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IMPULSIVE DIFFERENTIAL EQUATIONS WITH VARIABLE TIMES

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1. INTRODUCTION

Let k be a positive integer and $T \in (0, \infty]$. In this paper we establish existence results for the impulsive differential equation (IDE)

$$\begin{cases} y' = f(t, y) & \text{for a.e. } t \in [0, T),\ t \neq \tau_i(y(t)), \\ y(t^+) = I_i(y(t^-)) & \text{if } t = \tau_i(y(t)),\ i = 1, \dots, k, \\ y(0) = y_0. \end{cases} \tag{1.1}$$

Here $f: [0, T) \times \mathbf{R}^n \to \mathbf{R}^n$ is a $L^1_{\text{loc}}[0, T)$-Carathéodory function, I_i maps $\mathbf{R}^n$ into $\mathbf{R}^n$ and $\tau_i: \mathbf{R}^n \to \mathbf{R}$ for $i \dots, 1, \dots, k$.

We let $\Omega_a = \{u: [a, T) \to \mathbf{R}^n : u$ is continuous except for a countable number of points $t_i \in (a, T)$, $u(t_i^+) = \lim_{h \to 0^+} u(t_i + h)$ and $u(t_i^-)$ exist and $u(t_i) = u(t_i^-)\}$ and $\Omega_a^1 = \{u \in \Omega_a : u$ is differentiable almost everywhere on (a, T) and $u' \in L^1_{\text{loc}}[a, T)\}$; here $0 \leq a < T$. By a solution to (1.1) we mean a function $y \in \Omega_0^1$ with $y(t^+) = I_i(y(t^-))$ if $t = \tau_i(y(t))$, $i = 1, \dots, k$, $y(0) = y_0$ and y satisfying the differential equation $y' = f(t, y)$ for a.e. $t \in [0, T)$, $t \neq \tau_i(y(t))$, $i = 1, \dots, k$.

Most of the literature to date, see [1–4] and their references, examine impulsive differential equations with fixed moments, i.e. when τ_i is a constant for each i and $\tau_{i+1} > \tau_i$ for $i = 1, \dots, k$. Only very recently [5, 6] have attempts been made to establish existence results for the more general problem (1.1), i.e. for impulsive differential equations with variable times. In [6] Lakshmikantham *et al.* establish some interesting results for the (IDE)

$$\begin{cases} y' = f(t, y),\ t \in [0, T_1],\ t \neq \tau(y(t)) \text{ and } T_1 \in (0, \infty), \\ y(t^+) = y(t^-) + I(y(t^-)),\ t = \tau(y(t)), \\ y(0) = y_0, \end{cases} \tag{1.2}$$

using the idea of upper and lower solutions. In particular they show that if

(i) α, β are lower and upper solutions of (1.2) such that $\alpha(t) \leq \beta(t)$ on $[0, T_1]$;

(ii) $\beta(t)$ hits the surface $S: t = \tau(x)$ only once at t_0, $t_0 \in (0, T_1]$ and $\beta(t_0) < \beta(t_0^+)$;

(iii) $f \in C([0, T_1] \times \mathbf{R}, \mathbf{R})$, $I: \mathbf{R} \to \mathbf{R}$, $\tau \in C^1(\mathbf{R}, (0, \infty))$ and τ is increasing for $\alpha(t) \leq x \leq \beta(t)$, $t \in [0, T_1]$;

(iv) $\tau_x(x + sI(x))I(x) < 0$, $0 \leq s \leq 1$, $t = \tau(x)$, $\alpha(t) \leq x \leq \beta(t)$, $t \in [0, T_1]$;

(v) $\tau_x(x)f(t,x) < 1$, $t = \tau(x)$, $\alpha(t) \le x \le \beta(t)$, $t \in [0, T_1]$; and

(vi) for any (t, x) such that $t = \tau(x)$, $\alpha(t) \le x \le \beta(t)$ implies $\alpha(t) \le x^+ \le \beta(t)$ for $t \in [0, T_1]$ are satisfied, then (1.2) has a solution y with $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0, T_1]$.

The ideas in [6] together with "continuation of solution" ideas are used by Kaul *et al.* [5] to establish other existence type results to IDEs with variable times when there is one or more surfaces (barriers) $t = \tau_i(x)$. One of the features of [5, 6] is that the conditions put on I_i and τ_i guarantee that the solution to the IDE meets each barrier at most once.

In this paper two types of existence results are established for (1.1). Our first result uses the notion of upper and lower solutions. Conditions are put on f, I_i and τ_i to guarantee that (1.1) has a solution $y \in \Omega_0^1$ with $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0, T)$. It is interesting to note that our solution y may hit each barrier $t = \tau_i(x)$ more than once, also we have no monotonicity assumption on the barrier. Our second result establishes existence of a solution to (1.1) when our nonlinearity f satisfies a growth condition of Wintner type [7, 8].

It is worthwhile to mention that our results contain as a particular case the problem with fixed moments; this is not the case in [5, 6].

2. PRELIMINARIES

In this section we establish two existence results for the initial value problem

$$\begin{cases} y' = f(t, y) & \text{a.e. on } [a, T), \\ y(a) = a_0, \end{cases} \tag{2.1}$$

which will be needed when we examine the IDE (1.1); here $0 \le a < T \le \infty$. We will establish the existence of a solution to (2.1) in $AC_{\mathrm{loc}}([a, T), \mathbf{R}^n)$. Recall $AC_{\mathrm{loc}}([a, T), \mathbf{R}^n)$ is the set of functions $u \in C([a, T), \mathbf{R}^n)$ which are absolutely continuous on every compact subset of $[a, T)$. Also recall if $u \in C([a, T), \mathbf{R}^n)$ then for every $m \in \{1, 2, \ldots\} = N^+$, we define the seminorm $\rho_m(u)$ by

$$\rho_m(u) = \sup_{[a, t_m]} |u(t)|$$

where $t_m \uparrow T$. Notice $C([a, T), \mathbf{R}^n)$ is a locally convex linear topological space. The topology on $C([a, T), \mathbf{R}^n)$, induced by $\{\rho_m\}_{m \in N^+}$, is the topology of uniform convergence on every compact interval of $[a, T)$. From the Arzela–Ascoli theorem, a set $\Omega \subseteq C([a, T), \mathbf{R}^n)$ is compact iff Ω is uniformly bounded and equicontinuous on each compact interval of $[a, T)$.

We use the Schauder–Tychonoff theorem [9] to establish existence results of (2.1). For completeness we state the fixed point result.

THEOREM 2.1. Let K be a closed convex subset of a locally convex linear topological space E. Assume that $f: K \to K$ is continuous and that $F(K)$ is relatively compact in E. Then f has at least one fixed point in K.

We will assume throughout this section that $f: [a, T) \times \mathbf{R}^n \to \mathbf{R}^n$ is an $L^1_{\mathrm{loc}}[a, T)$-Carathéodory function; by this we mean

(a) the map $s \mapsto f(s, y)$ is measurable for all $y \in \mathbf{R}^n$,

(b) the map $y \mapsto f(s, y)$ is continuous for almost all $s \in [a, T)$,

(c) for each $r > 0$ there exists $\mu_r \in L^1_{\mathrm{loc}}[a, T)$ such that $|y| \le r$ implies $|f(s, y)| \le \mu_r(s)$ for almost all $s \in [a, T)$.

THEOREM 2.2. Let $f\colon [a,T) \times \mathbf{R}^n \to \mathbf{R}^n$ be a $L^1_{\text{loc}}[a,T)$-Carathéodory function. Assume the following conditions are satisfied

$$\begin{cases} |f(t,y)| \le q(t)\psi(|y|) \text{ for almost all } t \in [a,T) \text{ with} \\ \psi\colon [0,\infty) \to (0,\infty) \text{ a Borel measurable function} \\ \text{with } \dfrac{1}{\psi} \in L^1_{\text{loc}}[a,\infty) \text{ and } q \in L^1_{\text{loc}}[a,T); \end{cases} \tag{2.2}$$

$$\psi \text{ is nondecreasing} \tag{2.3}$$

and

$$\int_a^{t^*} q(s)\,\mathrm{d}s < \int_{|a_0|}^{\infty} \frac{\mathrm{d}u}{\psi(u)} \qquad \text{for any } t^* < T. \tag{2.4}$$

Then (2.1) has a solution $y \in AC_{\text{loc}}([a,T),\mathbf{R}^n)$.

Proof. A solution to (2.1) is a fixed point of the operator $N\colon C([a,T),\mathbf{R}^n) \to C([a,T),\mathbf{R}^n)$ defined by

$$Ny(t) = a_0 + \int_a^t f(s,y(s))\,\mathrm{d}s.$$

Let

$$K = \{y \in C([a,T),\mathbf{R}^n) : |y(t)| \le b(t),\ t \in [a,T)\}$$

where

$$b(t) = J^{-1}\left(\int_a^t q(x)\,\mathrm{d}x\right) \tag{2.5}$$

and

$$J(z) = \int_{|a_0|}^{z} \frac{\mathrm{d}x}{\psi(x)}.$$

Notice K is a closed, convex, bounded subset of $C([a,T),\mathbf{R}^n)$. We next claim that N maps K into K. To see this let $y \in K$. Notice for $t < T$ that

$$\begin{aligned} |Ny(t)| &\le |a_0| + \int_a^t q(s)\psi(|y(s)|)\,\mathrm{d}s \le |a_0| + \int_a^t q(s)\psi(b(s))\,\mathrm{d}s \\ &= |a_0| + \int_a^t b'(s)\,\mathrm{d}s = b(t) \end{aligned}$$

since

$$\int_{|a_0|}^{b(t)} \frac{\mathrm{d}x}{\psi(x)} = \int_a^t q(x)\,\mathrm{d}x.$$

Thus, $Ny \in K$ and so $N\colon K \to K$.

It remains to show that $N\colon C([a,T),\mathbf{R}^n) \to C([a,T),\mathbf{R}^n)$ is continuous and completely continuous. To see continuity let $y_n \to y$ in $C([a,T),\mathbf{R}^n)$, i.e. $\rho_m(y_n) \to \rho_m(y)$ for each m. The Lebesgue dominated convergence theorem together with the fact that $f\colon [a,T) \times \mathbf{R}^n \to \mathbf{R}^n$ is a

$L^1_{loc}[a, T)$-Carathéodory function implies $Ny_n \to Ny$ uniformly on $[a, t_m]$ for each t_m. Consequently, $N: C([a, T), \mathbf{R}^n) \to C([a, T), \mathbf{R}^n)$ is continuous. To show N is completely continuous, let $A \subseteq C([a, T), \mathbf{R}^n)$ be bounded. It is easy to see that $N(A)$ is uniformly bounded and equicontinuous on $[a, t_m]$ for each t_m. Hence, $N: C([a, T), \mathbf{R}^n) \to C([a, T), \mathbf{R}^n)$ is completely continuous.

The Schauder–Tychonoff theorem implies that N has a fixed point in K, i.e. (2.1) has a solution $y \in AC_{loc}([a, T), \mathbf{R}^n)$. ■

Remark. We note that assumption (2.3) can be removed in theorem 2.2; see [10].

Theorem 2.2 immediately yields the following corollary which is useful when we know that there exists upper and lower solutions to (2.1).

COROLLARY 2.3. Let $f: [a, T) \times \mathbf{R}^n \to \mathbf{R}^n$ be a $L^1_{loc}[a, T)$-Carathéodory function with the following condition satisfied

$$\begin{cases} \text{there exists } q \in L^1_{loc}[a, T) \text{ with } |f(t, y)| \le q(t) \\ \text{for almost all } t \in [a, T) \text{ and all } y \in \mathbf{R}^n. \end{cases} \tag{2.6}$$

Then (2.1) has a solution $y \in AC_{loc}([a, T), \mathbf{R}^n)$.

We now use corollary 2.3 to obtain an extra existence result for (2.1) when $f: [a, T) \times \mathbf{R} \to \mathbf{R}$ is a $L^1_{loc}[a, T)$-Carathéodory function. A function $\alpha \in \Omega^1_a$ is called a *lower solution* of (2.1) if

$$\begin{cases} \alpha'(t) \le f(t, \alpha(t)) & \text{a.e. } t \in [a, T) \\ \alpha(t^+) \le \alpha(t^-) & \text{for all } t \in [a, T), \\ \alpha(a) \le a_0. \end{cases}$$

Similarly we define an *upper solution* of (2.1) by reversing the inequalities.

THEOREM 2.4. Let $f: [a, T) \times \mathbf{R} \to \mathbf{R}$ be a $L^1_{loc}[a, T)$-Carathéodory function. Assume the following conditions are satisfied

$$\begin{cases} \text{there exists } \alpha, \beta \in \Omega^1_a, \text{ respectively, lower and upper solutions} \\ \text{of (2.1) with } \alpha(t) \le \beta(t) \text{ for } t \in [a, T); \end{cases} \tag{2.7}$$

and

$$\begin{cases} \text{there exists } s_i \in (a, T),\ i = 1, 2, \ldots,\ a < s_1 < s_2 < \cdots \text{ and } s_i \uparrow T \\ \text{with } \alpha(t^+) = \alpha(t^-),\ \beta(t^+) = \beta(t^-) \text{ for all } t \in [a, T)/\{s_1, s_2, \ldots\}. \end{cases} \tag{2.8}$$

Then (2.1) has a solution $y \in AC_{loc}([a, T), \mathbf{R}^n)$ with $\alpha(t) \le y(t) \le \beta(t)$ on $[a, T)$.

Proof. Consider the modified problem

$$\begin{cases} y' = f^*(t, y) & \text{a.e. on } [a, T), \\ y(a) = a_0, \end{cases} \tag{2.9}$$

where

$$f^*(t, y) = \begin{cases} f(t, \beta(t)), & y > \beta(t), \\ f(t, y), & \alpha(t) \le y \le \beta(t), \\ f(t, \alpha(t)), & y < \alpha(t). \end{cases}$$

Notice that f^* is a $L^1_{\mathrm{loc}}[a, T)$-Carathéodory function since α, β are continuous a.e. on $[0, T)$. Corollary 2.3 implies that (2.9) has a solution $y \in AC_{\mathrm{loc}}([a, T), \mathbf{R}^n)$. We now claim that $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [a, T)$.

First consider $t \in [a, s_1]$, where s_1 is as described in (2.8). We show $y(t) \le \beta(t)$ for $t \in [a, s_1]$. Assume this is false. Then since $y(a) = a_0 \le \beta(a)$ there exists $x_1 < x_2 \in [a, s_1]$ such that $y(x_1) \le \beta(x_1)$, $y(x_2) > \beta(x_2)$ and $y(t) \ge \beta(t)$ for $t \in (x_1, x_2)$. Consequently,

$$\beta(x_2) - \beta(x_1) < y(x_2) - y(x_1) = \int_{x_1}^{x_2} f(t, \beta(t))\, \mathrm{d}t \le \beta(x_2) - \beta(x_1),$$

a contradiction. Thus, $y(t) \le \beta(t)$ for $t \in [a, s_1]$ and a similar argument yields $y(t) \ge \alpha(t)$ for $t \in [a, s_1]$. Thus,

$$\alpha(t) \le y(t) \le \beta(t) \qquad \text{for } t \in [a, s_1]. \tag{2.10}$$

This together with the fact that α, β are, respectively, lower and upper solutions of (2.1) implies

$$\alpha(s_1^+) \le \alpha(s_1^-) \le y(s_1) \le \beta(s_1^-) \le \beta(s_1^+).$$

That is

$$\alpha(s_1^+) \le y(s_1) \le \beta(s_1^+). \tag{2.11}$$

Next consider $t \in [s_1, s_2]$. Essentially the same reasoning (since (2.11) is true) establishes that $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [s_1, s_2]$. Continue this process and so the theorem is established. ∎

3. IMPULSIVE DIFFERENTIAL EQUATIONS

In this section various existence results are established for the impulsive differential equation

$$\begin{cases} y' = f(t, y) & \text{for a.e. } t \in [0, T),\ t \ne \tau_i(y(t)), \\ y(t^+) = I_i(y(t)) & \text{if } t = \tau_i(y(t)),\ i = 1, \dots, k, \\ y(0) = y_0. \end{cases} \tag{3.1}$$

We first present a result for scalar IDEs based on the notion of upper and lower solutions. A function $\alpha \in \Omega_0^1$ is called a *lower solution* of (3.1) if

$$\begin{cases} \alpha'(t) \le f(t, \alpha(t)) & \text{for a.e. } t \in [0, T), \\ \alpha(t^+) \le \alpha(t^-) & \text{for all } t \in [0, T), \\ \alpha(0) \le y_0. \end{cases}$$

Similarly, we define an *upper solution* of (3.1) by reversing the inequalities.

THEOREM 3.1. Let $f: [0, T) \times \mathbf{R} \to \mathbf{R}$ be a $L^1_{\mathrm{loc}}[0, T)$-Carathéodory function. Also let $I_i: \mathbf{R} \to \mathbf{R}$ and $\tau_i: \mathbf{R} \to \mathbf{R}$ with $\tau_i \in C^1(\mathbf{R})$ for each $i = 1, \dots, k$. Assume the following conditions are satisfied:

$$\tau_i(y_0) \neq 0 \qquad \text{for } i = 1, \ldots, k; \tag{3.2}$$

$$\begin{cases} \text{there exists } \alpha, \beta \in \Omega_0^1, \text{ respectively, lower and upper solutions of (3.1)} \\ \text{with } \alpha(t) \leq \beta(t) \text{ for } t \in [0, T); \text{ also for any } (t, x) \text{ and } i \in \{1, 2, \ldots, k\} \\ \text{such that } t = \tau_i(x),\ \alpha(t) \leq x \leq \beta(t) \text{ we have } \alpha(t^+) \leq I_i(x) \leq \beta(t^+); \end{cases} \tag{3.3}$$

$$\begin{cases} \text{there exist } \varepsilon > 0,\ a_0 \geq 0,\ b_0 > 1 \text{ such that for all } t_1 < t_2\ (t_1, t_2 \in (0, T)) \\ \text{and for all } y \in C[t_1, t_2] \text{ such that } \tau_j(y(t_2)) = t_2 \text{ for some } j \text{ and} \\ y(t_1) \in \bigcup_{i=1}^k I_i(\tau_i^{-1}(t_1)),\ T(t_1, t_2, \tau_j, y) = \int_{t_1}^{t_2} f(t, y(t))\tau_j'(y(t))\,\mathrm{d}t + \tau_j(y(t_1)) \\ \text{satisfies one of the following conditions} \\ \text{(i) } T(t_1, t_2, \tau_j, y) < t_2, \\ \text{(ii) } T(t_1, t_2, \tau_j, y) > t_2, \\ \text{(iii) } T(t_1, t_2, \tau_j, y) \geq \varepsilon + t_1 - a_0(t_2 - t_1), \\ \text{(iv) } T(t_1, t_2, \tau_j, y) \leq -\varepsilon + t_1 + b_0(t_2 - t_1); \end{cases} \tag{3.4}$$

and

$$\begin{cases} \text{either } \tau_i(x) \neq \tau_j(x) \text{ for all } x \in \mathbf{R} \text{ and } i \neq j \text{ with } i, j \in \{1, \ldots, k\} \text{ or if} \\ \tau_i(x) = \tau_j(x) \text{ for some } x \in \mathbf{R} \text{ and } i \neq j \text{ then } I_i(x) = I_j(x). \end{cases} \tag{3.5}$$

Then (3.1) has a solution $y \in \Omega_0^1$ with $\alpha(t) \leq y(t) \leq \beta(t)$ for $t \in [0, T)$.

Remark. In (3.4), τ_i^{-1} denotes the inverse image.

Proof. Let $y_1 \in C[0, T)$ be a solution (guaranteed by theorem 2.4) of the IVP

$$\begin{cases} y' = f(t, y) \text{ a.e. on } [0, T), \\ y(0) = y_0, \end{cases} \tag{3.6}$$

with $\alpha(t) \leq y_1(t) \leq \beta(t)$ for $t \in [0, T)$. Define

$$r_{i,1}(t) = \tau_i(y_1(t)) - t \qquad \text{for } t \geq 0.$$

Now (3.2) implies $r_{i,1}(0) \neq 0$ for $i = 1, \ldots, k$. If $r_{i,1}(t) \neq 0$ on $[0, T)$ for $i = 1, \ldots, k$ (i.e. $t \neq \tau_i(y_1(t))$ on $[0, T)$ for $i = 1, \ldots, k$) then the result of the theorem follows, i.e. y_1 is a solution of (3.1). It remains to consider the case when $r_{i,1}(t) = 0$ for some t and some i. Now since $r_{i,1}(0) \neq 0$ for $i = 1 \ldots, k$ and $r_{i,1}$ are continuous, there exists $t_1 > 0$ such that $r_{i,1}(t_1) = 0$ for some $i_1 \in \{1, \ldots, k\}$ and $r_{j,1}(t) \neq 0$ for all $t \in [0, t_1)$ and $j = 1, \ldots, k$. Assumption (3.3) implies

$$\beta(t_1^+) \geq I_{i_1}(y_1(t_1)) \geq \alpha(t_1^+). \tag{3.7}$$

Now let $y_2 \in C[t_1, T)$ be a solution (guaranteed from (3.7) and theorem 2.4) of the IVP

$$\begin{cases} y' = f(t, y) \text{ a.e. on } [t_1, T), \\ y(t_1) = I_{i_1}(y_1(t_1)), \end{cases} \tag{3.8}$$

with $\alpha(t) \leq y_2(t) \leq \beta(t)$ for $t \in [t_1, T)$. Observe that $y_2(t_1) \in I_{i_1}(\tau_{i_1}^{-1}(t_1))$. Define

$$r_{i,2}(t) = \tau_i(y_2(t)) - t \qquad \text{for } t \geq t_1.$$

If $r_{i,2}(t) \neq 0$ on (t_1, T) for all $i = 1, \ldots, k$ then

$$y = \begin{cases} y_1, & \text{on } [0, t_1], \\ y_2, & \text{on } (t_1, T), \end{cases}$$

is a solution of (3.1). It remains to consider the case when there exists $t^* > t_1$ with $r_{i,2}(t^*) = 0$ for some i. Now since $y_2' = f(t, y_2)$ a.e. on $[t_1, T)$ we have

$$\begin{aligned} t^* &= [\tau_i(y_2(t^*)) - \tau_i(y_2(t_1))] + \tau_i(y_2(t_1)) \\ &= \int_{t_1}^{t^*} f(t, y_2(t))\tau_i'(y_2(t))\,\mathrm{d}t + \tau_i(y_2(t_1)) = T(t_1, t^*, \tau_i, y_2). \end{aligned}$$

By assumption (3.4), if (i) or (ii) is satisfied, we get a contradiction (and we are finished). If (iii) is satisfied then

$$t^* \geq \varepsilon + t_1 - a_0(t_2 - t_1) \qquad \text{and so } t^* - t_1 \geq \frac{\varepsilon}{1 + a_0}.$$

If (iv) is satisfied then $t^* - t_1 \geq \varepsilon/(b_0 - 1)$. Consequently,

$$t^* - t_1 \geq \min\left\{\frac{\varepsilon}{1 + a_0}, \frac{\varepsilon}{b_0 - 1}\right\} \equiv \varepsilon^*. \tag{3.9}$$

Now (3.9) together with the continuity of $r_{i,2}$ for $i = 1, \ldots, k$ implies that there exists $t_2 \geq t_1 + \varepsilon^*$ such that $r_{i,2}(t) \neq 0$ for all $t \in (t_1, t_2)$ and $i = 1, \ldots, k$ with $r_{i_2,2}(t_2) = 0$ for some i_2. Assumption (3.3) implies

$$\beta(t_2^+) \geq I_{i_2}(y_2(t_2)) \geq \alpha(t_2^+).$$

Let $y_3 \in C[t_2, T)$ be a solution of the IVP

$$\begin{cases} y' = f(t, y) & \text{a.e. on } [t_2, T), \\ y(t_2) = I_{i_2}(y_2(t_2)), \end{cases} \tag{3.10}$$

with $\alpha(t) \leq y_3(t) \leq \beta(t)$ for $t \in [t_2, T)$. Define

$$r_{i,3}(t) = \tau_i(y_3(t)) - t \qquad \text{for } t \geq t_2.$$

If $r_{i,3}(t) \neq 0$ on (t_2, T) for all $i = 1, \ldots, k$ then

$$y = \begin{cases} y_1, & \text{on } [0, t_1], \\ y_2, & \text{on } (t_1, t_2], \\ y_3, & \text{on } (t_2, T), \end{cases}$$

is a solution of (3.1). It remains to consider the case when there exists $\bar{t} > t_2$ with $r_{i,3}(\bar{t}) = 0$ for some i. Then, as above, $\bar{t} = T(t_2, \bar{t}, \tau_i, y_3)$. By assumption (3.4), if (i) or (ii) is satisfied, we get a contradiction. If (iii) or (iv) is satisfied then $\bar{t} - t_2 \geq \varepsilon^*$. Hence there exists $t_3 \geq t_2 + \varepsilon^*$ such that $r_{i,3}(t) \neq 0$ for all $t \in (t_2, t_3)$ and $i = 1, \ldots, k$ with $r_{i_3,3}(t_3) = 0$ for some i_3.

Continue this process and the result of the theorem follows. Observe that if $T < \infty$ the process will stop after a finite number of steps. ■

Remark. In assumption (3.4) we may replace "for all $y \in C[t_1, t_2]$" by "for all $y \in C[t_1, t_2]$ which satisfy $\alpha(t) \le y(t) \le \beta(t)$, $t \in [t_1, t_2]$". Also, we can replace ε by $\varepsilon(t_1)$ with ε an appropriate function.

Examples. (i) A problem with fixed moments satisfies (3.4), indeed suppose $\tau_i(x) = \overline{t_i}$ for each $x \in \mathbf{R}$ with $\overline{t_1} < \overline{t_2} < \cdots < \overline{t_k}$. We claim that (3.4)(iii) is satisfied.

To see this, since $\overline{t_1} < \overline{t_2} < \cdots < \overline{t_k}$ notice there exists $\varepsilon > 0$ with $\text{dist}(\overline{t_i}, \overline{t_j}) \ge \varepsilon$ for all $i, j \in \{1, \ldots, k\}$ and $i \ne j$. Notice also that $\tau_j'(x) = 0$ so

$$T(t_1, t_2, \tau_j, y) = \tau_j(y(t_1)) = \overline{t_j};$$

here $t_1, t_2 \in (0, T)$, $t_1 < t_2$ and $y \in C[t_1, t_2]$ with $\tau_j(y(t_2)) = t_2$ for some j and $y(t_1) \in \bigcup_{i=1}^k I_i(\tau_i^{-1}(t_1))$. Notice

$$\overline{t_j} = t_2 \quad \text{and} \quad t_1 \in \{\overline{t_1}, \ldots, \overline{t_k}\}.$$

This follows since

$$\tau_i^{-1}(t_1) = \begin{cases} \emptyset, & \text{if } t_1 \ne \overline{t_i}, \\ \mathbf{R}, & \text{if } t_1 = \overline{t_i}, \end{cases}$$

and so $y(t_1) \in \bigcup_{i=1}^k I_i(\tau_i^{-1}(t_1))$ if and only if $t_1 \in \{\overline{t_1} \ldots, \overline{t_k}\}$. Thus if $t_1 < t_2$ with $t_2 = \overline{t_j}$ and $t_1 = \overline{t_i}$ for some i then $j > i$ (since $t_1 < t_2$) and so

$$T(t_1, t_2, \tau_j, y) = \overline{t_j} \ge \overline{t_{j-1}} + \varepsilon \ge \overline{t_i} + \varepsilon = t_1 + \varepsilon.$$

Consequently (3.4)(iii) is satisfied.

(ii) Suppose for each $x \in \mathbf{R}$ we have $\tau_i(x) = \tau(x)$ and $I_i(x) = I(x)$ for all $i = 1, \ldots, k$. In addition assume $\tau : \mathbf{R} \to \mathbf{R}$ is increasing and $I(x) \le x$ for all $x \in \mathbf{R}$. Also suppose $f(t, x)\tau'(x) < 1$ for all $t \in (0, T)$ and $x \in \mathbf{R}$ (need only for all x with $\alpha(t) \le x \le \beta(t)$, $t \in (0, T)$) and $j \in \{1, \ldots, k\}$. We claim that (3.4)(i) is satisfied.

To see this notice

$$T(t_1, t_2, \tau, y) = \int_{t_1}^{t_2} f(t, y(t))\tau'(y(t))\,dt + \tau(y(t_1)) < t_2 - t_1 + \tau(y(t_1));$$

here $t_1, t_2 \in (0, T)$, $t_1 < t_2$ and $y \in C[t_1, t_2]$ with $\tau(y(t_2)) = t_2$ and $y(t_1) \in I(\tau^{-1}(t_1))$. Now since τ is increasing $y(t_1) \in I(\tau^{-1}(t_1))$ means $y(t_1) = I(\tau^{-1}(t_1))$. Also, since $I(x) \le x$ we have

$$\tau(y(t_1)) = \tau(I(\tau^{-1}(t_1))) \le \tau(\tau^{-1}(t_1)) = t_1.$$

Thus

$$T(t_1, t_2, \tau, y) < t_2 - t_1 + \tau(y(t_1)) \le t_2$$

and so (3.4)(i) is satisfied.

Remark. Notice we could replace "$I(x) \le x$ and $f(t, x)\tau'(x) < 1$" by "$I(x) < x$ and $f(t, x)\tau'(x) \le 1$" in the above example.

We now establish a result when our nonlinearity f satisfies a growth condition of Wintner type. First we prove the result for scalar equations and then we state the analogue for systems of equations (since the proof is essentially the same we will omit it).

THEOREM 3.2. Let $f: [0, T) \times \mathbf{R} \to \mathbf{R}$ be a $L^1_{\mathrm{loc}}[0, T)$-Carathéodory function. Also let $I_i: \mathbf{R} \to \mathbf{R}$ and $\tau_i: \mathbf{R} \to \mathbf{R}$ with $\tau_i \in C^1(\mathbf{R})$ for each $i = 1, \dots, k$. Suppose (3.2), (3.4) and (3.5) are satisfied. In addition assume that following conditions hold:

$$I_i: \mathbf{R} \to \mathbf{R} \text{ is bounded on bounded sets, } i = 1, \dots, k; \tag{3.11}$$

$$\begin{cases} |f(t, y)| \le q(t)\psi(|y|) \text{ for almost all } t \in [0, T) \text{ with} \\ \psi: [0, \infty) \to (0, \infty) \text{ a Borel measurable function} \\ \text{with } 1/\psi \in L^1_{\mathrm{loc}}[0, \infty) \text{ and } q \in L^1_{\mathrm{loc}}[0, T); \end{cases} \tag{3.12}$$

$$\psi \text{ is nondecreasing;} \tag{3.13}$$

and

$$\int^{\infty} \frac{\mathrm{d}x}{\psi(x)} = \infty. \tag{3.14}$$

Then (3.1) has a solution $y \in \Omega^1_0$.

Proof. Let $y_1 \in C[0, T)$ be a solution (guaranteed by theorem 2.2) of (3.6) with

$$|y_1(t)| \le J^{-1}\left(\int_0^t q(x)\,\mathrm{d}x\right) \qquad \text{for } t \in [0, T) \tag{3.15}$$

where

$$J(z) = \int_{|y_0|}^{z} \frac{\mathrm{d}x}{\psi(x)}.$$

Define

$$r_{i,1}(t) = \tau_i(y_1(t)) - t \qquad \text{for } t \ge 0.$$

If $r_{i,1}(t) \ne 0$ on $[0, T)$ for all $i = 1, \dots, k$, we are finished. It remains to consider the case when there exists $t_1 > 0$ such that $r_{i_1,1}(t_1) = 0$ for some i_1 with $r_{j,1}(t) \ne 0$ for all $t \in [0, t_1)$ and $j = 1, \dots, k$. Observe that

$$|I_{i_1}(y_1(t_1))| \le \max_{[-Q_0, Q_0]} |I_{i_1}(x)| \qquad \text{where } Q_0 = J^{-1}\left(\int_0^{t_1} q(x)\,\mathrm{d}x\right).$$

Let $y_2 \in C[t_1, T)$ be a solution (guaranteed by theorem 2.2) of (3.8). Define

$$r_{i,2}(t) = \tau_i(y_2(t)) - t \qquad \text{for } t \ge t_1.$$

Essentially the same reasoning as in theorem 3.1 (theorem 2.2 is used instead of theorem 2.4) guarantees the result. ■

Remark. Assumption (3.13) could be removed in theorem 3.2 if we use a result in [10].

We now state the analogue of theorem 3.2 for systems of first order IDEs.

THEOREM 3.3. Let $f: [0, T) \times \mathbf{R}^n \to \mathbf{R}^n$ be a $L^1_{\mathrm{loc}}[0, T)$-Carathéodory function. Also let $I_i: \mathbf{R}^n \to \mathbf{R}^n$ and $\tau_i: \mathbf{R}^n \to \mathbf{R}$ with $\tau_i \in C^1(\mathbf{R}^n)$ for each $i = 1, \dots, k$. Suppose (3.2), (3.5) and

(3.11)–(3.14), with $\mathbf{R}$ replaced by $\mathbf{R}^n$, are satisfied. In addition assume

$$\begin{cases} \text{there exist } \varepsilon > 0,\ a_0 \geq 0,\ b_0 > 1 \text{ such that for all } t_1 < t_2 (t_1, t_2 \in (0, T)) \\ \text{and for all } y \in C[t_1, t_2] \text{ such that } \tau_j(y(t_2)) = t_2 \text{ for} \\ \text{some } j \text{ and } y(t_1) \in \bigcup_{i=1}^{k} I_i(\tau_i^{-1}(t_1)), \\ T(t_1, t_2, \tau_j, y) = \int_{t_1}^{t_2} f(t, y(t)) \,.\, \nabla\tau_j(y(t))\, dt + \tau_j(y(t_1)) \\ \text{satisfies one of the following conditions} \\ \quad \text{(i) } T(t_1, t_2, \tau_j, y) < t_2, \\ \quad \text{(ii) } T(t_1, t_2, \tau_j, y) > t_2, \\ \quad \text{(iii) } T(t_1, t_2, \tau_j, y) \geq \varepsilon + t_1 - a_0(t_2 - t_1), \\ \quad \text{(iv) } T(t_1, t_2, \tau_j, y) \leq -\varepsilon + t_1 + b_0(t_2 - t_1). \end{cases} \tag{3.16}$$

Then (3.1) has a solution $y \in \Omega_0^1$.

Remark. The ∇ in (3.16) denotes the gradient and . denotes the usual inner product in $\mathbf{R}^n$.

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