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Some Remarks on the Interval of Existence of Solutions to First Order Initial Value Problems

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Abstract. In this paper we obtain "weak solutions" via Topological Transversality to nonlinear first order initial value problems of the form $y' = f(t;y)$, $t \in [0,T]$, with $y(0) = r$, where $f: [0,T] \times R^n \to R^n$ satisfies the Caratheodory Conditions. Our analysis, which is based on the notions of an essential map and on a priori bounds on solutions, is used to examine the dependence of the interval of existence for the above stated problem on its initial data and the nonlinearity in the differential equation.

1. Introduction.

Many physical situations give rise to initial value problems of the form

$$\begin{cases} y' = f(t;y), & t \in [0,T] \\ y(0) = r \end{cases} \tag{1.1}$$

where in fact $f: [0,T] \times R^n \to R^n$ is discontinuous. In this paper we study the case where f satisfies the *Caratheodory Conditions*, that is,

(a) for fixed $y = (y_1, \dots, y_n) \in R^n$, $f(-;y)$ is measurable on $[0,T]$;
(b) for all $t \in [0,T]$, $f(t;-)$ is continuous on R^n.

Let R^n denote Euclidean n space and $|v|$ the Euclidean norm. Also let $L^2((0,T); R^n)$ denote the Banach space of Lebesgue measurable functions g on $(0,T)$ which are square integrable with the norm given by $\|g\|_{L^2} = \left(\int_0^T |g(t)|^2\, dt\right)^{\frac{1}{2}}$. Finally, $L^\infty((0,T); R^n)$ denotes the space of essentially bounded measurable functions on $(0,T)$.

By a weak solution to (1.1) we mean a function y which is absolutely continuous on $[0,T]$ with $y' \in L^2((0,T); R^n)$, $y(0) = r$ and $y' = f(t;y)$ almost everywhere on $[0,T]$. We shall establish under reasonable physical assumptions on f, that (1.1) has bounded weak solutions. This paper in fact extends results of Lee and O'Regan [**12**] which deals with the case where f is continuous. Our analysis is based on the Topological Transversality theorem and known results on Sobolev spaces.

Preliminary Notation.

Let $H^1((0,T);R^n)$ denote the space of functions u which are absolutely continuous on $[0,T]$ and whose derivative u' (which exists almost everywhere) is an element of $L^2((0,T);R^n)\cdot H^1((0,T);R^n)$ with norm.

$$||u||_{H^1} = ||u||_{L^2} + ||u'||_{L^2}$$

is a Banach space. Also we let

$$H^1_B((0,T);R^n) = \{u \in H^1((0,T);R^n): u(0) = r\}.$$

Theorem 1.1 (Sobolev Imbedding Theorem). *$H^1((0,T);R^n)$ is compactly imbedded into $C([0,T];R^n)$ that is, the imbedding operator j: $H^1((0,T);R^n) \to C([0,T];R^n)$ is continuous and completely continuous.*

Theorem 1.2 (Change of variable, [16] or [17]). *Let g be an absolutely continuous function on $[a,b]$ with $g' \in L(a,b)$ and $g(a) < g(b)$. If f is a measurable bounded function on $[g(a),g(b)]$ then*

$$\int_{g(a)}^{g(b)} f(y)\, dy = \int_a^b f(g(t))g'(t)\, dt.$$

Proof: (Sketch) Let $W^{1,1}(a,b)$ denote the space of absolutely continuous functions whose derivative is an element of $L^1((a,b);R^n) \equiv L^1(a,b)$. Put $G(y) = \int_{g(a)}^{y} f(x)\, dx$, so $G' = f$ and $G \in W^{1,1}(g(a),g(b))$. Then there exists a sequence G_n such that $G_n|_{[g(a),g(b)]} \to G$ in $W^{1,1}$ and $||G_n||_{L^\infty} \le M_1, ||G'_n||_{L^\infty} \le M_2$. Also, $G_n \circ g \in W^{1,1}(a,b)$, $(G_n \circ g)' = (G'_n \circ g)g'$ so $G_n(g(b)) - G_n(g(a)) = \int_a^b G'_n(g(t))g'(t)\, dt$. Now since $G_n(g(b)) - G_n(g(a)) \to G(g(b)) - G(g(a)) = \int_{g(a)}^{g(b)} f(x)\, dx$, it remains to show

$$\int_a^b G'_n(g(t))g'(t)\, dt \to \int_a^b f(g(t))g'(t)\, dt.$$

One has $G'_n|_{[g(a),g(b)]} \to f$ in $L^1(g(a),g(b))$ so there exists a subsequence (G'_{n_k}) such that $G'_{n_k}(x) \to f(x)$ a.e. for $x \in [g(a),g(b)]$. Put $E = \{t \in [a,b]: g'(t)$ exists and $g'(t) \neq 0\}$, $Z_0 = \{x \in [g(a),g(b)]: G'_{n_k}(x) \to f(x)\}$ and $Z = [g(a),g(b)]|Z_0$. Let $B \subset E$ such that $g(B) \subset Z$, then $\mathrm{mes}[g(B)] = 0$ and a little argument yields $\mathrm{mes}[B] = 0$. Thus,

$$G'_{n_k}(g(t))g'(t) \to f(g(t))g'(t) \quad \text{a.e.} \quad t \in [a,b],$$

with

$$|G'_{n_k}(g(t))g'(t)| \le M_2 |g'(t)| \in L^1(a,b).$$

The result follows from the Lebesgue dominated convergence theorem. ∎

2. Weak solutions to first order initial value problems.

We begin by considering the problems

$$\begin{cases} y' = f(t;y), & t \in [0,T] \\ y(0) = 0 \end{cases} \tag{2.1}$$

where $f\colon [0,T]\times R^n \to R^n$ is a function which satisfies the following hypothesis:

$$f \text{ satisfied the Caratheodory Conditions.} \tag{2.2}$$

$$\begin{gathered}\text{There exists a function } \psi\colon [0,\infty) \to (0,\infty);\ \ \psi, \frac{1}{\psi} \in L^\infty_{\text{loc}}[0,\infty) \\ \text{such that } |f(t;y)| \le \psi(|y|) \ \text{ for } \ t \in [0,T].\end{gathered} \tag{2.3}$$

Proposition 2.1. *Suppose f satisfies (2.2) and (2.3). Then $F\colon C([0,T];R^n) \to L^2((0,T);R^n)$ given by $(Fu)(t) = f(t;u(t))$ is defined and continuous.*

Proof: Let $\epsilon > 0$ be given and suppose $u_0 \in C([0,T];R^n)$. Consider

$$G_{vm} = \left\{ t \in [0,T]\colon |v - u_0(t)| < \frac{1}{m} \Rightarrow |f(t;v) - f(t;u_0(t))| < \frac{\epsilon}{L} \right\}$$

where $v \in R^n$ and L is a predetermined constant which will be described below. G_{vm} is measurable since f satisfies (2.2). Let $E_{m\epsilon} = \cap_{v\in R^n} G_{vm}$. Now $E_{m\epsilon}$ is measurable and $E_{1\epsilon} \subset E_{2\epsilon} \subset \dots$. Also $\cup_{m=1}^\infty E_{m\epsilon} = (0,T)$ for if $t_0 \in (0,T)$, then there exists m such that $|v - u_0(t_0)| < \frac{1}{m}$ and hence $|f(t_0;v) - f(t_0;u_0(t))| < \frac{\epsilon}{L}$ since $f(t_0;\ \ )$ satisfies (2.2) and so $t_0 \in E_{m_0\epsilon}$. Hence, there exists $m_0 \in N$ such that $\text{mes}(E_{m_0\epsilon}) > T - \frac{\epsilon}{L}$. Finally, let $K = \text{ess sup}_{x\le 1+\|u_0\|_\infty} \psi(x) < \infty$ and $\eta < \frac{\epsilon^2}{8K^2}$ where $\|u_0\|_\infty = \sup_{t\in[0,T]} |u_0(t)|$. Put $0 < \delta < \frac{1}{m_0}$ and $\max\{\frac{\epsilon}{\eta}, (2T)^{\frac{1}{2}}\} < L$. Let $u \in C([0,T];R^n)$ such that $\|u - u_0\|_\infty < \delta$. We will now show that $\|Fu - Fu_0\|_{L^2} < \epsilon$. If $t_0 \in E_{m_0\epsilon}$, then $|f(t;u(t)) - f(t;u_0(t))| < \frac{\epsilon}{L}$ and so $\int_{E_{m_0\epsilon}} |f(t;u(t)) - f(t;u_0(t))|^2\, dt$ $< \frac{\epsilon^2 T}{L^2} < \frac{\epsilon^2 T}{2T} = \frac{\epsilon^2}{2}$. However, $\text{mes}(E^C_{m_0\epsilon}) < \frac{\epsilon}{L} < \frac{\epsilon\eta}{\epsilon} = \eta$ and so

$$\begin{aligned}\int_{E^C_{m_0\epsilon}} |f(t;u(t)) - f(t;u_0(t))|^2\, dt &\le 2\int_{E^C_{m_0\epsilon}} \{[\psi(|u(t)|)]^2 + [\psi(|u_0(t)|)]^2\}\, dt \\ &\le 4K^2\eta < \frac{4K^2\epsilon^2}{8K^2} = \frac{\epsilon^2}{2}.\end{aligned}$$

Hence $\|Fu - Fu_0\|_{L^2} < \epsilon$, so $F\colon C([0,T];R^n) \to L^2((0,T);R^n)$ is continuous. ∎

The Sobolev Imbedding Theorem together with Proposition 2.1 are now used to extend Theorem 2.1 of [**8**] for the new class of problems (2.1).

Theorem 2.2. *Let $f: [0,T] \times R^n \to R^n$ satisfy (2.2), (2.3) and $0 \leq \lambda \leq 1$. Suppose there is a constant K independent of λ such that $||y||_{H^1} \leq K$ for each solution $y(t)$ to*

$$\begin{cases} y' = \lambda f(t;y), & t \in (0,T) \\ y(0) = 0. \end{cases} \tag{2.1)$_\lambda$ $$

Then (2.1) has a solution y in $H^1((0,T);R^n)$.

Proof: Let B denote the set of functions which satisfy the initial condition $y(0) = 0$. Let $\overline{V} = \{u \in H^1_B((0,T);R^n) : ||y||_{H^1} \leq K+1\}$ and define $F_\lambda: C([0,T];R^n) \to L^2((0,T);R^n)$ by $(F_\lambda v)\ (t) = \lambda f(t;v(t))$. Now F_λ is continuous by Proposition 2.1. We also have the imbedding $j : H^1_B((0,T);R^n) \to C([0,T];R^n)$ defined by $ju = u$ completely continuous by Theorem 1.1. Finally, we define N: $H^1_B((0,T);R^n) \to L^2((0,T);R^n)$ by $Ny = y'$. Clearly, N is linear and continuous. N is also one-to-one since if $Ny = 0$ the absolute continuity of y together with the initial condition yields $y \equiv 0$. To show N is onto let $f(t) \in L^2((0,T);R^n))$ and take $y(t) = \int_0^t f(u)\ du$. Clearly, y is absolutely continuous, $y(0) = 0$, $y' = f(t)$ almost everywhere and $y' \in L^2((0,T);R^n)$. It follows from Theorem 5.10 of [**14**] that N^{-1} is a bounded linear operator. Thus, $H_\lambda = N^{-1}F_\lambda j: \overline{V} \to H^1_B((0,T);R^n)$ defines a homotopy. It is clear that the fixed points of H_λ are precisely the solutions to $(2.1)_\lambda$. Now H_λ is fixed point free on ∂V. Moreover, the complete continuity of j together with the continuity of N^{-1} and F_λ imply that the homotopy H_λ is compact. Now H_0 is essential so Theorem 1.5 of [**9**] implies that H_1 is essential. Thus (2.1) has a solution. ∎

We are now in a position to prove our basic existence theorem for first order initial problems.

Theorem 2.3. *Suppose $f: [0,T] \times R^n \to R^n$ satisfied (2.2) and (2.3). Then the initial value problem (2.1) has a solution in $H^1((0,T);R^n)$ for*

$$T < \int_0^\infty \frac{dx}{\psi(x)}.$$

Proof: To prove existence of a solution $H^1((0,T);R^n)$ we apply Theorem 2.2. So to establish a priori bounds for $(2.1)_\lambda$, let $y(t)$ be a solution to $(2.1)_\lambda$. Then since

$$|\lambda f(t;y)| \leq \psi(|y|)$$

we have for each $t \in [0,T]$

$$\int_0^t \frac{|y'(u)|}{\psi(|y(u)|)}\ du \leq t \leq T.$$

Suppose $y(t) \neq 0$ for some $t \in [0,T]$. Since $y(0) = 0$ there is an interval $[a,t]$ in $[0,T]$ such that $|y(s)| > 0$ on $a < s \leq t$ and $y(a) = 0$. Then for any point t where $y(t) \neq 0$

$$\int_a^t |y(x)|' \, ds = \int_a^t \frac{y(s)\cdot y'(s)}{|y(s)|} \leq \int_a^t |y'(s)| \, ds$$

and this together with the above inequality yields

$$\int_a^t \frac{|y(s)|'}{\psi(|y(s)|)} \, ds \leq T < \int_0^\infty \frac{dx}{\psi(x)}.$$

Theorem 1.2 yields

$$\int_0^{|y(t)|} \frac{dx}{\psi(x)} \leq T < \int_0^\infty \frac{dx}{\psi(x)}.$$

So there exists a constant $M < \infty$ such that $|y(t)| < M$. Moreover,

$$||y||_{L^2} = \left(\int_0^T |y(t)|^2 \, dt\right)^{\frac{1}{2}} \leq MT^{\frac{1}{2}} \equiv M_1.$$

Also (2.3) and the differential equation yield

$$\left(\int_0^T |y'(t)|^2 \, dt\right)^{\frac{1}{2}} \leq \left(\int_0^T [\psi(|y(t)|)]^2 \, dt\right)^{\frac{1}{2}} \leq M_2 T^{\frac{1}{2}}$$

where $M_2 = \text{ess sup}_{x \leq M_1} \psi(x)$. Hence $||y||_{H^1} \leq K = M_1 + M_2 T^{\frac{1}{2}}$ and the existence of a solution to (2.1) is established. ∎

Theorem 2.4 (Maximal Interval of Existence). *Let $\psi\colon [0,\infty) \to (0,\infty)$ be continuous. Then the initial value problem*

$$\begin{cases} y' = \tilde{f}(t;y), & t \in [0,\hat{T}] \\ y(0) = 0 \end{cases} \tag{2.4}$$

where $\tilde{f}(t;y) = (\psi(|y|), 0, \ldots, 0)$, has a solution if and only if

$$\hat{T} < T_\infty = \int_0^\infty \frac{dx}{\psi(x)}.$$

Proof: The existence of a solution for $\hat{T} < T_\infty$ follows from Theorem 2.3. Finally, $y(t) = (y_1(t), \ldots, y_n(t))$ is a solution to (2.4) if and only if $y_2(t) = \ldots = y_n(t) \equiv 0$ and $y_1'(t) = \psi(|y_1|), y_1(t) > 0$. Clearly, $y_1' > 0$ and thus $y_1(t) \geq 0$ since $y_1(0) = 0$. Now integration yields

$$\hat{T} = \int_0^{\hat{T}} \frac{y_1'(t)}{\psi(y_1(t))} \, dt = \int_0^{y_1(\hat{T})} \frac{dx}{\psi(x)} < \int_0^\infty \frac{dx}{\psi(x)} = T_\infty.$$

∎

Theorem 2.2 also holds for the inhomogeneous initial condition $y(0) = r$, via ideas of Theorem 5.1 in [**10**]. So trivial adjustments in the proof of Theorem 2.3 yields:

Theorem 2.5. *Suppose $f\colon [0,T] \times R^n \to R^n$ satisfies (2.2) and (2.3). Then the initial value problem*

$$\begin{cases} y' = f(t;y), & t \in [0,T] \\ y(0) = r \end{cases} \tag{2.5}$$

has a solution in $H^1((0,T);R^n)$ for each

$$T < \int_{|r|}^{\infty} \frac{dx}{\psi(x)}.$$

Moreover, this result is best possible as described in Theorem 2.4.

Example 1: (Electrical Circuits) Let us consider an electrical circuit which contains a resistance R, a condenser of capacitance C, and switch S and a generator E. Suppose the switch is closed at $t = 0$. Then Kirchoff's, Ohm's, and Coulomb's Laws imply that the RC circuit satisfies

$$R\frac{dq}{dt} + \frac{1}{C}q(t) = E(t,q)$$

where $q(t)$ denotes the charge on the capacitor at time t and $E(t,q)$ the value of the voltage impressed on the circuit by E.

We also have $q(0) = q_0$ where q_0 is the charge on the capacitor at $t = 0$. Thus we are interested in the initial value problem

$$\begin{cases} \frac{dq}{dt} = -\frac{1}{RC}q + \frac{E(t,q)}{R} \\ q(0) = q_0 . \end{cases} \tag{2.6}$$

Suppose $|E(t,q)| \leq A(t)|q| + B(t)$ for bounded functions $A(t), B(t) \geq 0$ and that $E(t,q)$ satisfies the Caratheodory Conditions. Consequently, since

$$\int_{|q_0|}^{\infty} \frac{dq}{A_0 q + B_0} = \infty,$$

for any constants $A_0, B_0 > 0$, Theorem 2.5 implies that (2.5) has a solution in $H^1((0,T);R^n)$ for any $T > 0$. On the other hand suppose

$$|E(t,q)| \leq A(t)|q|^m + B(t), \quad m = 1,2,\ldots$$

for bounded functions $A(t), B(t), \geq 0$ and that $E(t,q)$ satisfies the Caratheodory Conditions. Then Theorem 2.5 implies that (2.5) has a solution in $H^1((0,T);R)$ for any $T < T_\infty$ where T_∞ is as described in Example 2 of [**12**].

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