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A Leray-Schauder Alternative for Mönch Maps on Closed Subsets of Fréchet Spaces

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Abstract. In this paper, a continuation principle is obtained for maps defined on a closed, convex subset which may have empty interior in a Fréchet space, and satisfying a condition of Mönch type. An application to first order systems of differential equations is presented to illustrate our theory.

Keywords: *Fixed points, Mönch maps, Leray-Schauder alternatives*

AMS subject classification: 54H25, 47H10, 34G20

1. Introduction

In [5], Mönch establised a generalization of the fixed point theorems of Schauder, Krasnoselskii, Darbo, and Sadovskii. He also obtained a continuation theorem of Leray-Schauder type. We state an analogue of these results to quasicomplete, metrizable, locally convex space. The proof is analogous to the ones presented in [5].

Theorem 1.1. *Let X be a closed, convex subset of the quasicomplete, metrizable, locally convex space E, $f : X \to X$ a continuous map, and $x_0 \in X$ such that for every countable set $C \subset X$ satisfying $\overline{C} = \operatorname{conv}(\{x_0\} \cup f(C))$ one has $\overline{C}$ compact. Then f has a fixed point*

Theorem 1.2. *Let U be an open subset of E, $x_0 \in U$, and $f : \overline{U} \to E$ a continuous map satisfying the following conditions:*

(i) *For every countable set $C \subset \overline{U}$ satisfying $\overline{C} \subset \operatorname{conv}(\{x_0\} \cup f(C))$ one has $\overline{C}$ compact.*

(ii) *$x \notin (1-\lambda)x_0 + \lambda f(x)$ for every $x \in \partial U$ and every $\lambda \in [0,1]$.*

Then f has a fixed point.

We note that in the Banach space setting, Theorem 1.2 is applicable to wide classes of problems (see [7]). However, in the non-normable situation, Theorem 1.2 is rarely of interest since in applications the set U constructed is usually bounded and

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so has empty interior. As a result from an application viewpoint, Theorem 1.2 needs to be adjusted.

In this paper, we obtain a continuation principle similar to Theorem 1.2 for a map f defined on a closed, convex subset of E which can have empty interior. Indeed, this situation occurs naturally in applications. For the sake of simplicity our results will be stated for Fréchet spaces but it is worthwhile to mention that they are also valid in quasicomplete, metrizable, locally convex spaces. In Section 5 we show how our theory can be applied in practice by considering first order infinite systems of differential equations.

2. Preliminaries

Let E be a Fréchet space with the topology generated by a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. For the sake of simplicity, we will assume that the following condition is satisfied:

$(\star)$ $\|x\|_n \leq \|x\|_m$ for every $x \in E$ and $n \leq m$.

We follow the construction in [3]. To E, we associate for every $n \in \mathbb{N}$ a normed space E_n as follows: For each $n \in \mathbb{N}$, we write

$$x \sim_n y \quad \text{if and only if} \quad \|x - y\|_n = 0. \tag{2.1}$$

This defines an equivalence relation on E. We denote by $\hat{E}_n = E/\sim_n$ the quotient space, and by E_n the completion of $\hat{E}_n$ with respect to $\|\cdot\|_n$ (the norm on $\hat{E}_n$ induced by $\|\cdot\|_n$ and its extension to E_n are still denoted by $\|\cdot\|_n$). This construction defines a continuous map $\mu_n : E \to E_n$. Now, observe that since condition $(\star)$ is satisfied, the semi-norm $\|\cdot\|_n$ induces a semi-norm on E_m for every $m \geq n$. Again, this semi-norm is still denoted by $\|\cdot\|_n$. Also, relation (2.1) defines an equivalence relation on E_m from which we obtain a continuous map $\mu_{n,m} : E_m \to E_n$ since $E_m \backslash \sim_n$ can be regarded as a subset of E_n. Observe that E is the projective limit of $(E_n)_{n\in\mathbb{N}}$.

For each subset $X \subset E$ and each $n \in \mathbb{N}$, we set $X_n = \mu_n(X)$, and we denote by $\overline{X_n}$ and ∂X_n the closure and the boundary of X_n with respect to $\|\cdot\|_n$ in E_n, respectively. We denote by diam_n the n-diameter induced by $\|\cdot\|_n$; that is,

$$\text{diam}_n(X) = \sup\{\|x - y\|_n : x, y \in X\}.$$

Define $S_n : X \to X$ by

$$S_n(x) = \{y \in X : \|x - y\|_n = 0\}.$$

Since the set X that we will consider can have empty interior, we recall the notion of *pseudo-interior* of X introduced in [3]

$$\text{pseudo-int}(X) = \{x \in X : \mu_n(x) \in \overline{X_n} \backslash \partial X_n \text{ for every } n \in \mathbb{N}\}.$$

A notion of admissible map was introduced in [3] for compact maps. Let us introduce an analogous definition which will permit us to consider non-compact maps.

Definition 2.1. Let $Y \subset X \subset E$ be such that X is convex. We say that a map $f : Y \to X$ belongs to the class $\mathcal{M}$ if, for every $n \in \mathbb{N}$:

(1) The multi-valued map $\hat{F}_n : Y_n \to \overline{X_n}$ defined by

$$\hat{F}_n(\mu_n(x)) = \overline{\operatorname{conv}}\big(\mu_n \circ f \circ S_n(x)\big)$$

admits an upper semi-continuous extension $F_n : \overline{Y_n} \to \overline{X_n}$ with convex, compact values.

(2) For every $\varepsilon > 0$ and $x \in Y$, there exists $m \geq n$ such that

$$\operatorname{diam}_n\big(f(S_m(x))\big) < \varepsilon.$$

Remark. We can state an analogous definition for maps $f : Y \times [0,1] \to X$ and we also say that f is in the class $\mathcal{M}$.

3. Fixed point result in class $\mathcal{M}$

Our first result is an analog of the Mönch fixed point theorem [5] for maps in class $\mathcal{M}$. The proof is similar to the one presented in [7].

Theorem 3.1. *Let X be a closed, convex subset of the locally convex space E, and $f : X \to X$ a map in $\mathcal{M}$. Assume that there exists $x_0 \in X$ satisfying:*

(C) *For every $n \in \mathbb{N}$ and every $Y \subset X$ with $Y = \operatorname{conv}\big(\{x_0\} \cup f(S_n(Y))\big)$, and for which there exists a countable set $C \subset Y$ satisfying $\overline{C_n} = \overline{Y_n}$, one has $\overline{Y_n}$ compact.*

Then f has a fixed point.

Proof. For every $n \in \mathbb{N}$, we define inductively a sequence of convex subsets of X,

$$Y^{n,0} = \{x_0\}$$
$$Y^{n,k} = \operatorname{conv}\big(\{x_0\} \cup f(S_n(Y^{n,k-1}))\big) \quad (k \geq 1).$$

It is immediately seen by induction that, for every $k \geq 1$ and every $m \geq n$, $Y^{n,k-1} \subset Y^{n,k}$ and $Y^{m,k} \subset Y^{n,k}$. For every $n \in \mathbb{N}$, we define

$$Y^n = \cup_{k \geq 0} Y^{n,k}.$$

Since $f \in \mathcal{M}$, for every $k \geq 0$, $\overline{Y_n^{n,k}}$ is compact. Thus, there exists a countable set $C^{n,k} \subset Y^{n,k}$ such that $\overline{C_n^{n,k}} = \overline{Y_n^{n,k}}$, we have by assumption that $\overline{Y_n^n}$ is compact. Moreover, it is clear that

$$Y^n = \operatorname{conv}\big(\{x_0\} \cup f(S_n(Y^n))\big).$$

Again we can notice that, for every $m \geq n$, $Y^m \subset Y^n$.

Now we claim that, for every $n \in \mathbb{N}$, $F_n(x) \cap \overline{Y_n^n} \neq \emptyset$ for every $x \in \overline{Y_n^n}$. Indeed, for every $x \in \overline{Y_n^n}$ there exists a sequence $\{x_i\}$ in Y^n such that $\mu_n(x_i) \to x$ as $i \to \infty$. Let $k_i \geq 1$ be such that $x_i \in Y^{n,k_i}$, and choose $y_i \in f(S_n(x_i))$. Observe that

$$y_i \in f(S_n(Y^{n,k_i})) \subset Y^{n,k_i+1} \subset Y^n.$$

The compactness of $\overline{Y_n^n}$ implies the existence of a subsequence still denoted $\{\mu_n(y_i)\}$ converging to $u \in \overline{Y_n^n}$ as $i \to \infty$. It follows from the upper semi-continuity of F_n that $u \in F_n(x)$. Therefore, for every $n \in \mathbb{N}$, we can define an upper semi-continuous map with non-empty, convex, compact values $G_n : \overline{Y_n^n} \to \overline{Y_n^n}$ by $G_n(x) = F_n(x) \cap \overline{Y_n^n}$. The Kakutani fixed point theorem implies the existence of $z_n \in G_n(z_n) \subset F_n(z_n)$.

Obviously, $\mu_{n,m}(z_m) \in G_n(\mu_{n,m}(z_m))$ for every $m \geq n$. By compactness, the sequence $(\mu_{1,m}(z_m))_{m\geq 1}$ has a subsequence $(\mu_{1,m}(z_m))_{m\in N_1}$ converging to $x_1 \in \overline{Y_1^1}$. It follows from the upper semi-continuity of F_1 that $x_1 \in F_1(x_1)$. Again, the sequence $(\mu_{2,m}(z_m))_{m\in N_1}$ has a subsequence $(\mu_{2,m}(z_m))_{m\in N_2}$ converging to $x_2 \in \overline{Y_2^2}$, with $x_2 \in F_2(x_2)$. By uniqueness of the limit, $\mu_{1,2}(x_2) = x_1$. In repeating this argument we obtain, for every $n \in \mathbb{N}$, $x_n \in \overline{Y_n^n}$ such that $x_n \in F_n(x_n)$ and $\mu_{n,m}(x_m) = x_n$ for every $m \geq n$. Since X is closed, we deduce the existence of $x \in X$ such that $\mu_n(x) = x_n$ for every $n \in \mathbb{N}$.

We claim that $x = f(x)$. If this is false, there exists $n \in \mathbb{N}$ and $\delta > 0$ such that $\|x - f(x)\|_n = \delta$. Let $\varepsilon < \frac{\delta}{2}$. By Definition 2.1/(2), there exists $m \geq n$ such that

$$\operatorname{diam}_n\big(f(S_m(x))\big) = \operatorname{diam}_n\big(\operatorname{conv}\big(f(S_m(x))\big)\big) < \varepsilon.$$

On the other hand, since $x_m \in F_m(x_m)$, we can take $y \in \operatorname{conv}\big(f(S_m(x))\big)$ such that $\|x - y\|_m < \varepsilon$. Thus,

$$\delta = \|x - f(x)\|_n \leq \|x - y\|_n + \|y - f(x)\|_n < \|x - y\|_m + \varepsilon < 2\varepsilon < \delta$$

which is a contradiction.

4. Fixed point result of Leray-Schauder type in class $\mathcal{M}$

We introduce the notion of homotopy in the class $\mathcal{M}$.

Definition 4.1. Let $f, g : X \to E$ be maps in $\mathcal{M}$. We say that f and g are *$\mathcal{M}$-homotopic* if there exists a map $h : X \times [0,1] \to E$ in $\mathcal{M}$ such that:

(1) $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$.

(2) For every $n \in \mathbb{N}$, $z \notin H_n(z, \lambda)$ for every $z \in \partial X_n$ and every $\lambda \in [0,1]$.

We write $f \approx g$ and we say that h is a *homotopy*.

Here is our main theorem.

Theorem 4.2. *Let X be a closed subset of E, $x_0 \in$ pseudo-int(X), and $f : X \to E$ in the class $\mathcal{M}$. Assume that f is $\mathcal{M}$-homotopical to the constant map x_0 with a homotopy h. Finally, let the following condition be satisfied:*

(C′) *For every $n \in \mathbb{N}$ and every $Y \subset X$ with $Y \subset \operatorname{conv}\big(h(S_n(Y) \times [0,1])\big)$, and for which there exists a countable set $C \subset Y$ satisfying $\overline{C_n} = \overline{Y_n}$, one has $\overline{Y_n}$ compact.*

Then f has a fixed point.

Proof. For $n \in \mathbb{N}$, we set

$$A_n = \big\{z \in \overline{X_n} : z \in H_n(z,\lambda) \text{ for some } \lambda \in [0,1]\big\}.$$

Observe that $\mu_n(x_0) \in A_n$. Also, A_n is closed and $A_n \cap \overline{X_n} \backslash X_n = \emptyset$, since $h \in \mathcal{M}$. Moreover, for $m \geq n$, $\mu_{n,m}(A_m) \subset A_n$. Let $\theta : \overline{X_n} \to [0,1]$ be an Urysohn's function such that $\theta(x) = 0$ on $\overline{X_n} \backslash X_n$ and $\theta(x) = 1$ on A_n. We define inductively a sequence of convex subsets of E by

$$Y^{n,0} = \{x_0\}$$
$$Y^{n,k} = \operatorname{conv}\big\{h(x,\theta(\mu_n(x))) : x \in S_n(Y^{n,k-1})\big\} \quad (k \geq 1)$$

and we set $Y^n = \cup_{k\geq 0} Y^{n,k}$. It is easy to see that $Y^n \subset \operatorname{conv}\big(h(S_n(Y^n) \times [0,1])\big)$. Assumption (C′) implies that $\overline{Y_n^n}$ is compact.

As in the proof of Theorem 3.1 we can show that the map $K_n : \overline{Y_n^n} \to \overline{Y_n^n}$ defined by $K_n(x) = H_n(x,\theta(x)) \cap \overline{Y_n^n}$ is upper semi-continuous with non-empty, convex, compact values. The Kakutani fixed point theorem implies the existence of

$$z_n \in K_n(z_n, \theta(z_n)) \subset H_n(z_n, [0,1]).$$

So, $z_n \in A_n$, and hence $\theta(z_n) = 1$. Thus $z_n \in F_n(z_n)$. Moreover, for every $m \geq n$, $\mu_{n,m}(z_m) \in A_n \cap \overline{Y_n^n}$ and $\mu_{n,m}(z_m) \in F_n(\mu_{n,m}(z_m))$. Arguing as in the proof of Theorem 3.1, we deduce the existence of $x \in X$ such that $x = f(x)$ ∎

Corollary 4.3. *Let X be a closed subset of E, $x_0 \in$ pseudo-int(X), and $f : X \to E$ in the class $\mathcal{M}$. Assume that the following conditions are satisfied:*

(i) *For every $n \in \mathbb{N}$ and every $Y \subset X$ with $Y \subset \operatorname{conv}\big(\{x_0\} \cup f(S_n(Y))\big)$, and for which there exists a countable set $C \subset Y$ satisfying $\overline{C_n} = \overline{Y_n}$, one has $\overline{Y_n}$ compact.*

(ii) *For every $n \in \mathbb{N}$, $z \notin (1-\lambda)\mu_n(x_0) + \lambda F_n(z)$ for every $z \in \partial X_n$ and every $\lambda \in [0,1]$.*

Then f has a fixed point.

Proof. It suffices to show that the map $h : X \times [0,1] \to E$ defined by $h(x,\lambda) = (1-\lambda)x_0 + \lambda f(x)$ is in $\mathcal{M}$. Then the conclusion follows directly from the previous theorem ∎

Remark. All the previous results can be generalized to a map $f : X \to K$ with K a closed, convex subset of E. In that case, one takes the closure and the boundary of X_n relative to $\overline{K_n}$.

5. Application

Consider the infinite system of differential equations

$$\left.\begin{array}{l} y'(t) = g(t, y(t)) \quad (t \in [0,T]) \\ y(0) = a = (a_1, a_2, \ldots) \in B = \prod_{n\in\mathbb{N}} B_n \end{array}\right\} \tag{5.1}$$

where $(B_n, \|\cdot\|_n)$ is a Banach space for every $n \in \mathbb{N}$. For any bounded subset Y of B, we define $\{\beta_n(Y)\}_{n\in\mathbb{N}}$ the family of *Hausdorff's measure of non-compactness* of Y as follows: for any $n \in \mathbb{N}$, $\beta_n(Y)$ denotes the infimum of all $\varepsilon > 0$ such that there exists a finite set $\{x_1, \ldots, x_k\} \subset A$ with

$$Y \subset \bigcup_{j=1}^{k} \{y \in B : \|x_j - y\|_i < \varepsilon \text{ for } i = 1, \ldots, n\}.$$

For properties of $\{\beta_n(Y)\}_{n\in\mathbb{N}}$, the reader is referred to [1, 5, 6]. Observe that β_n corresponds to the measure of non-compactness of $B_1 \times \cdots \times B_n$.

Theorem 5.1. *Let B be as above and $g : [0,T] \times B \to B$ a continuous map satisfying:*

(i) *For every $n \in \mathbb{N}$, there exist $q_n \in C([0,T], B)$, and $\phi_n : [0,\infty) \to (0,\infty)$ a non-increasing function such that*

$$\|g_n(t,y)\|_n \le q_n(t)\phi_n(\|y\|_n) \qquad \forall t \in [0,T], y \in B$$

and

$$\int_0^T q_n(t)\,dt < \int_{\|a_n\|_n}^{\infty} \frac{ds}{\phi_n(s)}.$$

(ii) *For every $n \in \mathbb{N}$, there exists a continuous map $w_n : [0,\infty) \to [0,\infty)$ such that $w_n(0) = 0$, $w_n(s) > 0$ for all $s > 0$, $\int_0^\infty \frac{ds}{w_n(s)} = \infty$ and*

$$\beta_n(g(t,Y)) \le w_n(\beta_n(Y)) \qquad \forall t \in [0,T], Y \subset B \text{ bounded.}$$

(iii) *For every $n \in \mathbb{N}$, and for every bounded subset $Y \subset B$, there exists a sequence $(k_m^n)_{m\ge n}$ converging to 0 such that, for every $m \ge n$,*

$$\max_{1\le i\le n} \left\{ \left\| g_i(t,x) - g_i(t,y) \right\|_i \right\} \le k_m^n$$

for every $t \in [0,T]$, $x, y \in Y$ such that $\|x_i - y_i\|_i = 0$ for $i = 1, \ldots, m$.

Then problem (5.1) has a solution.

Proof. Let us consider $E = C([0,T], B)$ endowed with the family of semi-norms $\{|\cdot|_n\}_{n\in\mathbb{N}}$ defined by

$$|u|_n = \max_{1\le i\le n} \left\{ \max_{t\in[0,T]} \|u_i(t)\|_i \right\}.$$

For every $n \in \mathbb{N}$ we set

$$I_n(z) = \int_{\|a_n\|_n}^{z} \frac{ds}{\phi_n(s)} \qquad \text{and} \qquad \zeta_n(t) = I_n^{-1}\left(\int_0^t q_n(\tau)\,d\tau\right) + 1.$$

Consider the closed set

$$X = \big\{u \in E : \|u_n(t)\|_n \le \zeta_n(t) \text{ for all } t \in [0,T]\big\}$$

and define $f : X \to E$ by

$$f(u)(t) = \int_0^t g(\tau, u(\tau))\,d\tau.$$

Essentially the same reasoning as in [3: Theorem 5.2] guarantees that f is in the class $\mathcal{M}$. Also, by standard arguments (see [4]) it can be shown that, for every $n \in \mathbb{N}$,

$$z \notin (1-\lambda)(a_1,\ldots,a_n) + \lambda F_n(z) \qquad \forall\, z \in \partial X_n, \forall\, \lambda \in [0,1].$$

Indeed, to see this we need only to note that if there exists $n \in \mathbb{N}$, $z \in \partial X_n$ and $\lambda \in [0,1]$ with $z \in (1-\lambda)(a_1,...,a_n) + \lambda\, F_n(z)$, then

$$\|z(t)\|_n \le \|a_n\|_n + \int_0^t q_n(t)\,\phi_n(\|z(s)\|_n)\,ds \equiv u(t) \qquad (t \in [0,T])$$

and so

$$\zeta_n(t) = \|z(t)\|_n \le u(t) \le I_n^{-1}\left(\int_0^t q_n(s)\,ds\right) = \zeta_n(t) - 1,$$

which is a contradiction.

Now, we claim that $h(x,\lambda) = (1-\lambda)x_0 + \lambda f(x)$ satisfies condition (C$'$). Indeed, let $n \in \mathbb{N}$ and $Y \subset X$ such that $Y \subset \operatorname{conv}\big(\{a\} \cup f(S_n(Y))\big)$ and for which there exists a countable set $C \subset Y$ with $\overline{C_n} = \overline{Y_n}$. The assumptions imply the existence of $M_n \ge 0$ such that $\|g_i(t,u(t))\|_i \le M_n$ for every $u \in S_n(X)$ and every $i \in \{1,\ldots,n\}$. So, Y_n is bounded. Moreover, for $y \in Y$, there exists $u_1,\ldots,u_m$ in $S_n(Y)$ and $\lambda_0,\ldots,\lambda_m$ in $[0,1]$ such that

$$y = \lambda_0 a + \sum_{k=1}^m \lambda_k f(u_k)$$

with $\lambda_0 + \cdots \lambda_m = 1$. Thus, for $i = 1,\ldots,n$,

$$\big\|y_i(t) - y_i(s)\big\|_i = \left\|\sum_{k=1}^m \lambda_k \int_s^t g_i(\tau, u_k(\tau))\,d\tau\right\|_i \le M_n|t-s|.$$

Therefore, Y_n is equicontinuous.

To show that $\overline{Y_n}$ is compact, by Arzela-Ascoli's theorem, one needs to show that $\overline{Y_n(t)}$ is compact for every $t \in [0,T]$. Fix $t \in [0,T]$. Since C is countable, there exists $D \subset S_n(Y)$ countable such that $C \subset \operatorname{conv}(\{a\} \cup f(D))$. The properties of the measure of non-compactness and [6: Theorem 1] imply that

$$t \mapsto \psi(t) = \beta_n\left(\left\{\int_0^t g(\tau, v(\tau))\, d\tau : v \in D\right\}\right)$$

is absolutely continuous, and

$$\begin{aligned}\beta_n(D(t)) \le \beta_n(Y(t)) &= \beta_n(C(t)) \le \beta_n\big(\operatorname{conv}(\{a\} \cup f(D)(t))\big) = \beta_n(f(D)(t))\\ &= \psi(t) = \textstyle\int_0^t \psi'(\tau)\, dt \le 2\int_0^t \beta_n\big(\{g(\tau, v(\tau))\, d\tau : v \in D\}\big)\\ &\le 2\textstyle\int_0^t \beta_n\big(g(\tau, D(\tau))\big)\, d\tau \le 2\int_0^t w_n\big(\beta_n(D(\tau))\big)\, d\tau.\end{aligned}$$

It follows from [10: Lemma 2] that $w_n\big(\beta_n(D(\tau))\big) = 0$ for every $\tau \le t$. This implies that $\beta_n(Y(t)) = 0$, so $\overline{Y_n(t)}$ is compact for each $t \in [0, T]$. Therefore $\overline{Y_n}$ is compact. Now the conclusion follows from Corollary 4.3 ∎

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