{\infty} \frac{1}{\varphi(x)}\,dx$.

Let $J(z) = \int_{\|a\|}^{z} \frac{1}{\varphi(x)}\,dx$ and notice $J : [\,\|a\|, \infty) \to [0, \infty)$ is strictly increasing. Define $M = J^{-1}(\int_0^T r(s)\,ds)$ and assume (3.2) and (3.3) hold with $H = E$. Then (1.1) has a solution.

Proof. Consider the initial value problem (3.4). First, notice that in that case, the radial retraction p is Lipschitz and not necessarily nonexpansive. More precisely,

$$\|p(y_1) - p(y_2)\| \le 2\|y_1 - y_2\| \quad \text{for all } \; y_1, y_2 \in E.$$

This together with (3.5) implies

$$\|h_1(t, u) - h_1(t, v)\| \le q(t)\|p(u) - p(v)\| \le 2q(t)\|u - v\|;$$

so (2.2) is satisfied with $h^* = h_1$ and $q^* = q$. Notice as well that $g^* = g_1$ satisfies (2.1). Consequently, Theorem 2.1 implies that (3.4) has a solution y. For such a y, $\|y(t)\| = \|y(0) + \int_0^t y'(s)\,ds\| \le \|a\| + \int_0^t \|y'(s)\|\,ds \equiv \rho(t)$. Clearly, $\rho(t)$ is absolutely continuous and $\rho'(t) = \|y'(t)\|$ almost everywhere. Since φ is nondecreasing,

$$\rho'(t) = \|y'(t)\| \le r(t)\varphi(\|y(t)\|) \le r(t)\varphi(\rho(t))$$

almost everywhere. Lemma 3.2 applied with $R = \|a\|$, $\psi(x) = \varphi(x)$, and $z = \rho$ implies that $\|y(t)\| \le \rho(t) \le M$ for all $t \in [0, T]$ and consequently that y is a solution to the original problem (1.1).

3.2 Existence results without a growth condition

In this section, we establish existence results without a growth restriction. To our knowledge, these results are new even in a finite dimensional context. Let $E = H$ be a real Hilbert space.

Theorem 3.4. *Assume $f : [0,T] \times H \to H$ has the decomposition $f = g + h$ with g and h Carathéodory functions such that*

(3.6) there exist $v \in W^{1,1}([0,T], H)$ and $M \in W^{1,1}([0,T],[0,\infty))$ such that $\langle y - v(t), f(t,y) - v'(t)\rangle \le M(t)M'(t)$ for a.e. $t \in [0,T]$ and all $y \in H$ with $\|y - v(t)\| = M(t)$; $f(t, v(t)) = v'(t)$ a.e. on $\{t \in [0,T] : M(t) = 0\}$; $\|a - v(0)\| \le M(0)$;

(3.7) for each $t \in [0,T]$, the set $\{\int_0^t g(s,u(s))\,ds : u \in C_a([0,T],H)$ with $\|u(s) - v(s)\| \le M(s)$ for all $s \in [0,T]\}$ is relatively compact;

(3.8) there exists $q \in L^1([0,T],R)$ with $\|h(t,u_1) - h(t,u_2)\| \le q(t)\|u_1 - u_2\|$ a.e. $t \in [0,T]$ and all $u_i \in H$ with $\|u_i - v(t)\| \le M(t)$, $i = 1,2$.

Then (1.1) has a solution such that $\|y(t) - v(t)\| \le M(t)$ for all $t \in [0,T]$.

Remark 3.5. Observe that in the scalar case, if $\alpha \le \beta \in W^{1,1}([0,T],R)$ are respectively lower and upper solutions to (1.1) i.e. $\alpha(0) \le a \le \beta(0)$, $\alpha'(t) \le f(t,\alpha(t))$, $\beta'(t) \ge f(t,\beta(t))$, then $v = (\alpha+\beta)/2$ and $M = (\beta - \alpha)/2$ satisfy (3.6).

Proof. Consider the initial value problem:

$$y'(t) = f_2(t, y(t)) \quad t \in [0,T], \quad y(0) = a; \tag{3.9}$$

with $f_2(t,y) = f(t, \hat{p}(t,y))$ and

$$\hat{p}(t,y) = \begin{cases} y, & \text{if } \|y - v(t)\| \le M(t) \\ M(t)\left(\frac{y - v(t)}{\|y - v(t)\|}\right) + v(t), & \text{if } \|y - v(t)\| > M(t) \end{cases}$$

is the radial retraction of H onto $\overline{B(v(t), M(t))} = \{y : \|y - v(t)\| \le M(t)\}$. It is easy to check that $\hat{p}$ is continuous and nonexpansive i.e.

$$\|\hat{p}(t,y_1) - \hat{p}(t,y_2)\| \le \|y_1 - y_2\| \quad \text{for all } \; y_1, y_2 \in H;$$

and condition (2.2) is satisfied with $h^*(t,y) = h(t,\hat{p}(t,y))$ and $q^* = q$. Notice as well that $g^*(t,y) = g(t,\hat{p}(t,y))$ satisfies (2.1). Consequently, Theorem 2.1 implies that (3.9) has a solution y. We now claim that $\|y(t) - v(t)\| \le M(t)$ for all $t \in [0,T]$, and consequently, y is a solution to the original problem (1.1).
Suppose there exists $t_1 \in (0,T]$ with $\|y(t_1) - v(t_1)\| > M(t_1)$. Since $\|y(0) - v(0)\| \le M(0)$, there exists $t_0 \in [0,t_1)$ such that $\|y(t_0) - v(t_0)\| = M(t_0)$ and $\|y(t) - v(t)\| > M(t)$ for $t \in (t_0,t_1)$. Now, (3.6) implies that a.e. on (t_0,t_1),

$$\begin{aligned} M(t)M'(t) &\ge \langle \hat{p}(t, y(t)) - v(t), f_2(t, y(t)) - v'(t)\rangle \\ &= \frac{M(t)}{\|y(t) - v(t)\|}\langle y(t) - v(t), f_2(t,y(t)) - v'(t)\rangle. \end{aligned}$$

Thus

$$M'(t) \geq \frac{\langle y(t) - v(t), f_2(t, y(t)) - v'(t)\rangle}{\|y(t) - v(t)\|} \quad \text{a.e. } t \in (t_0, t_1). \tag{3.10}$$

On the other hand, $\|y(t) - v(t)\|$ is differentiable a.e. on (t_0, t_1) and

$$\|y(t_1) - v(t_1)\| = \|y(t_0) - v(t_0)\| + \int_{t_0}^{t_1} \frac{\langle y(t) - v(t), y'(t) - v'(t)\rangle}{\|y(t) - v(t)\|}\, dt. \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$M(t_1) < \|y(t_1) - v(t_1)\| \leq M(t_0) + \int_{t_0}^{t_1} M'(t)\, dt = M(t_1),$$

a contradiction.

Corollary 3.6. Assume $f : [0,T] \times H \to H$ has the decomposition $f = g + h$ with g and h Carathéodory functions such that

(3.12) there exists a positive constant $M \geq \|a\|$ such that $\langle y, f(t,y)\rangle \leq 0$ for a.e. $t \in [0,T]$ and all $y \in H$ with $\|y\| = M$;

and satisfying (3.2) and (3.3). Then (1.1) has a solution such that $\|y(t)\| \leq M$ for all $t \in [0,T]$. We next give an existence result for a real Banach space E.

Theorem 3.7. *Assume $f : [0,T] \times E \to E$ has the decomposition $f = g + h$ with g and h Carathéodory functions such that*

(3.13) there exist $v \in W^{1,1}([0,T], E)$ and $M \in W^{1,1}([0,T], [0,\infty))$ a nondecreasing function such that $\|a - v(0)\| \leq M(0)$ and $\|f(t,y) - v'(t)\| \leq M'(t)$ for a.e. $t \in [0,T]$ and all $y \in H$ with $\|y - v(t)\| = M(t)$;

and satisfying (3.7) and (3.8) with $H = E$. Then (1.1) has a solution such that $\|y(t) - v(t)\| \leq M(t)$ for all $t \in [0,T]$.

Proof. Consider the initial value problem (3.9). As before, Theorem 2.1 implies the existence of a solution y to (3.9).
Suppose there exists $t_1 \in (0,T]$ with $\|y(t_1) - v(t_1)\| > M(t_1)$. The initial conditions imply the existence of $t_0 \in [0, t_1)$ such that $\|y(t_0) - v(t_0)\| = M(t_0)$ and $\|y(t) - v(t)\| > M(t)$ for $t \in (t_0, t_1)$. By condition (3.13), we get

$$\begin{aligned} M(t_1) < \|y(t_1) - v(t_1)\| &= \|y(t_0) - v(t_0) + \int_{t_0}^{t_1} [y'(t) - v'(t)]\, dt\,\| \\ &\leq M(t_0) + \int_{t_0}^{t_1} M'(t)\, dt = M(t_1), \end{aligned}$$

a contradiction. Therefore $\|y(t) - v(t)\| \leq M(t)$ for all $t \in [0,T]$ and hence y is a solution to (1.1).

References

[1] R.R.Akhmerov, M.I.Kamenskii, A.S.Potapov, A.E.Rodkina and B.N.Sadovskii, "Measures of Noncompactness and Condensing Operators", Birkhäuser, Basel, 1992.

[2] K. Deimling, "Ordinary Differential Equations in Banach Spaces", Springer-Verlag, New York/Berlin, 1977.

[3] N. Dunford and J.T. Schwartz, "Linear Operators Part.I: General theory", Interscience, New York, 1967.

[4] M. Frigon and J.W. Lee, Existence principles for Carathéodory differential equations in Banach spaces, Topological Methods in Nonlinear Analysis, 1(1993), 91–106.

[5] A. Granas, R.B. Guenther and J.W. Lee, Some general existence principles in the Carathéodory theory of nonlinear differential systems, J. Math. pures et appl. 70(1991), 153–196.

[6] M.A. Krasnoselskii, Two remarks on the method of successive approximations, Uspehi Mat. Nauk, 10(1955), 123–127.

[7] V. Lakshmikantham and S. Leela, "Nonlinear Differential Equations in Abstract Spaces", Pergamon, NewYork, 1981.

[8] J.W. Lee and D. O'Regan, Existence results for differential equations in Banach spaces, Commentat. Math. Univ. Carol., to appear.

[9] J.W. Lee and D. O'Regan, Topological transversality. Applications to initial-value problems, Ann. Polonici Math. 48(1988), 247–252.

[10] R.H. Martin, "Nonlinear Operators and Differential Equations in Banach Spaces", Wiley, New York, 1976.

[11] D.R. Smart, "Fixed Point Theorems", Cambridge University Press, 1980.

[12] A. Wintner, The nonlocal existence problem for ordinary differential equations, Am. J. Math. 67(1945), 227–284.