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First Order Impulsive Initial and Periodic Problems with Variable Moments

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1. INTRODUCTION

In this paper we will consider the following system of differential equations with impulses at variable times

$$y'(t) = f(t, y(t)) \quad \text{a.e. } t \in [0, T], \tag{1.1}$$

$$y(t^+) = I_i(y(t)) \quad \text{if } t = \tau_i(y(t)), \qquad i = 1, \dots, k \tag{1.2}$$

with the periodic or the initial value conditions

$$y(0) = y(T), \tag{1.3}$$

$$y(0) = y_0, \tag{1.4}$$

where $f\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$.

Most papers in the literature on initial and periodic impulsive problems concern the case of fixed moments (i.e., when τ_i is constant for each i). However, in the last five years a number of papers [7, 9, 10] have appeared

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on initial value impulsive problems with variable times. Unfortunately, the periodic problem with variable moments has received very little attention. A very interesting result on this problem was obtained by Bajo and Liz [3]. In this paper, the authors considered the problem (1.1)–(1.3) in the scalar case with one barrier ($n = 1$ and $k = 1$). They imposed monotonicity conditions on I_1 and τ_1, and strong assumptions on f under which, for every $y_0 \in \mathbb{R}$, the initial value problem (1.1), (1.2), (1.4) has a unique solution $y(\cdot, y_0)$, and the Poincaré operator

$$P: \mathbb{R} \to \mathbb{R} \text{ defined by } P(y_0) = y(T, y_0)$$

is continuous. From the existence of $\alpha \leq \beta$, respectively lower and upper solutions of (1.1)–(1.3), they deduced that P maps $[\alpha(0), \beta(0)]$ into itself. The existence of a solution followed from the Brouwer fixed point Theorem.

In this paper many of the assumptions of Bajo and Liz are generalized, weakened, or removed. In particular, we make no monotonicity assumptions. Moreover, the function f is continuous. Therefore, we can not expect to get uniqueness of the solution of the initial value problem (1.1), (1.2), (1.4). In other words, the Poincaré operator P is multivalued.

Cellina [4] was the first to obtain the existence of a solution of periodic problems for differential inclusions $[x' \in F(t, x)]$ via the Poincaré operator, and hence to work with a multivalued Poincaré operator. His proof relies on the existence of a sequence of single-valued Lipschitz functions $\{f_n\}$ such that the graph of f_n becomes closer and closer to the graph of F. It seems likely that our results could be obtained via this approach. However, we choose a different approach which we describe below.

Our main theorem gives the existence of a solution to the periodic problem (1.1)–(1.3), and relies on two important results: (1) a generalization of Aronszajn's result [1] on the topological structure of solution sets of initial value problems for systems of first order differential equations, see [5, 13]; and (2) a fixed point theorem for compositions of acyclic maps due to Powers [11] based on an idea of Eilenberg and Montgomery [6]. Those two results were used by Plaskacz [12] to obtain existence results for periodic problems for differential inclusions.

Let us mention that a general existence theorem for the initial value problem with impulses (1.1), (1.2), (1.4) is presented as well.

2. PRELIMINARIES

We recall some definitions and results which will be used throughout this paper. We denote $\Omega^1([0,1], \mathbb{R}^k) = \{u\colon [0,T] \to \mathbb{R}^k$: there exist $0 < t_1$

$< \cdots < t_m = T$ such that u is C^1 on $[0, t_1]$ and on $]t_{i-1}, t_i]$, $i = 2, \dots, m$, and $\lim_{h \to 0^+} u(t_i + h)$ exists for $i = 1, \dots, m-1\}$. The closed ball in $\mathbb{R}^n$ centered in x and of radius r is denoted by $B(x, r)$.

DEFINITION 1. A couple $(v, M) \in \Omega^1([0,1], \mathbb{R}^n) \times \Omega^1([0,1], [0, \infty[)$ is called a *solution-tube of* (1.1), (1.3) [resp. (1.1), (1.4)] if

(i) $\langle x - v(t), f(t,x) - v'(t) \rangle \leq M(t)M'(t)$ for $\|x - v(t)\| = M(t)$, a.e. $t \in [0, T]$;

(ii) $f(t, v(t)) = v'(t)$ a.e. on $\{t: M(t) = 0\}$;

(iii) $B(v(t), M(t)) \subset B(v(t^+), M(t^+))$ for all $t \in [0, T]$;

(iv) $\|v(0) - v(T)\| \leq M(0) - M(T)$ [resp. $\|v(0) - y_0\| \leq M(0)$].

DEFINITION 2. Let X, Y be two metric spaces, a multivalued mapping $\phi: X \to Y$ is *upper semi-continuous* if $\{x: \phi(x) \cap K \neq \emptyset\}$ is closed for every closed subset K of Y. It is *acyclic* if it is upper semi-continuous with compact values and for every $x \in X$, $H^m(\phi(x)) = \delta_{0m}\mathbb{Z}$, where $\{H^m\}_{m \in \mathbb{N}}$ denote the Čech cohomology functor with integer coefficients.

DEFINITION 3. A compact metric space X is an R_δ*-set* if it is the intersection of a decreasing sequence of compact contractible metric spaces.

LEMMA. *An upper semi-continuous multivalued mapping with compact R_δ-values is acyclic.*

3. MAIN THEOREM

Throughout this section we will assume that the following conditions hold:

(H1) The function $f: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $I_i: \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $\tau_i: \mathbb{R}^n \to \mathbb{R}$ is of class C^1 for $i = 1, \dots, k$. Moreover,

$$0 < \tau_1(x) < \cdots < \tau_k(x) < \tau_{k+1}(x) \equiv T \qquad \text{for all } x \in \mathbb{R}^n.$$

(H2) There exists $(v, M) \in \Omega^1([0,1], \mathbb{R}^n) \times \Omega^1([0,1], [0, \infty[)$ a solution-tube of (1.1), (1.3) such that for $i = 1, \dots, k$,

$$\| I_i(x) - v(t^+) \| \leq M(t^+)$$

for any (t, x) such that $t = \tau_i(x)$ and $\|x - v(t)\| \leq M(t)$.

(H3) For $i = 1, \ldots, k$, for all $t \in [0, T]$, and for all x such that $\|x - v(t)\| \leq M(t)$, we have

$$\langle \tau_i'(x), f(t, x) \rangle \neq 1.$$

(H4) For $i = 1, \ldots, k$ and for every $x \in \mathbb{R}^n$,

$$\tau_i(I_i(x)) \leq \tau_i(x) < \tau_{i+1}(I_i(x)).$$

MAIN THEOREM. *Under the assumptions* (H1)–(H4), *the impulsive periodic problem* (1.1)–(1.3) *has a solution.*

In the proof we will use the following notations:

$$X = \{(t, x) \in [0, T] \times \mathbb{R}^n : \|x - v(t^+)\| \leq M(t^+)\};$$
$$Y = \{(t, u) \in [0, T] \times C([0, T], \mathbb{R}^n) : \|u(s) - v(s)\| \leq M(s) \text{ for all } s \geq t\};$$
$$Z_i = \{(t, u) \in Y : t < \tau_i(u(t)) \text{ and } u'(s) = f(s, u(s)) \text{ for all } s \geq t\}, \quad i = 1, \ldots, k.$$

We define the following maps

$\mathscr{I}: B(v(0), M(0)) \to X$ given by $x \mapsto (0, x)$;
$\mathscr{E}: Y \to B(v(T), M(T))$ given by $(t, u) \mapsto u(T)$;
$\mathscr{J}_i: Y \to [0, T] \times \mathbb{R}^n$ given by $(t, u) \mapsto (t, I_i(u(t)))$, $\quad i = 1, \ldots, k$;
$\mathscr{T}_i: Z_i \to Y$ given by $(t, u) \mapsto (t_i(t, u), u)$, $\quad i = 1, \ldots, k$;
$\mathscr{S}: X \to C([0, T], \mathbb{R}^n)$ given by $(t, x) \mapsto (t, S(t, x))$,

where

$$t_i(t, u) = \inf\{s > t : s = \tau_i(u(s))\};$$
$$S(t, x) = \{u \in C([0, T], \mathbb{R}^n) : u'(s) = f^*(s, u(s)) \text{ a.e. on } [0, T] \text{ and } u(t) = x\}$$

with

$$f^*(s,y) = \begin{cases} f(s,y), & \\ & \text{if } \|y - v(s)\| \le M(s), \\ f\left(s, \frac{M(s)}{\|y - v(s)\|}(y - v(s)) + v(s)\right), & \\ & \text{if } \|y - v(s)\| > M(s). \end{cases}$$

Obviously, $\mathscr{I}$, $\mathscr{E}$, and $\mathscr{J}_i$, $i = 1, \dots, k$, are continuous. Moreover, $\mathscr{S}$ is upper semi-continuous with R_δ-values and hence $\mathscr{S}$ is an acyclic map, see [5 or 13].

PROPOSITION 1. *If* (H1) *and* (H2) *are satisfied, then* $\mathscr{S}(X) \subset Y$.

Proof. Let $0 = s_0 < s_1 < s_2 < \cdots < s_m < s_{m+1} = T$ such that v and M are continuous on $[0,T] \setminus \{s_1, \dots, s_m\}$. Let $(t,x) \in X$ and $(t,u) \in S(t,x)$; that is $u \in \Omega^1([0,1], \mathbb{R}^n)$ is such that

$$u(t) = x \quad \text{and} \quad u'(s) = f^*(t, u(s)) \text{ on } [0,T] \setminus \{s_1, \dots, s_m\}.$$

If $s_{i-1} \le t < s_i$ for some $i \in \{1, \dots, m+1\}$, then $\|u(s) - v(s)\| \le M(s)$ on $[t, s_i]$. Indeed, if this is not the case there exist $t \le \tau_1 < \tau_2 \le s_i$ such that $\|u(\tau_1) - v(\tau_1)\| = M(\tau_1)$ and $\|u(s) - v(s)\| > M(s)$ for $s \in]\tau_1, \tau_2]$. So

$$\begin{aligned} 0 &< \|u(\tau_2) - v(\tau_2)\| - M(\tau_2) \\ &= \int_{\tau_1}^{\tau_2} \frac{\langle u(s) - v(s), f^*(s, u(s)) - v'(s)\rangle}{\|u(s) - v(s)\|} - M'(s)\, ds \\ &= \int_{\tau_1}^{\tau_2} \frac{\langle u(s) - v(s), f(s, z(s)) - v'(s)\rangle}{\|u(s) - v(s)\|} - M'(s)\, ds \\ &= \int_{\tau_1}^{\tau_2} \frac{1}{M(s)}\langle z(s) - v(s), f(s, z(s)) - v'(s)\rangle - M'(s)\, ds \le 0, \end{aligned}$$

where $z(s) = v(s) + M(s)(u(s) - v(s))/\|u(s) - v(s)\|$, which is a contradiction. So $\|u(s) - v(s)\| \le M(s)$ on $[t, s_i]$. If $i = m+1$, we have the result. If not, since

$$B(v(s_i), M(s_i)) \subset B(v(s_i^+), M(s_i^+))$$

the same argument implies that $\|u(s) - v(s)\| \le M(s)$ on $[s_i, s_{i+1}]$. Thus this inequality holds on $[t, T]$. ∎

PROPOSITION 2. *Assume* (H1) *and* (H3) *hold, then for each* $i \in \{1, \dots, k\}$ *the function* $\mathcal{T}_i$ *is continuous.*

Proof. We need to show that $(t, u) \mapsto t_i(t, u)$ is continuous. Let $(\bar{t}, \bar{u}) \in Z_i$. Since $\tau_i(\bar{u}(\bar{t})) > \bar{t}$ and $\tau_i(\bar{u}(T)) < T$, it follows from the intermediate value theorem that $t_i(\bar{t}, \bar{u})$ is well defined. For simplicity, let us denote $\bar{t}_i = t_i(\bar{t}, \bar{u})$.

Let $\varepsilon > 0$. Take $0 < \gamma < \varepsilon$ such that

$$1 - \left\langle \tau_i'\left(\bar{u}\left(\bar{t}_i\right)\right), f\left(\bar{t}_i, \bar{u}\left(\bar{t}_i\right)\right)\right\rangle > \gamma \quad \text{and} \quad \bar{t}_i > \bar{t} + 2\gamma.$$

By continuity there exists $0 < \eta \leq \gamma$ such that for every $t \in [\bar{t}_i - \eta, \bar{t}_i + \eta] \cap [0, T]$, we have

$$1 - \left\langle \tau_i'(\bar{u}(t)), f(t, \bar{u}(t))\right\rangle > \gamma/2 \quad \text{and} \quad \left\|\bar{u}(t) - \bar{u}\left(\bar{t}_i\right)\right\| \leq 1.$$

Therefore,

$$\begin{aligned} \bar{t}_i + \eta - \tau_i\left(\bar{u}\left(\bar{t}_i + \eta\right)\right) &= \int_{\bar{t}_i}^{\bar{t}_i + \eta} 1 - \left\langle \tau_i'(\bar{u}(t)), f(t, \bar{u}(t))\right\rangle dt \\ &> \frac{\gamma\eta}{2}, \\ \bar{t}_i - \eta - \tau_i\left(\bar{u}\left(\bar{t}_i - \eta\right)\right) &= -\int_{\bar{t}_i - \eta}^{\bar{t}_i} 1 - \left\langle \tau_i'(\bar{u}(t)), f(t, \bar{u}(t))\right\rangle dt \\ &< -\frac{\gamma\eta}{2}. \end{aligned} \tag{3.1}$$

On the other hand, τ_i is uniformly continuous on $B(\bar{u}(\bar{t}_i), 1 + \gamma)$. So there exists $0 < \delta < \gamma$ such that for every $x, y \in B(\bar{u}(\bar{t}_i), 1 + \gamma)$ with $\|x - y\| < \delta$, we have $|\tau_i(x) - \tau_i(y)| < \gamma\eta/4$.

Therefore, if $\|u - \bar{u}\|_0 < \delta$, for $s = \bar{t}_i \pm \eta$, $\|u(s) - \bar{u}(s)\| < \delta$, and hence

$$\left|\tau_i(u(s)) - \tau_i(\bar{u}(s))\right| < \frac{\gamma\eta}{4}. \tag{3.2}$$

By combining (3.1) and (3.2) we get

$$\bar{t}_i - \eta - \tau_i\left(u\left(\bar{t}_i - \eta\right)\right) < -\frac{\gamma\eta}{4} < \frac{\gamma\eta}{4} < \bar{t}_i + \eta - \tau_i\left(u\left(\bar{t}_i + \eta\right)\right).$$

The intermediate value theorem implies that there exists $s \in]\bar{t}_i - \eta, \bar{t}_i + \eta[$ such that $s = \tau_i(u(s))$. Moreover, if $(t, u) \in Z_i$ and $|t - \bar{t}| < \delta$, then

$$t < \bar{t} + \gamma < \bar{t}_i - \gamma < \bar{t}_i - \eta < s.$$

Thus, if there exists $\hat{s} \in]t, s[$ such that $\hat{s} = \tau_i(u(\hat{s}))$, then by Rolle's Theorem, there exists $\tilde{s} \in]\hat{s}, s[$ such that $\langle \tau_i'(u(\tilde{s})), f(\tilde{s}, u(\tilde{s})) \rangle = 1$, which is a contradiction. So $t_i(t, u) = s$.

Consequently, if $(t, u) \in Z_i$ with $\|u - \bar{u}\|_0 < \delta$ and $|t - \bar{t}| < \delta$, we have

$$\left| t_i(t, u) - t_i(\bar{t}, \bar{u}(t)) \right| < \eta \leq \varepsilon$$

and the proof is complete. ∎

Now we can prove our main result.

Proof of the Main Theorem. Consider the operator

$$\mathscr{P} = \mathscr{E} \circ \mathscr{S} \circ \mathscr{J}_k \circ \mathscr{T}_k \circ \cdots \circ \mathscr{S} \circ \mathscr{J}_1 \circ \mathscr{T}_1 \circ \mathscr{S} \circ \mathscr{I}.$$

First of all, we want to show that $\mathscr{P}$ is well defined. Take $y_0 \in B(v(0), M(0))$, and take $y_1 \in S(0, y_0)$. By Proposition 1, $y_1 \in Y$. Since $\tau_1(y_1(0)) = \tau_1(y_0) > 0$, $y_1 \in Z_1$, and hence

$$\mathscr{S} \circ \mathscr{I}(B(v(0), M(0))) \subset Z_1.$$

Denote $\hat{t}_1 = t_1(0, y_1)$. By definition, $t < \tau_1(y_1(t))$ for every $t \in [0, \hat{t}_1[$. This and (H1) imply that $t < \tau_i(y_1(t))$ for every $t \in [0, \hat{t}_1]$ and every $i = 2, \ldots, k$. On the other hand, it follows from Assumption (H2) that $\|I_1(y_1(\hat{t}_1)) - v(\hat{t}_1^+)\| \leq M(\hat{t}_1^+)$. So

$$\mathscr{J}_1 \circ \mathscr{T}_1 \circ \mathscr{S} \circ \mathscr{I}(B(v(0), M(0))) \subset X.$$

Now take $y_2 \in S(\hat{t}_1, I_1(y_1(\hat{t}_1)))$, which is in Y by Proposition 1. By (H4) $\hat{t}_1 < \tau_2(I_1(y_1(\hat{t}_1))) = \tau_2(y_2(\hat{t}_1))$ and hence $y_2 \in Z_2$. So

$$\mathscr{S} \circ \mathscr{J}_1 \circ \mathscr{T}_1 \circ \mathscr{S} \circ \mathscr{I}(B(v(0), M(0))) \subset Z_2.$$

Denote $\hat{t}_2 = t_2(\hat{t}_1, y_2)$. The definition of t_2 and (H1) imply that

$$t < \tau_i(y_2(t)) \qquad \text{for every } t \in \left[\hat{t}_1, \hat{t}_2\right[, \qquad i = 2, \ldots, k.$$

Moreover,

$$t > \tau_1(y_2(t)) \qquad \text{for every } t \in \left]\hat{t}_1, \hat{t}_2\right].$$

Indeed, if this is false, the function $t \mapsto -t + \tau_1(y_2(t))$ attains a nonnegative maximum at $s \in]\hat{t}_1, T]$, since by (H4), $\tau_1(y_2(\hat{t}_1)) \leq \hat{t}_1$. Also $s \neq T$. Therefore, $\langle \tau_1'(y_2(s)), f(s, y_2(s)) \rangle - 1 = 0$, which contradicts (H3). Again (H2) implies that

$$\mathscr{J}_2 \circ \mathscr{T}_2 \circ \mathscr{S} \circ \mathscr{J}_1 \circ \mathscr{T}_1 \circ \mathscr{S} \circ \mathscr{I}(B(v(0), M(0))) \subset X.$$

In repeating this argument we get in starting with $y_0 \in B(v(0), M(0))$

$$(0, y_1), (\hat{t}_1, y_2), \ldots, (\hat{t}_k, y_{k+1}) \in [0, T] \times C([0, T], \mathbb{R}^n),$$

such that

$$\begin{aligned}
&y_i'(t) = f^*(t, y_i(t)) \quad \text{for a.e. } t \in [0, T], \\
&y_1(0) = y_0, \\
&y_{i+1}(\hat{t}_i) = I_i(y_i(\hat{t}_i)), \qquad i = 1, \ldots, k, \\
&\hat{t}_i = \tau_i(y_i(\hat{t}_i)), \qquad i = 1, \ldots, k, \\
&\| y_i(t) - v(t) \| \le M(t) \qquad \text{for } t \ge \hat{t}_{i-1}, i = 1, \ldots, k+1, \\
&t \ne \tau_j(y_i(t)) \quad \text{on }]\hat{t}_{i-1}, \hat{t}_i[, i = 1, \ldots, k+1, j = 1, \ldots, k;
\end{aligned} \tag{3.3}$$

here $\hat{t}_0 = 0$, $\hat{t}_{k+1} = T$. Also we deduce that

$$\mathcal{S} \circ \mathcal{J}_k \circ \mathcal{T}_k \circ \cdots \circ \mathcal{S} \circ \mathcal{J}_1 \circ \mathcal{T}_1 \circ \mathcal{S} \circ \mathcal{I}(B(v(0), M(0))) \subset Y.$$

Consequently the Poincaré operator

$$\mathcal{P}: B(v(0), M(0)) \to B(v(T), M(T)) \subset B(v(0), M(0))$$

is well defined and it is a composition of acyclic maps. Therefore, $\mathcal{P}$ has a fixed point $y_0 \in B(v(0), M(0))$, see [11, Theorem 5.6]. So there exists

$$(\hat{t}_k, y_{k+1}) \in \mathcal{S} \circ \mathcal{J}_k \circ \mathcal{T}_k \circ \cdots \circ \mathcal{S} \circ \mathcal{J}_1 \circ \mathcal{T}_1 \circ \mathcal{S} \circ \mathcal{I}(y_0)$$

with $y_0 = \mathcal{E}(\hat{t}_k, y_{k+1})$. As a result there exists

$$(\hat{t}_{k-1}, y_k) \in \mathcal{S} \circ \mathcal{J}_{k-1} \circ \mathcal{T}_{k-1} \circ \cdots \circ \mathcal{S} \circ \mathcal{J}_1 \circ \mathcal{T}_1 \circ \mathcal{S} \circ \mathcal{I}(y_0)$$

with $(\hat{t}_k, y_{k+1}) = \mathcal{S} \circ \mathcal{J}_k \circ \mathcal{T}_k(\hat{t}_{k-1}, y_k)$. Continue this process to obtain for $i = 1, \ldots, k-1$,

$$(\hat{t}_i, y_{i+1}) \in \mathcal{S} \circ \mathcal{J}_i \circ \mathcal{T}_i \circ \cdots \circ \mathcal{S} \circ \mathcal{J}_1 \circ \mathcal{T}_1 \circ \mathcal{S} \circ \mathcal{I}(y_0)$$

with $(\hat{t}_{i+1}, y_{i+2}) = \mathcal{S} \circ \mathcal{J}_{i+1} \circ \mathcal{T}_{i+2}(\hat{t}_i, y_{i+1})$. Finally, there exists

$$(0, y_1) \in \mathcal{S} \circ \mathcal{I}(y_0) \quad \text{with } (\hat{t}_1, y_2) = \mathcal{S} \circ \mathcal{J}_1 \circ \mathcal{T}_1(0, y_1).$$

It is clear that $(0, y_1), (\hat{t}_1, y_2), \ldots, (\hat{t}_k, y_{k+1})$ satisfies (3.3). Therefore, the function $y \in \Omega^1([0,1], \mathbb{R}^n)$ defined by

$$y = \begin{cases} y_1, & \text{on } [0, \hat{t}_1], \\ y_2, & \text{on }]\hat{t}_1, \hat{t}_2], \\ \vdots & \\ y_{k+1}, & \text{on }]\hat{t}_k, T] \end{cases} \tag{3.4}$$

is a solution of the impulsive problem (1.1)–(1.3) with $y(0) = y(T) = y_0$. ∎

Obviously from the previous proof we can deduce an existence result for the initial value problem with impulses. For that we make the following assumption.

(H2)* There exist $(v, M) \in \Omega^1([0,1], \mathbb{R}^n) \times \Omega^1([0,1], [0, \infty[)$ a solution-tube of (1.1), (1.4) such that for $i = 1, \ldots, k$,

$$\| I_i(x) - v(t^+) \| \leq M(t^+)$$

for any (t, x) such that $t = \tau_i(x)$ and $\|x - v(t)\| \leq M(t)$.

THEOREM. *Assume* (H1), (H2)*, (H3), *and* (H4), *then the problem* (1.1), (1.2), (1.4) *has a solution.*

Proof. Let $(0, y_1), (\hat{t}_1, y_2), \ldots, (\hat{t}_k, y_{k+1})$ be as in the proof of the previous theorem and satisfying (3.3). Then the function y defined by (3.4) is a solution of (1.1), (1.2), (1.4) satisfying $\|y(t) - v(t)\| \leq M(t)$ for $t \in [0, T]$. ∎

Remarks. (i) In the previous theorem, we can rewrite the proof without mentioning any operators. Moreover, Assumption (H4) can be replaced by (H4)* for $i = 1, \ldots, k$, and for every $x \in \mathbb{R}^n$,

$$\tau_i(I_i(x)) \leq \tau_i(x).$$

In this case, the solution does not need to hit each surface at least once. Also the functions I_i do not need to be continuous. Notice that this generalizes a result of [10].

(ii) Suppose that for $i \in \{1, \ldots, k\}$, $\langle \tau_i'(y), I_i(x) - x \rangle \leq 0$ for all $x \in \mathbb{R}$ and all y lying on the segment joining $I_i(x)$ and x, see [9]. The mean value theorem implies that $\tau_i(I_i(x)) \leq \tau_i(x)$. This is the case, for example, if $n = 1$, τ_i is increasing, and $I_i(x) \leq x$ as in [3].

(iii) In (H4), it is enough to assume that the inequalities are satisfied only for $x \in \tau_i^{-1}(t) \cap B(v(t), M(t))$ for some $t \in [0, T]$.

(iv) Assumption (H1) can be weakened if we ask $0 < \tau_1(x) < \cdots < \tau_k(x) < T$ only for $x \in B(v(0), c)$ where $c = \max\{\|v(t) - v(0)\| + M(t): t \in [0, T]\}$.

(v) If $n = 1$, then (v, M) is a solution-tube if and only if $\alpha = v - M$ is a lower solution and $\beta = v + M$ is an upper solution.

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