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Existence Results for First-Order Impulsive Differential Equations

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1. INTRODUCTION

Let p be a positive integer and $T \in (0, \infty)$. Suppose the sequence $\{t_k\}_{k=1}^p$ is such that $0 < t_1 < t_2 < \cdots < t_p < T$. This paper establishes existence results for the impulsive initial value problem

$$\begin{aligned} y' &= f(t, y), \qquad 0 < t < T, t \neq t_k; k = 1, \ldots, p, \\ y(t_k^+) &= I_k(y(t_k^-)), \qquad k = 1, \ldots, p, \\ y(0) &= a, \end{aligned} \tag{1.1}$$

and the impulsive periodic problem

$$\begin{aligned} y' &= f(t, y), \qquad 0 < t < T, t \neq t_k; k = 1, \ldots, p \\ y(t_k^+) &= I_k(y(t_k^-)), k = 1, \ldots, p, \\ y(0) &= y(T), \end{aligned} \tag{1.2}$$

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where $f:[0, T] \times \mathbf{R} \to \mathbf{R}$ is an L^1-Carathéodory function and I_k maps $\mathbf{R}$ into $\mathbf{R}$ for $k = 1, \ldots, p$.

Let $\Omega = \{u:[0, T] \to \mathbf{R}: u$ is continuous for $t \neq t_k$, $u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k) = u(t_k^-)$, $k = 1, \ldots, p\}$; $\Omega^1 = \{u \in \Omega: u$ is differentiable almost everywhere on $(0, T)$ and $u' \in L^1[0, T]\}$. Here $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$. Denote Ω_p (respectively Ω_p^1) $= \{u \in \Omega$ (respectively Ω^1): $u(0) = u(T)\}$. By a solution to (1.1) we mean a function $y \in \Omega^1$ with $y(t_k^+) = I_k(y(t_k^-))$, $k = 1, \ldots, p$, with $y(0) = a$ and y satisfying the differential equation almost everywhere on $(0, T)/\{t_k\}$, $k = 1, \ldots, p$. Similarly we can define a solution to (1.2).

The paper will be divided into two main sections. In Section 2 we establish four existence theorems for (1.1). The results of this section provide an easy way to recognize just by looking at the differential equation when (1.1) has a solution on $[0, \infty)$ or on $[0, T]$ for $T < \infty$. In fact two of the results are new even for the initial value problem without impulses [3, 10]; i.e., $I_k(x) = x$ for $k = 1, \ldots, p$. The impulsive periodic problem is treated in Section 3 using the idea of upper and lower solutions (see Section 3 for the appropriate definitions). Our results complement the theory in the literature, see [1, 7–9] and their references; in particular our results do not assume any type of monotonicity condition on f and I_k, $k = 1, \ldots, p$, which is usually the situation in the literature.

We use a fixed point approach to establish our existence principles. In particular we use a nonlinear alternative of Leray–Schauder type [5] which is an immediate consequence of the topological transversality theorem [2, 4, 5] of Granas. For completeness we state the result. By a map being *compact* we mean it is continuous with relatively compact range. A map is *completely continuous* if it is continuous and the image of every bounded set in the domain is contained in a compact set of the range.

THEOREM 1.1. *Assume U is a relatively open subset of a convex set K in a Banach space E. Let $G: \overline{U} \to K$ be a compact map, $p^* \in U$ and $N_\lambda(u) = N(u, \lambda): \overline{U} \times [0, 1] \to K$ a family of compact maps (i.e., $N(\overline{U} \times [0, 1])$ is contained in a compact subset of K and $N: \overline{U} \times [0, 1] \to K$ is continuous) with $N_1 = G$ and $N_0 = p^*$, the constant map to p^*. Then either*

(i) *G has a fixed point in $\overline{U}$ (respectively U)*; or

(ii) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ (respectively $\lambda \in (0, 1]$) such that $u = N_\lambda u$.*

Also the following definition from the literature will be used throughout this paper. Let k be a positive integer and $-\infty < a < b < \infty$. A function $g:[a, b] \times \mathbf{R}^k \to \mathbf{R}$ is an L^1-Carathéodory function provided: if

$g(t, z)$, then

(a). the map $z \to g(t, z)$ is continuous for almost all $t \in [a, b]$,

(b). the map $t \to g(t, z)$ is measurable for all $z \in \mathbf{R}^k$,

(c). for a given $r > 0$ there exists $h_r \in L^1[a, b]$ such that $|z| \leq r$ implies $|g(t, z)| \leq h_r(t)$ for almost all $t \in [a, b]$.

We remark here that conditions (a) and (b) implies for $t \in [a, b]$ that $g(t, u(t))$ is measurable for any measurable $u(t)$. This is a result of Carathéodory; see [6, 11]. Also (c) implies that $g(t, u(t))$ is integrable.

2. Impulsive Initial Value Problems

We first recall an existence principle from the literature of ordinary differential equations.

THEOREM 2.1. *Let $f: [b, c] \times \mathbf{R} \to \mathbf{R}$ be an L^1-Carathéodory function.*

(i) *Suppose there is a constant M, independent of λ, with*

$$|y|_0 = \sup_{[b,c]} |y(t)| \leq M$$

for any solution y to

$$\begin{aligned} y' &= \lambda f(t, y), \qquad b < t < c, \\ y(b) &= d \end{aligned} \tag{2.1}_\lambda$$

for any $\lambda \in (0, 1)$. (By a solution to $(2.1)_\lambda$ we mean a function $y \in AC[b, c]$ with y satisfying the differential equation almost everywhere on $[b, c]$ and $y(b) = d$). Then $(2.1)_1$ has at least one solution.

(ii) *Suppose there is a function $h \in L^1[b, c]$ with*

$$|f(t, u)| \leq h(t) \qquad \text{a.e. } t \in (b, c) \text{ and for all } u \in \mathbf{R}.$$

Then $(2.1)_1$ has at least one solution.

Proof. (i) See [5].

(ii) Observe that a solution to $(2.1)_1$ is a fixed point of the operator $N: C_d[b, c] \to C_d[b, c]$ defined by

$$(Ny)(t) = d + \int_b^t f(\tau, y(\tau))\, d\tau.$$

Here $C_d[b, c] = \{u \in C[b, c]: u(b) = d\}$. Clearly N is continuous by the Lebesgue dominated convergence theorem (we also use the fact that f is an L^1-Carathéodory function and that we have an h as described above). N is compact by the Arzela–Ascoli theorem (and the fact that we have an h as described above). The Schauder fixed point theorem then gives the existence of a fixed point to N, i.e., a solution to $(2.1)_1$. ∎

This existence principle will now be used to obtain four existence theorems for the impulsive initial value problem (1.1).

THEOREM 2.2. *Let $t_0 = 0$, $t_{p+1} = T$ and assume*

$$f: [0, T] \times \mathbf{R} \to \mathbf{R} \text{ is an } L^1\text{-Carathéodory function with } I_k \text{ mapping } \mathbf{R} \text{ into } \mathbf{R} \text{ and bounded on bounded sets, } k = 1, \ldots, p \tag{2.2}$$

$$|f(t, u)| \le q(t)\psi(|u|) \text{ for almost all } t \in [0, T] \text{ with } \psi: [0, \infty) \to (0, \infty) \text{ a Borel function and } q \in L^1[0, T] \tag{2.3}$$

and

$$\int_{t_{k-1}}^{t_k} q(s)\, ds < \int_{N_{k-1}}^{\infty} \frac{du}{\psi(u)}, \qquad k = 1, \ldots, p + 1. \tag{2.4}$$

Here $N_0 = |a|$ and for $k = 2, \ldots, p + 1$ we have

$$N_{k-1} = \max|I_{k-1}(x)|, \qquad x \in [-A_{k-2}, A_{k-2}], \qquad A_{k-2} = \Gamma_{k-1}^{-1}\left(\int_{t_{k-2}}^{t_{k-1}} q(s)\, ds\right)$$

with

$$\Gamma_m(z) = \int_{N_{m-1}}^{z} \frac{du}{\psi(u)}, \qquad z \ge N_{m-1}; m = 1, \ldots, p + 1.$$

Then (1.1) *has at least one solution on* $[0, T]$.

Proof. Let y be a solution to

$$\begin{aligned} y' &= \lambda f(t, y), \qquad 0 < t < t_1, \\ y(0) &= a \end{aligned} \tag{$2.5)_\lambda$}$$

for $\lambda \in (0, 1)$. Suppose $|y(t)| > |a|$ for some $t \in (0, t_1]$. Then there exists $\mu \in [0, t)$ with $|y(s)| > |a|$ for $s \in (\mu, t]$ and $|y(\mu)| = |a|$. Note since

$|y|^2 = y^2$ then

$$|y|' = \frac{yy'}{|y|} \leq |y'|$$

almost everywhere on (μ, t). Thus we have

$$|y(s)|' \leq q(s)\psi(|y(s)|)$$

almost everywhere on (μ, t) and integration from μ to t yields

$$\Gamma_1(|y(t)|) = \int_{|a|}^{|y(t)|} \frac{du}{\psi(u)} \leq \int_{\mu}^{t} q(s)\,ds \leq \int_0^{t_1} q(s)\,ds$$

using the change of variables formula [3, 5]. Consequently,

$$|y(t)| \leq \Gamma_1^{-1}\left(\int_0^{t_1} q(s)\,ds\right)$$

and so

$$\sup_{[0,t_1]} |y(s)| \leq \Gamma_1^{-1}\left(\int_0^{t_1} q(s)\,ds\right) = A_0.$$

Now Theorem 2.1 implies that problem $(2.5)_1$ has at least one solution $y \in AC\,[0, t_1]$. Pick **one** such solution; call it y_1 say.

Let y be a solution to

$$\begin{aligned} y' &= \lambda f(t, y), \qquad t_1 < t < t_2, \\ y(t_1^+) &= I_1(y_1(t_1^-)) \end{aligned} \tag{2.6}_\lambda$$

for $\lambda \in (0, 1)$. Note first that

$$|y(t_1^+)| \leq \max |I_1(x)| = N_1, \qquad \text{where } x \in [-A_0, A_0].$$

Suppose $|y(t)| > N_1$ for some $t \in (t_1, t_2]$. Then there exists $\eta \in [t_1, t)$ with $|y(s)| > N_1$ for $s \in (\eta, t]$ and $|y(\eta)| = N_1$. Thus we have

$$|y(s)|' \leq q(s)\psi(|y(s)|)$$

almost everywhere on (η, t). Integration from η to t yields

$$\Gamma_2(|y(t)|) = \int_{N_1}^{|y(t)|} \frac{du}{\psi(u)} \leq \int_{\eta}^{t} q(s)\,ds \leq \int_{t_1}^{t_2} q(s)\,ds$$

and so

$$\sup_{[t_1,t_2]} |y(s)| \le \Gamma_2^{-1}\left(\int_{t_1}^{t_2} q(s)\,ds\right) = A_1.$$

Now Theorem 2.1 implies that problem $(2.6)_1$ has at least one solution $y \in AC[t_1, t_2]$. Pick **one** such solution; call it y_2, say.

Continue this process and construct solutions $y_k \in AC[t_{k-1}, t_k]$, with $k = 3, \ldots, p+1$, to

$$y' = f(t, y), \qquad t_{k-1} < t < t_k$$
$$y(t_{k-1}^+) = I_{k-1}(y_{k-1}(t_{k-1}^-)).$$

Then

$$y(t) = \begin{cases} y_1(t), & t \in [0, t_1] \\ y_2(t), & t \in (t_1, t_2] \\ \vdots & \\ y_p(t), & t \in (t_{p-1}, t_p] \\ y_{p+1}(t), & t \in (t_p, T] \end{cases}$$

is a solution of (1.1). ∎

THEOREM 2.3. *Let $t_0 = 0$, $t_{p+1} = T$ and assume that* (2.2) *is satisfied. In addition, suppose*

$$\begin{aligned} &\text{there exist } \{r_i\}_{i=0}^p \text{ and } \{s_i\}_{i=0}^p \text{ with } s_0 \le a \le r_0 \text{ and} \\ &r_{i+1} \ge \max_{[s_i, r_i]} I_{i+1}(x) \ge \min_{[s_i, r_i]} I_{i+1}(x) \ge s_{i+1},\ i = 0, \ldots, p-1 \end{aligned} \tag{2.7}$$

and

$$f(t, r_i) \le 0, f(t, s_i) \ge 0 \qquad \text{for } t \in [t_i, t_{i+1}],\ i = 0, \ldots, p. \tag{2.8}$$

Then (1.1) *has at least one solution on* $[0, T]$.

Proof. Let y be a solution to

$$y' = \lambda f_1(t, y), \qquad 0 < t < t_1, \tag{$2.9)_\lambda$}$$
$$y(0) = a$$

for $\lambda \in (0, 1)$ with

$$f_1(t, y) = \begin{cases} f(t, r_0), & y \geq r_0, \\ f(t, y), & s_0 \leq y \leq r_0, \\ f(t, s_0), & y \leq s_0. \end{cases}$$

Suppose there exists $t \in (0, t_1]$ with $y(t) > r_0$. Then there exists $\mu \in [0, t)$ with $y(s) > r_0$ for $s \in (\mu, t]$ and $y(\mu) = r_0$. Now since $f(t, r_0) \leq 0$ for $t \in [0, t_1]$ we have

$$0 < y(t) - y(\mu) = \int_\mu^t y'(s)\, ds \leq 0,$$

a contradiction. Thus $y(t) \leq r_0$ for $t \in [0, t_1]$. Similarly, $y(t) \geq s_0$ for $t \in [0, t_1]$. Now Theorem 2.1 implies that $(2.9)_1$ has at least one solution $y \in AC[0, t_1]$ with $s_0 \leq y(t) \leq r_0$, $t \in [0, t_1]$. Pick **one** such solution; call it y_1, say. Since $s_0 \leq y_1(t) \leq r_0$, $t \in [0, t_1]$, then y_1 is a solution of

$$\begin{aligned} y' &= f(t, y), \qquad 0 < t < t_1, \\ y(0) &= a. \end{aligned}$$

Let y be a solution to

$$\begin{aligned} y' &= \lambda f_2(t, y), \qquad t_1 < t < t_2, \\ y(t_1^+) &= I_1(y_1(t_1^-)) \end{aligned} \tag{$2.10)_\lambda$}$$

for $\lambda \in (0, 1)$ with

$$f_2(t, y) = \begin{cases} f(t, r_1), & y \geq r_1 \\ f(t, y), & s_1 \leq y \leq r_1 \\ f(t, s_1), & y \leq s_1. \end{cases}$$

Since $y_1(t_1^-) \in [s_0, r_0]$ then (2.7) implies that

$$r_1 \geq I_1(y(t_1^-)) \geq s_1, \qquad \text{i.e., } s_1 \leq y(t_1^+) \leq r_1.$$

Suppose there exists $t \in (t_1, t_2]$ with $y(t) > r_1$. Then there exists $\eta \in [t_1, t)$ with $y(s) > r_1$ for $s \in (\eta, t]$ and $y(\eta) = r_1$. Now since $f(t, r_1) \leq 0$

for $t \in [t_1, t_2]$, we have

$$0 < y(t) - y(\eta) = \int_{\eta}^{t} y'(s)\, ds \leq 0,$$

a contradiction. Thus $y(t) \leq r_1$ for $t \in [t_1, t_2]$. Similarly, $y(t) \geq s_1$ for $t \in [t_1, t_2]$. Now Theorem 2.1 implies that $(2.10)_1$ has at least one solution $y \in AC[t_1, t_2]$ with $s_1 \leq y(t) \leq r_1$, $t \in [t_1, t_2]$. Pick **one** such solution; call it y_2, say. Since $s_1 \leq y_2(t) \leq r_1$, $t \in [t_1, t_2]$ then y_2 is a solution of

$$\begin{aligned} y' &= f(t, y), \qquad t_1 < t < t_2, \\ y(t_1^+) &= I_1(y_1(t_1^-)). \end{aligned}$$

Continue this process and construct solutions $y_k \in AC[t_{k-1}, t_k]$, with $k = 3, \ldots, p + 1$, to

$$\begin{aligned} y' &= f(t, y), \qquad t_{k-1} < t < t_k, \\ y(t_{k-1}^+) &= I_{k-1}(y_{k-1}(t_{k-1}^-)) \end{aligned}$$

with $s_{k-1} \leq y_k(t) \leq r_{k-1}$, $t \in [t_{k-1}, t_k]$. Then

$$y(t) = \begin{cases} y_1(t), & t \in [0, t_1], \\ y_2(t), & t \in (t_1, t_2] \\ \cdot & \\ \cdot & \\ \cdot & \\ y_p(t), & t \in (t_{p-1}, t_p], \\ y_{p+1}(t), & t \in (t_p, T], \end{cases}$$

is a solution of (1.1). ∎

It is possible to improve Theorem 2.3 if we use Theorem 2.1 (ii).

THEOREM 2.4. *Let $t_0 = 0$, $t_{p+1} = T$ and assume that* (2.2) *is satisfied. In addition, suppose*

$$\begin{aligned} &\text{there are functions } \{r_i\}_{i=0}^{p} \text{ and } \{s_i\}_{i=0}^{p} \text{ with } r_i,\ s_i \in C[t_i, t_{i+1}] \text{ and} \\ &s_i(t) \leq r_i(t) \text{ for } t \in [t_i, t_{i+1}],\ i = 0, \ldots, p. \text{ Also, } s_0(0) \leq a \leq r_0(0) \text{ and} \\ &r_{i+1}(t_{i+1}^+) \geq \max_{[b_i, c_i]} I_{i+1}(x) \geq \min_{[b_i, c_i]} I_{i+1}(x) \geq s_{i+1}(t_{i+1}^+), \\ &i = 0, \ldots, p - 1. \text{ Here } [b_i, c_i] = [s_i(t_{i+1}^-), r_i(t_{i+1}^-)], \end{aligned} \tag{2.11}$$

and

$$\int_{z_i}^{w_i} f(t, s_i(t))\,dt \geq s_i(w_i) - s_i(z_i), \int_{z_i}^{w_i} f(t, r_i(t))\,dt \leq r_i(w_i) - r_i(z_i), \tag{2.12}$$
$$i = 0, \ldots, p \text{ with } z_i < w_i \text{ and } z_i, w_i \in [t_i, t_{i+1}].$$

Then (1.1) *has at least one solution on* $[0, T]$.

Proof. From Theorem 2.1 (ii) there exists a solution $y_1 \in AC[0, t_1]$ to

$$y' = f_1(t, y), \qquad 0 < t < t_1,$$
$$y(0) = a$$

with

$$f_1(t, y) = \begin{cases} f(t, r_0(t)), & y > r_0(t), \\ f(t, y), & s_0(t) \leq y \leq r_0(t), \\ f(t, s_0(t)), & y < s_0(t). \end{cases}$$

We now claim that $s_0(t) \leq y_1(t) \leq r_0(t)$ for $t \in [0, t_1]$.

First we show that $y_1(t) \leq r_0(t)$ for $t \in [0, t_1]$. Assume this is false. Then since $y_1(0) = a \leq r_0(0)$, there exists $x_1 < x_2 \in [0, t_1]$ such that $y_1(x_1) \leq r_0(x_1)$, $y_1(x_2) > r_0(x_2)$, and $y_1(t) \geq r_0(t)$, $t \in (x_1, x_2)$. Consequently,

$$r_0(x_2) - r_0(x_1) < y_1(x_2) - y_1(x_1) = \int_{x_1}^{x_2} f(t, r_0(t))\,dt \leq r_0(x_2) - r_0(x_1),$$

a contradiction. Thus $y_1(t) \leq r_0(t)$ for $t \in [0, t_1]$ and a similar argument yields $y_1(t) \geq s_0(t)$ for $t \in [0, t_1]$. Hence y_1 is a solution of

$$y' = f(t, y), \qquad 0 < t < t_1,$$
$$y(0) = a.$$

Next consider the problem

$$y' = f_2(t, y), \qquad t_1 < t < t_2 \tag{2.13}$$
$$y(t_1^+) = I_1(y_1(t_1^-))$$

with

$$f_2(t, y) = \begin{cases} f(t, r_1(t)), & y > r_1(t) \\ f(t, y), & s_1(t) \leq y \leq r_1(t) \\ f(t, s_1(t)), & y < s_1(t). \end{cases}$$

Now since $y_1(t_1^-) \in [s_0(t_1^-), r_0(t_1^-)]$, then

$$r_1(t_1^+) \geq I_1(y_1(t_1^-)) \geq s_1(t_1^+), \qquad \text{i.e., } s_1(t_1^+) \leq y(t_1^+) \leq r_1(t_1^+).$$

Also Theorem 2.1(ii) implies that (2.13) has a solution $y_2 \in AC[t_1, t_2]$.

We now show that $y_2(t) \leq r_1(t)$ for $t \in [t_1, t_2]$. Assume this is false, then since $y_2(t_1^+) \leq r_1(t_1^+)$, there exists $x_3 < x_4 \in [t_1, t_2]$ such that $y_2(x_3) \leq r_1(x_3)$, $y_2(x_4) > r_1(x_4)$ and $y_2(t) \geq r_1(t)$, $t \in (x_3, x_4)$. Consequently,

$$r_1(x_4) - r_1(x_3) < \int_{x_3}^{x_4} f(t, r_1(t))\, dt \leq r_1(x_4) - r_1(x_3),$$

a contradiction. Thus $y_2(t) \leq r_1(t)$ for $t \in [t_1, t_2]$ and a similar argument yields $y_2(t) \geq s_1(t)$ for $t \in [t_1, t_2]$. Hence, y_2 is a solution of

$$\begin{aligned} y' &= f(t, y), \qquad t_1 < t < t_2, \\ y(t_1^+) &= I_1(y_1(t_1^-)). \end{aligned}$$

Continue this process and the result follows as in Theorem 2.3. ∎

THEOREM 2.5. *Let* $t_0 = 0$, $t_{p+1} = T$ *and assume that* (2.2) *and* (2.3) *hold. In addition, assume there exists* $\{s_i\}_{i=0}^{p}$ *with* $s_0 \leq a$, *such that*

$$\min_{[s_i, A_i]} I_{i+1}(x) \geq s_{i+1}, \qquad i = 0, \ldots, p-1, \tag{2.14}$$

$$f(t, s_i) \geq 0 \qquad \text{for } t \in [t_i, t_{i+1}],\ i = 0, \ldots, p, \tag{2.15}$$

and

$$\int_{t_{k-1}}^{t_k} q(s)\, ds < \int_{H_{k-1}}^{\infty} \frac{du}{\psi(u)}, \qquad k = 1, \ldots, p+1. \tag{2.16}$$

Here $H_0 = \max\{0, a\}$ *and for* $k = 2, \ldots, p+1$ *we have*

$$H_{k-1} = \max\{0, \max I_{k-1}(x)\}, \qquad x \in [-A_{k-2}, A_{k-2}],$$

with

$$A_{j-2} = J_{j-1}^{-1}\left(\int_{t_{j-2}}^{t_{j-1}} q(s)\, ds\right), \qquad j = 2, \ldots, p+1,$$

and

$$J_m(z) = \int_{H_{m-1}}^{z} \frac{du}{\psi(u)}, \qquad z \geq H_{m-1},\ m = 1, \ldots, p+1.$$

Then (1.1) *has at least one solution on* $[0, T]$.

Remark. In fact (2.3) could be replaced by $f(t, u) \leq q(t)\psi(u)$ for almost all $t \in [0, T]$ and all $u \geq 0$ with $\psi : [0, \infty) \to (0, \infty)$, a Borel function, and $q \in L^1[0, T]$, and the result of Theorem 2.5 is again true.

Proof. Let y be a solution to

$$\begin{aligned} y' &= \lambda f_1(t, y), \qquad 0 < t < t_1, \\ y(0) &= a \end{aligned} \tag{$2.17)_\lambda$}$$

for $\lambda \in (0, 1)$ with

$$f_1(t, y) = \begin{cases} f(t, y), & y \geq s_0, \\ f(t, s_0), & y \leq s_0. \end{cases}$$

The same reasoning as in Theorem 2.3 implies that $y(t) \geq s_0$ for $t \in [0, t_1]$.

Now suppose $y(t) > \max\{0, a\} = H_0$ for some $t \in (0, t_1]$. Then there exists $\mu \in [0, t)$ with $y(s) > H_0$ for $s \in (\mu, t]$ and $y(\mu) = H_0$. As in Theorem 2.2 we have

$$s_0 \leq y(t) \leq J_1^{-1}\left(\int_0^{t_1} q(s)\, ds\right) = A_0 \qquad \text{for } t \in [0, t_1].$$

Now Theorem 2.1 implies that $(2.17)_1$ has at least one solution, $y \in AC[0, t_1]$ with $s_0 \leq y(t) \leq A_0$, $t \in [0, t_1]$. Pick **one** such solution; call it y_1, say. Since $s_0 \leq y_1(t) \leq A_0$, $t \in [0, t_1]$, then y_1 is a solution of

$$\begin{aligned} y' &= f(t, y), \qquad 0 < t < t_1, \\ y(0) &= a. \end{aligned}$$

Let y be a solution to

$$\begin{aligned} y' &= \lambda f_2(t, y), \qquad t_1 < t < t_2, \\ y(t_1^+) &= I_1(y_1(t_1^-)) \end{aligned} \tag{$2.18)_\lambda$}$$

for $\lambda \in (0, 1)$ with

$$f_2(t, y) = \begin{cases} f(t, y), & y \geq s_1 \\ f(t, s_1), & y \leq s_1. \end{cases}$$

Since $y_1(t_1^-) \in [s_0, A_0]$, then (2.14) implies that

$$s_1 \le I_1(y(t_1^-)) = y(t_1^+).$$

Also note that

$$y(t_1^+) \le \max_{[s_0, A_0]} I_1(x) \equiv K_1.$$

The same reasoning as in Theorem 2.3 implies that $y(t) \ge s_1$ for $t \in [t_1, t_2]$.

Now suppose $y(t) > \max\{0, K_1\} = H_1$ for some $t \in (t_1, t_2]$. Then there exists $\eta \in [t_1, t)$ with $y(s) > H_1$ for $s \in (\eta, t]$ and $y(\eta) = H_1$. Consequently,

$$s_1 \le y(t) \le J_2^{-1}\left(\int_{t_1}^{t_2} q(s)\,ds\right) = A_1 \qquad \text{for } t \in [t_1, t_2].$$

Now Theorem 2.1 implies that $(2.18)_1$ has at least one solution $y \in AC[t_1, t_2]$ with $s_1 \le y(t) \le A_1$, $t \in [t_1, t_2]$. Pick **one** such solution; call it y_2, say. Since $s_1 \le y_2(t) \le A_1$, $t \in [t_1, t_2]$ then y_2 is a solution of

$$\begin{aligned} y' &= f(t, y), \qquad t_1 < t < t_2, \\ y(t_1^+) &= I_1(y_1(t_1^-)). \end{aligned}$$

Continue this process and the result follows as in Theorem 2.3. ∎

Remark. We note here that it is possible to obtain an analogue version of Theorem 2.5 if we have $\{r_i\}_{i=0}^{p}$, $r_0 \ge a$, with $f(t, r_i) \le 0$ for $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, p$.

EXAMPLES. (i) Let $a_0, b_0 : [0, \infty) \to \mathbf{R}$ and $\kappa : (0, \infty) \to \mathbf{R}$ be continuous with $\int_0^c |\kappa(s)|\,ds < \infty$ for any $c > 0$. Also I_k maps $\mathbf{R}$ into $\mathbf{R}$ and is bounded on bounded sets, $k = 1, 2, \ldots$. Then the impulsive differential equation

$$\begin{aligned} y'(t) &= \kappa(t)[a_0(t)y^\alpha + b_0(t)], \qquad 0 < t < T, t \ne n; n = 1, 2, \ldots, \\ y(0) &= a, \\ y(n^+) &= I_n(y(n^-)), \qquad n = 1, 2, \ldots, \end{aligned} \tag{2.19}$$

with $0 \le \alpha \le 1$, has a solution for any $T > 0$.

To see this fix $T > 0$. Then there exists $p \in \{1, 2, \ldots\}$ with $p < T \le p + 1$. Let $L_0 = \sup_{[0,T]} |a_0(t)|$ and $L_1 = \sup_{[0,T]} |b_0(t)|$. Clearly (2.2) holds and (2.3) is satisfied with $\psi(x) = L_0 x^\alpha + L_1$ and $q(t) = |\kappa(t)|$. In addition (2.4) is true since $\int_d^\infty (du/\psi(u)) = \infty$ for any $d \ge 0$. The existence of a solution to (2.19) is now guaranteed by Theorem 2.2.

(ii) The impulsive differential equation

$$\begin{aligned} y' &= 1 + y^2, \qquad 0 < t < T, t \neq n\pi/4; n = 1, 2, \ldots, \\ y(0) &= 0, \\ y((n\pi/4)^+) &= y((n\pi/4)^-) - 1, \qquad n = 1, 2, \ldots, \end{aligned} \tag{2.20}$$

has a solution for any $T > 0$.

This can be seen by explicitly constructing the solution. However, the existence of a solution follows immediately from Theorem 2.5. Fix $T > 0$. Then there exists $p \in \{1, 2, \ldots\}$ with $p\pi/4 < T \leq (p + 1)\pi/4$. Let $q = 1, f(t, u) = \psi(u) = 1 + u^2, I_k(x) = x - 1, k = 1, \ldots, p$, and $s_i = -i$, $i = 0, \ldots, p$. Clearly (2.2), (2.3), and (2.15) are true, since $f(t, s_i) = 1 + s_i^2 > 0$. Also, for $k = 0, \ldots, p$ we have

$$\int_{t_k}^{t_{k+1}} q(s)\, ds = \frac{\pi}{4}, \qquad \int_{t_p}^{t_{p+1}} q(s)\, ds = T - \left(\frac{p\pi}{4}\right) \leq \frac{\pi}{4}.$$

We **claim** for $k = 0, \ldots, p$ that $H_k = 0$ and $A_k = 1$. If the claim is true then (2.16) holds, since for $k = 1, \ldots, p + 1$,

$$\int_{H_{k-1}}^{\infty} \frac{du}{\psi(u)} = \int_0^{\infty} \frac{du}{1 + u^2} = \frac{\pi}{2}.$$

Also (2.14) is true, since for $i = 0, \ldots, p - 1$ we have

$$\min_{[s_i, A_i]} I_{i+1}(x) = \min_{[-i,1]}(x - 1) = -(i + 1) = s_{i+1}.$$

Theorem 2.5 now guarantees a solution to (2.20).

It remains to prove the claim. Note that $H_0 = 0$, $J_1(z) = \arctan(z)$ so $A_0 = 1$. Then since $\max_{[s_0, A_0]} I_1(x) = \max_{[0,1]}(x - 1) = 0$, we have $H_1 = 0$, $J_2(z) = \arctan(z)$ and so $A_1 = 1$. Continue this process and our claim is established.

3. Impulsive Periodic Problems

We first give an existence principle which generalizes one given by Granas, Guenther, and Lee [4] for ordinary differential equations.

THEOREM 3.1. *Let $f: [0, T] \times \mathbf{R} \times [0, 1] \to \mathbf{R}$ be an L^1-Carathéodory function such that $f(\cdot, \cdot, 0) = 0$, $I_k: \mathbf{R} \to \mathbf{R}$ be continuous, $m_k \in \mathbf{R}$, $k = 1, \ldots, p$, and $a \in L^1[0, T]$, such that $m_1 \ldots m_p \exp\{\int_0^T a(s)\, ds\} \neq 1$.*

Assume there exists an open set $0 \in U \subset \Omega$ *such that for all* $\lambda \in [0, 1)$, *there is no solution* y *to*

$$\begin{aligned} y'(t) - a(t)y(t) &= f(t, y(t), \lambda), \qquad a.e.\ t \in [0, T], \\ y(t_k^+) &= \lambda I_k(y(t_k^-)) + (1 - \lambda)m_k y(t_k^-), \qquad k = 1, \ldots, p, \quad (3.1)_\lambda \\ y(0) &= y(T), \end{aligned}$$

such that $y \in \partial U$. *Then* $(3.1)_1$ *has at least one solution* $y \in \overline{U}$.

Proof. Define the linear operator $L : \Omega_p^1 \to L^1[0, T] \times \mathbf{R}^p$ by

$$L(y) = (y' - ay, y(t_1^+) - m_1 y(t_1^-), \ldots, y(t_p^+) - m_p y(t_p^-)).$$

It is easy to show that L is invertible since $m_1 \ldots m_p e^{\int_0^T a(s)\,ds} \neq 1$. Let $N : \Omega \times [0, 1] \to L^1[0, T] \times \mathbf{R}^p$ be defined by

$$N(y, \lambda) = (t \mapsto f(t, y(t), \lambda), \lambda(I_1(y(t_1^-)) - m_1 y(t_1^-)), \ldots, \lambda(I_p(y(t_p^-)) - m_p y(t_p^-))).$$

Since I_k are continuous and f is Carathéodory, N is continuous, and for every bounded set $B \subset \Omega$ there exist $h \in L^1[0, T]$, $M_k \geq 0$, $k = 1, \ldots, p$, such that

$$\begin{aligned} N(B \times [0, 1]) \subset \{(g, r_1, \ldots, r_p) : &|g(t)| \leq h(t) \\ \text{a.e. } t \in [0, T], r_k \in [-M_k, M_k], &k = 1, \ldots, p\}. \end{aligned}$$

Therefore, $j \circ L^{-1} \circ N : \Omega \times [0, 1] \to \Omega$ is continuous and completely continuous, where $j : \Omega_p^1 \to \Omega$ is the continuous inclusion. Moreover, y is a solution of $(3.1)_\lambda$ if and only if $y = j \circ L^{-1} \circ N(y, \lambda)$. The result follows by applying Theorem 1.1 with $p^* = 0$. ∎

Now we recall the definition of upper and lower solutions of the impulsive periodic problem (1.2) (see [1, 7, 9]).

A function $\alpha \in \Omega^1$ is called a *lower solution* of (1.2) if

$$\begin{aligned} \alpha'(t) &\leq f(t, \alpha(t)), \qquad \text{a.e. } t \in [0, T], \\ \alpha(t_k^+) &\leq I_k(\alpha(t_k^-)), \qquad k = 1, \ldots, p, \\ \alpha(0) &\leq \alpha(T). \end{aligned}$$

Similarly, we define an *upper solution* of (1.2) by reversing the inequalities.

In the literature, most of the results assume that I_k are nondecreasing and concern the existence of a solution y to (1.2) such that $\alpha \leq y \leq \beta$, where α and β are respectively lower and upper solutions of (1.2) (see [1, 7, 9]). Our first result establishes the existence of a solution when $\beta < 0 < \alpha$ and I_k are not necessarily monotone.

THEOREM 3.2. *Assume that*

$$\begin{aligned} &f:[0,T]\times\mathbf{R}\to\mathbf{R} \text{ is an } L^1\text{–Carathéodory function,}\\ &I_k:\mathbf{R}\to\mathbf{R} \text{ are continuous,} \qquad k=1,\dots,p, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\text{there exist } \alpha,\beta\in\Omega^1 \text{ respectively lower and upper solutions of (1.2)}\\ &\text{such that } \beta(t)<0<\alpha(t) \text{ for all } t\in[0,T]. \end{aligned} \tag{3.3}$$

Then (1.2) *possesses a solution* y *such that* $\beta(t)\le y(t)\le\alpha(t)$ *on* $[0,T]$.

Proof. Let $\varepsilon>0$ be such that $\beta(t)+\varepsilon<0<\alpha(t)-\varepsilon$ on $[0,T]$ and let $a:[0,T]\to(0,\infty)$ integrable such that $\alpha(t)(\beta(t)+\varepsilon)<\beta'(t)$, $a(t)(\alpha(t)-\varepsilon)>\alpha'(t)$, a.e. $t\in[0,T]$. Define $f_3:[0,T]\times\mathbf{R}\times[0,1]\to\mathbf{R}$ by

$$f_3(t,y,\lambda)=\begin{cases} \lambda(f(t,\beta(t))-a(t)y), & y\le\beta(t)+(1-\lambda)\varepsilon/2,\\ \lambda(\delta f(t,\beta(t))+(1-\delta)f(t,y)-a(t)y, & \\ \qquad y=\beta(t)+(1-\lambda)(1-\delta/2)\varepsilon,\ \delta\in(0,1),& \\ \lambda(\delta f(t,\alpha(t))+(1-\delta)f(t,y)-a(t)y), & \\ \qquad y=\alpha(t)-(1-\lambda)(1-\delta/2)\varepsilon,\ \delta\in(0,1),& \\ \lambda(f(t,\alpha(t))-a(t)y), & y\ge\alpha(t)-(1-\lambda)\varepsilon/2,\\ \lambda(f(t,y)-a(t)y), & \text{otherwise,} \end{cases}$$

and $m_k\ge\max\{1,\alpha(t_k^+)/\alpha(t_k^-),\beta(t_k^+)/\beta(t_k^-)\}$. Consider the family of problems

$$\begin{aligned} y'(t)-a(t)y(t)&=f_3(t,y(t),\lambda), \qquad \text{a.e. } t\in[0,T]\\ y(t_k^+)&=\lambda I_k(y(t_k^-))+(1-\lambda)m_k y(t_k^-), \qquad k=1,\dots,p\\ y(0)&=y(T). \end{aligned} \tag{3.4}_\lambda$$

We will prove the result by applying Theorem 3.1 to $(3.4)_\lambda$ with

$$U=\{y\in\Omega:\beta(t)<y(t)<\alpha(t) \text{ on } [0,T]\}.$$

Remark. The function f_3 is an L^1–Carathéodory function and $m_1\ldots m_p\exp\{\int_0^T a(s)\,ds\}\ne 1$.

Assume $y\in\partial U$ is a solution to $(3.4)_\lambda$ for some $\lambda\in[0,1)$. Without loss of generality assume there exists $\bar t\in[0,T]$ such that $y(\bar t)=\alpha(\bar t)$. If $\bar t\in[t_k^+,t_{k+1})$, for some $k=0,\dots,p$, where $t_0=0$, $t_{p+1}=T$, then there exists $t^*\in(\bar t,t_{k+1})$ such that $\alpha(t)\ge y(t)\ge\alpha(t)-(1-\lambda)\varepsilon/2$ for all $t\in(\bar t,t^*)$.

Then the definition of f_3, (3.3), and $a(t)\ (\alpha(t) - \varepsilon) > \alpha'(t)$, a.e. $t \in [0, T]$, yields

$$\alpha(t^*) - \alpha(\bar{t}) \geq y(t^*) - y(\bar{t}) = \int_{\bar{t}}^{t^*} [\lambda f(s, \alpha(s)) + (1 - \lambda)a(s)y(s)]\, ds$$

$$\geq \int_{\bar{t}}^{t^*} (\lambda\alpha'(s) + (1 - \lambda)[a(s)(\alpha(s) - \varepsilon) + \lambda a(s)\varepsilon/2])\, ds > \alpha(t^*) - \alpha(\bar{t}),$$

a contradiction.

If $\bar{t} = t_k^-$ for some $k \in \{1, \ldots, p + 1\}$, then we have

$$y(0) = y(T) = \alpha(T) \geq \alpha(0) \qquad \text{if } \bar{t} = t_{p+1} = T,$$

and if $k \in \{1, \ldots, p\}$ then the definition of m_k, together with the fact that α is a lower solution, gives

$$y(t_k^+) = \lambda I_k(\alpha(t_k^-)) + (1 - \lambda)m_k\alpha(t_k^-) \geq \lambda\alpha(t_k^+) + (1 - \lambda)\alpha(t_k^+) = \alpha(t_k^+).$$

We get a contradiction by the previous argument. Consequently $(3.4)_1$ has a solution $y \in \overline{U}$; i.e., $\beta(t) \leq y(t) \leq \alpha(t)$ on $[0, T]$. Thus y is a solution of (1.2). ∎

COROLLARY 3.3. *Assume that* (3.2) *is satisfied and there exist* $r_k > 0$ *such that*

$$yf(t, y) \geq 0, \qquad \text{a.e. } t \in [0, T], |y| = r_k.$$

Here $r_k \leq \min\{I_k(r_{k-1}), -I_k(-r_{k-1})\}$, $k = 0, \ldots, p$, *and* $r_0 \leq r_p$. *Then* (1.2) *has a solution.*

The following theorem generalizes some results in [1, 7], since there is no monotonicity condition on I_k and f.

THEOREM 3.4. *Assume* (3.2) *and*

$$\begin{aligned} &\text{there exist } \alpha, \beta \in \Omega^1 \text{ respectively lower and upper solutions} \\ &\text{of (1.2) such that } \alpha(t) \leq \beta(t) \text{ for all } t \in [0, T] \text{ and} \\ &\beta(t_k^+) \geq \max_{[b_k, c_k]} I_k(y) \geq \min_{[b_k, c_k]} I_k(y) \geq \alpha(t_k^+),\ k = 1, \ldots, p. \\ &\text{Here } [b_k, c_k] = [\alpha(t_k^-), \beta(t_k^-)] \end{aligned} \tag{3.5}$$

are satisfied. Then (1.2) *has a solution such that* $\alpha(t) \leq y(t) \leq \beta(t)$ *on* $[0, T]$.