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Fuzzy contractive maps and fuzzy fixed points

Marlène Frigon[a, *], Donal O'Regan[b]

[a] Département de Mathématiques et de Statistique, Université de Montréal, C.P. 6128, Succursale Centre-ville, Montréal, Canada H3C 3J7

[b] Department of Mathematics, National University of Ireland, Galway, Ireland

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Abstract

This paper presents a variety of fuzzy fixed point theorems for contractive type maps. Our theory can be derived directly from results in the literature related to multivalued contractive maps with closed values; this observation seems to have been overlooked in the literature. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we present fuzzy fixed point theorems for fuzzy contractive maps. Our analysis is based on the simple observation that fuzzy fixed point results can be deduced immediately from the fixed point theory of multivalued maps with closed values. This elementary observation seems to have been overlooked in the literature. We note here also that the claim made in [1,4,9] show that their theorems are generalizations, are not justified. In Section 2, we begin by presenting a local version of Heilpern's fuzzy fixed point theorem. This automatically leads to a generalization of Heilpern's theorem [4]; we assume a weaker contractive condition and our α level sets are not assumed to be convex and compact. Also, in this section we present nonlinear alternatives of Leray–Schauder type for fuzzy contractive and fuzzy nonexpansive maps. Section 3 presents a fuzzy fixed point theory for generalized contractive maps of Kulshreshtha type. In addition, we present a homotopy result for maps of this type.

Let (X,d) be a metric space. By $B(x,r)$ we denote the open ball in X centered at x of radius r and by $B(C,r)$ we denote $\bigcup_{x\in C} B(x,r)$ where C is a subset of X. For C and K two nonempty closed subsets of X, we define the generalized Hausdorff distance H by

$$H(C,K)=\inf\{\varepsilon>0:\ C\subseteq B(K,\varepsilon),\quad K\subseteq B(C,\varepsilon)\}\in[0,\infty].$$

A fuzzy set in X is a function with domain X and values in $[0,1]$. We let $\mathscr{F}(X)$ denote the collection of

* Corresponding author.

E-mail addresses: frigon@dms.umontreal.ca (M. Frigon), donal.oregan@nuigalway.ie (D. O'Regan).

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fuzzy sets in X. Let $A\in\mathscr{F}(X)$ and $\alpha\in[0,1]$. The α level set of A, denoted $[A]_\alpha$, is

$$[A]_\alpha=\{x\colon A(x)\geqslant\alpha\}\quad\text{if }\alpha\in(0,1]$$

$$\text{and } A_0=\overline{\{x\colon A(x)>0\}}.$$

We say that

(i) $A\in FC(X)$ if $A\in\mathscr{F}(X)$ and $[A]_1$ is nonempty and closed;
(ii) $A\in FK(X)$ if $A\in\mathscr{F}(X)$ and $[A]_1$ is nonempty and compact; and
(iii) $A\in FW(X)$ if $A\in\mathscr{F}(X)$ and $[A]_\alpha$ is nonempty, closed and bounded for each $\alpha\in[0,1]$.

For $A,B\in FC(X)$, we define

$$D_1(A,B)=H([A]_1,[B]_1),$$

and for $A,B\in FW(X)$, we define

$$D_\alpha(A,B)=H([A]_\alpha,[B]_\alpha)\quad\text{for }\alpha\in[0,1]$$

and

$$D(A,B)=\sup_\alpha D_\alpha(A,B).$$

For $A,B\in\mathscr{F}(X)$, $A\subseteq B$ means that $A(x)\leqslant B(x)$ for each $x\in X$. If A is a subset of X, its characteristic function χ_A is a fuzzy set. Thus we note that a subset of X can be seen as a fuzzy set if we denote with the same symbol the subset and its characteristic function.

2. Fixed point theory for contractive maps

We begin this section by presenting a local version of a fixed point result for contractive maps. Heilpern's fuzzy fixed point result [4] will then be generalized from our result. We also show how standard fixed point results for multivalued contractions (in particular Nadler's fixed point theorem) could be used to deduce immediately Heilpern's fuzzy fixed point result; this elementary idea seems to have been overlooked in the literature [1,4,9].

Theorem 2.1. *Let (X,d) be a complete metric space, $x_0\in X$ and $T:\overline{B(x_0,r)}\to FC(X)$ (here $r>0$). Suppose there exists a constant $k\in(0,1)$ with*

$$D_1(Tx,Ty)\leqslant kd(x,y)\quad\textit{for all } x,y\in\overline{B(x_0,r)}$$

and

$$\operatorname{dist}(x_0,[Tx_0]_1)<(1-k)r$$

holding. Then T has a fuzzy fixed point. That is there exists $x\in\overline{B(x_0,r)}$ with $\{x\}\subseteq Tx$ (i.e. $Tx(x)=1$).

Proof. Choose $x_1\in X$ such that

$$\{x_1\}\subseteq Tx_0\quad\text{and}\quad d(x_0,x_1)<(1-k)r;$$

this is possible since

$$[Tx_0]_1\neq\emptyset\quad\text{and}\quad\operatorname{dist}(x_0,[Tx_0]_1)<(1-k)r.$$

Now choose $\varepsilon>0$ such that

$$kd(x_0,x_1)+\varepsilon<k(1-k)r.$$

Then choose $x_2\in X$ such that $\{x_2\}\subseteq Tx_1$ and

$$d(x_1,x_2)\leqslant D_1(Tx_1,Tx_0)+\varepsilon.$$

As a result we have

$$d(x_1,x_2)\leqslant kd(x_1,x_0)+\varepsilon<k(1-k)r$$

and note that $x_2\in\overline{B(x_0,r)}$ since

$$\begin{aligned}d(x_0,x_2)&\leqslant(1-k)r[1+k]\\&\leqslant(1-k)r[1+k+k^2+\cdots]=r.\end{aligned}$$

Continue this process to obtain $\{x_n\}\subseteq Tx_{n-1}$ with $d(x_n,x_{n-1})<k^{n-1}(1-k)r$, for $n=3,4,\ldots$. Notice that (x_n) is a Cauchy sequence and, since X is complete, there exists $x\in\overline{B(x_0,r)}$ with $\lim_{n\to\infty}x_n=x$. It remains to show $\{x\}\subseteq Tx$. Notice

$$\begin{aligned}\operatorname{dist}(x,[Tx]_1)&\leqslant d(x,x_n)+\operatorname{dist}(x_n,[Tx]_1)\\&\leqslant d(x,x_n)+D_1(Tx_{n-1},Tx)\\&\leqslant d(x,x_n)+kd(x_{n-1},x)\\&\to0\quad\text{as }n\to\infty.\end{aligned}$$

Thus $x\in\overline{[Tx]_1}=[Tx]_1$ and so $\{x\}\subseteq Tx$. □

Next we present a generalization of Heilpern's fuzzy fixed point theorem. Notice here we assume a weaker contractive condition and our α level sets are not assumed to be convex and compact.

Theorem 2.2. *Let (X, d) be a complete metric space, $T : X \to FC(X)$ and suppose there exists a constant $k \in (0, 1)$ with*

$$D_1(Tx, Ty) \leqslant kd(x, y) \quad \text{for all } x, y \in X.$$

Then T has a fuzzy fixed point.

Proof. Fix $x_0 \in X$. Choose $r > 0$ so that $\mathrm{dist}(x_0, [Tx_0]_1) < (1 - k)r$. Now Theorem 2.1 guarantees that there exists $x \in \overline{B(x_0, r)}$ with $\{x\} \subseteq Tx$. □

In fact, Theorem 2.2 could easily be deduced from Nadler's fixed point theorem [2]: if (X, d) is a complete metric space and $F : X \to C(X)$ is k-contractive (here $k \in (0, 1)$), then there exists $x \in X$ with $x \in Fx$ (here $C(X)$ denotes the family of nonempty, closed subsets of X and F being k-contractive means $H(Fx, Fy) \leqslant kd(x, y)$ for all $x, y \in X$).

Another proof of Theorem 2.2. Let $F : X \to C(X)$ be given by $Fx = [Tx]_1$. Notice F satisfies the conditions of Nadler's fixed point theorem. As a result there exists $x \in X$ with $x \in Fx = [Tx]_1$. □

Once one realises the above elementary idea, then many results from the literature on contractive or indeed nonexpansive multifunctions have fuzzy analogues. For completeness, we present fuzzy analogues of the nonlinear alternative of Leray–Schauder type for contractive and nonexpansive multifunctions. We note here that we could present a more general version of Theorem 2.3 if we used the notion of contractive homotopy [2] (see Section 3 for a generalization). First recall the following two results from [2,3].

Theorem 2.3. *Let E be a Banach space, U an open subset of E, $0 \in U$ and $F : \bar{U} \to C(E)$ a k-contraction (here $k \in (0, 1)$) such that $F(\bar{U})$ is bounded. Then either*

(A1) *F has a fixed point, or*
(A2) *there exists $x \in \partial U$ and $\lambda \in (0, 1)$ with $x \in \lambda Fx$.*

Theorem 2.4. *Let $E = (E, \|\cdot\|)$ be a uniformly convex Banach space and U a bounded, convex, open subset of E with $0 \in U$. Suppose $F : \bar{U} \to K(E)$ is nonexpansive with $F(\bar{U})$ bounded (here $K(E)$ denotes the family of nonempty, compact subsets of E and F being nonexpansive means $H(Fx, Fy) \leqslant \|x - y\|$ for all $x, y \in \bar{U}$). Then either*

(A1) *F has a fixed point, or*
(A2) *there exists $x \in \partial U$ and $\lambda \in (0, 1)$ with $x \in \lambda Fx$.*

We now establish immediately the fuzzy analogue's of Theorems 2.3 and 2.4.

Theorem 2.5. *Let $E = (E, \|\cdot\|)$ be a Banach space, U an open subset of E, $0 \in U$ and $T : \bar{U} \to FC(E)$. Suppose there exists $k \in (0, 1)$ with*

$$D_1(Tx, Ty) \leqslant k\|x - y\| \quad \text{for all } x, y \in \bar{U} \tag{2.1}$$

and

$$Tx\left(\frac{x}{\lambda}\right) \neq 1 \quad \text{for all } x \in \partial U \text{ and } \lambda \in (0, 1) \tag{2.2}$$

holding, and assume $[T\bar{U}]_1$ is bounded. Then T has a fuzzy fixed point. That is there exists $x \in \bar{U}$ with $\{x\} \subseteq Tx$.

Proof. Let $F : \bar{U} \to C(E)$ be given by $Fx = [Tx]_1$. We will apply Theorem 2.3. Suppose (A2) holds. Then there exists $x \in \partial U$ and $\lambda \in (0, 1)$ with $x \in \lambda Fx = \lambda[Tx]_1$ (i.e. $Tx(x/\lambda) = 1$). This contradicts (2.2) so (A1) must hold. That is there exists $x \in \bar{U}$ with $x \in Fx = [Tx]_1$. □

Theorem 2.4 and a similar argument yields the following result.

Theorem 2.6. *Let $E = (E, \|\cdot\|)$ be a uniformly convex Banach space and U a bounded, convex, open subset of E with $0 \in U$, and $T : \bar{U} \to FK(E)$. Suppose*

$$D_1(Tx, Ty) \leqslant \|x - y\| \quad \text{for all } x, y \in \bar{U}$$

and

$$Tx\left(\frac{x}{\lambda}\right) \neq 1 \quad \text{for all } x \in \partial U \text{ and } \lambda \in (0, 1)$$

hold, and assume $[T\bar{U}]_1$ is bounded. Then there exists $x \in \bar{U}$ with $\{x\} \subseteq Tx$.

Remark 1. In the above spirit it is easily seen that the nonexpansive results in [1,10] can be generalized.

Next we present some fixed point theory for maps $T : FW(X) \to FW(X)$. As we shall see a stronger contractive condition will be needed to guarantee the existence of a fixed point.

Lemma 2.7. *Let (X, d) be a complete metric space. Then $(FW(X), D)$ is a complete metric space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $FW(X)$. Then $[u_n]_\alpha$, for every α, is a Cauchy sequence in $CB(X)$ (here $CB(X)$ denotes the family of nonempty, closed, bounded subsets of X). Recalling that $(CB(X), H)$ is complete, see for example [5, p. 24], then for every α there exists $A_\alpha \in CB(X)$ with

$$[u_n]_\alpha \to A_\alpha = \bigcap_n \overline{\bigcup_{m \geqslant n} [u_m]_\alpha}.$$

Define

$$u(x) = \sup_{x \in A_\alpha} \alpha.$$

Essentially the same analysis as in [8, p. 420–421], shows that $[u]_\alpha = A_\alpha$. □

Theorem 2.8. *Let (X, d) be a complete metric space and $T : FW(X) \to FW(X)$. Suppose there exists $k \in (0, 1)$ with*

$$D(Tx, Ty) \leqslant kD(x, y) \quad \textit{for all } x, y \in FW(X)$$

holding. Then there exists $x \in FW(X)$ with $x = Tx$.

Proof. This is immediate from Lemma 2.7 and the Banach contraction principle. □

We now present a result where the map is not defined on all of $FW(X)$. It is an immediate consequence of Lemma 2.7 and a result on contractive homotopy [3].

Theorem 2.9. *Let (X, d) be a complete metric space. Suppose U is an open subset of $FW(X)$ and $N : \bar{U} \times [0, 1] \to FW(X)$ is such that*

(a) *there exists $k \in (0, 1)$ with*

$$D(N(x, t), N(y, t)) \leqslant kD(x, y)$$

for all $x, y \in \bar{U}$ and $t \in [0, 1]$;

(b) *there exists $\phi : [0, 1] \to \mathbf{R}$ with*

$$D(N(x, t), N(x, s)) \leqslant |\phi(t) - \phi(s)|$$

for all $t, s \in [0, 1]$ and $x \in \bar{U}$;

(c) *$x \neq N(x, t)$ for all $x \in \partial U$ and $t \in [0, 1]$.*

Then $N(\cdot, 0)$ has a fixed point if and only if $N(\cdot, 1)$ has a fixed point.

3. Fixed point theory for maps of Kulshrestha type

In this section we begin by presenting fixed point results for Kulshreshtha contractive maps with closed values defined on complete metric spaces. From these results we deduce immediately some fuzzy fixed point theorems which generalize results of Singh and Talwar [9].

Theorem 3.1. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $F : \overline{B(x_0, r)} \to C(X)$. Suppose there exists $q \in (0, 1)$ such that for $x, y \in \overline{B(x_0, r)}$ we have*

$$H(Fx, Fy) \leqslant q \max\{d(x, y), \mathrm{dist}(x, Fx), \mathrm{dist}(y, Fy), \tfrac{1}{2}[\mathrm{dist}(x, Fy) + \mathrm{dist}(y, Fx)]\}$$

and

$$\mathrm{dist}(x_0, Fx_0) < (1 - q)r.$$

Then F has a fixed point (i.e. there exists $x \in \overline{B(x_0, r)}$ with $x \in Fx$).

Proof. Choose $x_1 \in Fx_0$ with $d(x_1, x_0) < (1 - q)r$, so $x_1 \in \overline{B(x_0, r)}$. Now choose $\varepsilon > 0$ such that

$$qd(x_1, x_0) + \frac{\varepsilon}{1 - q} < q(1 - q)r. \tag{3.1}$$

Then choose $x_2 \in Fx_1$ with

$$\begin{aligned} d(x_1, x_2) &\leqslant H(Fx_0, Fx_1) + \varepsilon \\ &\leqslant q \max\{d(x_0, x_1), \mathrm{dist}(x_0, Fx_0), \mathrm{dist}(x_1, Fx_1), \tfrac{1}{2}[\mathrm{dist}(x_0, Fx_1) + \mathrm{dist}(x_1, Fx_0)]\} + \varepsilon \\ &\leqslant qd(x_0, x_1) + \frac{\varepsilon}{2 - q}; \end{aligned}$$

this is immediate since if say the maximum of the right-hand side of the above displayed equation is $\frac{1}{2}[\mathrm{dist}(x_0, Fx_1) + \mathrm{dist}(x_1, Fx_0)]$, then

$$d(x_1, x_2) \leqslant (q/2)d(x_0, x_1) + \varepsilon$$

and so

$$d(x_1,x_2) \leqslant \frac{q}{2-q}d(x_0,x_1) + \varepsilon\left(\frac{2}{2-q}\right)$$

(note $q/(2-q)<q$ and $2/(2-q)<1/(1-q)$). The other cases are easier. Now with ε chosen as in (3.1) we have

$$d(x_1,x_2) < q(1-q)r.$$

Notice $x_2 \in \overline{B(x_0,r)}$ since

$$d(x_0,x_2) \leqslant (1-q)r + q(1-q)r$$
$$\leqslant (1-q)r[1+q+q^2+\cdots] = r.$$

Next choose $\delta>0$ such that

$$qd(x_1,x_2) + \frac{\delta}{1-q} < q^2(1-q)r.$$

Then choose $x_3 \in Fx_2$ with

$$d(x_2,x_3) \leqslant H(Fx_1,Fx_2) + \delta \leqslant qd(x_1,x_2) + \frac{\delta}{1-q}$$
$$< q^2(1-q)r.$$

Notice as well that $x_3 \in \overline{B(x_0,r)}$. Proceed inductively to obtain $x_n \in Fx_{n-1}$, $n=4,5,\ldots$ with $d(x_{n+1},x_n) < q^n(1-q)r$ and $x_n \in \overline{B(x_0,r)}$. Now since $q\in(0,1)$ we have that (x_n) is Cauchy and so there exists $x\in\overline{B(x_0,r)}$ with $\lim_{n\to\infty} x_n = x$. It remains to show $x\in Fx$. Notice

$$\begin{aligned}\operatorname{dist}(x,Fx) &\leqslant d(x,x_n) + \operatorname{dist}(x_n,Fx)\\ &\leqslant d(x,x_n) + q\max\{d(x,x_{n-1}),\\ &\qquad \operatorname{dist}(x,Fx), \operatorname{dist}(x_{n-1},Fx_{n-1}),\\ &\qquad \tfrac{1}{2}[\operatorname{dist}(x,Fx_{n-1}) + \operatorname{dist}(x_{n-1},Fx)]\}\\ &\leqslant d(x,x_n) + q\max\{d(x,x_{n-1}),\\ &\qquad \operatorname{dist}(x,Fx), d(x_{n-1},x_n),\\ &\qquad \tfrac{1}{2}[d(x,x_n) + d(x_{n-1},x) + \operatorname{dist}(x,Fx)]\}.\end{aligned}$$

Letting $n\to\infty$ gives

$$\operatorname{dist}(x,Fx) \leqslant q\operatorname{dist}(x,Fx) \quad \text{i.e. } \operatorname{dist}(x,Fx)=0.$$

Thus $x\in\overline{Fx}=Fx$. □

We next note that we obtain Kulshrestha's [6] fixed point result as a Corollary of Theorem 3.1, see also [7].

Theorem 3.2. *Let (X,d) be a complete metric space and $F:X\to C(X)$. Suppose there exists $q\in(0,1)$ such that for $x,y\in X$ we have*

$$H(Fx,Fy) \leqslant q\max\{d(x,y), \operatorname{dist}(x,Fx), \operatorname{dist}(y,Fy),$$
$$\tfrac{1}{2}[\operatorname{dist}(x,Fy) + \operatorname{dist}(y,Fx)]\}.$$

Then F has a fixed point.

Proof. Fix $x_0\in X$. Choose $r>0$ so that

$$\operatorname{dist}(x_0,Fx_0) < (1-q)r.$$

Now Theorem 3.1 guarantees that there exists $x\in\overline{B(x_0,r)}$ with $x\in Fx$. □

Next we extend the homotopy results in [2,3] for generalized contractive homotopy of Kulshrestha type.

Theorem 3.3. *Let (X,d) be a complete metric space and U open in X. Suppose $N:\bar{U}\times[0,1]\to C(X)$ is a closed map (i.e. has closed graph) with the following satisfied:*

(a) *$x\notin N(x,t)$ for $x\in\partial U$ and $t\in[0,1]$;*
(b) *there exists $q\in(0,1)$ such that for all $t\in[0,1]$ and $x,y\in\bar{U}$ we have*

$$H(N(x,t),N(y,t))$$
$$\leqslant q\max\{d(x,y), \operatorname{dist}(x,N(x,t)), \operatorname{dist}(y,N(y,t)),$$
$$\tfrac{1}{2}[\operatorname{dist}(x,N(y,t)) + \operatorname{dist}(y,N(x,t))]\};$$

(c) *there exists a continuous increasing function $\phi:[0,1]\to\mathbf{R}$ such that*

$$H(N(x,t),N(x,s)) \leqslant |\phi(t)-\phi(s)|$$
for all $t,s\in[0,1]$ and $x\in\bar{U}$.

Then $N(\cdot,0)$ has a fixed point if and only if $N(\cdot,1)$ has a fixed point.

Proof. Suppose $N(\cdot,0)$ has a fixed point. Consider

$$Q = \{(t,x)\in[0,1]\times U:\ x\in N(x,t)\}.$$

Now Q is nonempty since $N(\cdot,0)$ has a fixed point. On Q define the partial order

$$(t,x) \leqslant (s,y) \quad \text{iff } t \leqslant s \quad \text{and} \quad d(x,y) \leqslant \frac{2[\phi(s)-\phi(t)]}{1-q}.$$

Let P be a totally ordered subset of Q and let

$$t^\star = \sup\{t:\ (t,x) \in P\}.$$

Take a sequence $\{(t_n,x_n)\}$ in P such that $(t_n,x_n) \leqslant (t_{n+1},x_{n+1})$ and $t_n \to t^\star$. We have

$$d(x_m,x_n) \leqslant \frac{2}{1-q}[\phi(t_m)-\phi(t_n)] \quad \text{for all } m>n,$$

so (x_m) is a Cauchy sequence, which converges to some $x^\star \in \bar{U}$. Now since N is a closed map we have $x^\star \in N(x^\star,t^\star)$ and also (a) implies $x^\star \in U$. Thus $(t^\star,x^\star) \in Q$. It is also immediate from the definition of $t^\star$ and the fact that P is totally ordered that

$$(t,x) \leqslant (t^\star,x^\star) \quad \text{for every } (t,x) \in P.$$

Thus $(t^\star,x^\star)$ is an upper bound of P. By Zorn's lemma Q admits a maximal element $(t_0,x_0) \in Q$.

We claim $t_0 = 1$ (if our claim is true then we are finished). Suppose our claim is false. Then, choose $r>0$ and $t \in (t_0,1]$ with

$$\overline{B(x_0,r)} \subseteq U \quad \text{and} \quad r = \frac{2[\phi(t)-\phi(t_0)]}{1-q}.$$

Notice

$$\begin{aligned}&\operatorname{dist}(x_0,N(x_0,t))\\ &\quad\leqslant \operatorname{dist}(x_0,N(x_0,t_0)) + H(N(x_0,t_0),N(x_0,t))\\ &\quad\leqslant \phi(t)-\phi(t_0) = \left(\frac{1-q}{2}\right) r < (1-q)r.\end{aligned}$$

Now Theorem 3.1 guarantees that $N(\cdot,t)$ has a fixed point $x \in \overline{B(x_0,r)}$. Thus $(x,t) \in Q$ and notice since

$$d(x_0,x) \leqslant r = \frac{2[\phi(t)-\phi(t_0)]}{1-q} \quad \text{and} \quad t_0 < t,$$

we have $(t_0,x_0) < (t,x)$. This contradicts the maximality of (t_0,x_0). □

We now establish the fuzzy analogue of Theorems 3.1, 3.2 and 3.3.

Theorem 3.4. *Let (X,d) be a complete metric space, $x_0 \in X$, $r>0$ and $T:\overline{B(x_0,r)} \to FC(X)$. Suppose there exists $q \in (0,1)$ such that for $x,y \in \overline{B(x_0,r)}$ we have*

$$\begin{aligned}&D_1(Tx,Ty)\\ &\quad\leqslant q\ \max\{d(x,y), \operatorname{dist}(x,[Tx]_1), \operatorname{dist}(y,[Ty]_1),\\ &\qquad \tfrac{1}{2}[\operatorname{dist}(x,[Ty]_1)+\operatorname{dist}(y,[Tx]_1)]\}\end{aligned}$$

and

$$\operatorname{dist}(x_0,[Tx_0]_1) < (1-q)r.$$

Then T has a fuzzy fixed point. That is there exists $x \in \overline{B(x_0,r)}$ with $\{x\} \subseteq Tx$.

Proof. Let $F:X \to C(X)$ be given by $Fx = [Tx]_1$. Now apply Theorem 3.1. □

Next we present a generalization of the main result in [9] which follows directly from Theorem 3.2. Notice we assume a weaker contractive type condition and the α level sets are not assumed to be convex and compact.

Theorem 3.5. *Let (X,d) be a complete metric space and $T:X \to FC(X)$. Suppose there exists $q \in (0,1)$ such that for $x,y \in X$ we have*

$$\begin{aligned}&D_1(Tx,Ty)\\ &\quad\leqslant q\ \max\{d(x,y), \operatorname{dist}(x,[Tx]_1), \operatorname{dist}(y,[Ty]_1),\\ &\qquad \tfrac{1}{2}[\operatorname{dist}(x,[Ty]_1)+\operatorname{dist}(y,[Tx]_1)]\}.\end{aligned}$$

Then F has a fuzzy fixed point.

As an immediate consequence of Theorem 3.3 we have the following fuzzy result.

Theorem 3.6. *Let (X,d) be a complete metric space and U open in X. Suppose $T:\bar{U}\times[0,1] \to FC(X)$ is a closed map with the following satisfied:*

(a) *$x \notin [T(x,t)]_1$ for $x \in \partial U$ and $t \in [0,1]$;*
(b) *there exists $q \in (0,1)$ such that for all $t \in [0,1]$ and $x,y \in \bar{U}$ we have*

$$D_1(T(x,t),T(y,t))$$

$$\leqslant q \max\{d(x,y), \mathrm{dist}(x,[T(x,t)]_1), \mathrm{dist}(y,[T(y,t)]_1), \tfrac{1}{2}[\mathrm{dist}(x,[T(y,t)]_1) + \mathrm{dist}(y,[T(x,t)]_1)]\};$$

(c) *there exists a continuous increasing function* $\phi:[0,1]\to\mathbf{R}$ *such that*

$$D_1(T(x,t),T(x,s)) \leqslant |\phi(t)-\phi(s)|$$

for all $t,s\in[0,1]$ *and* $x\in\bar{U}$.

Then $T(\cdot,0)$ *has a fuzzy fixed point if and only if* $T(\cdot,1)$ *has a fuzzy fixed point.*

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