Recent Advances on Metric Fixed Point Theory

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ON CONTINUATION METHODS FOR CONTRACTIVE AND NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we present results concerning the homotopy invariance of the property of having a fixed point for contractive, weakly contractive, or nonexpansive, single-valued or multivalued mappings. The notion of measure of noncompactness is not used.

0. INTRODUCTION

It is well known that either f has a fixed point, or λf has a fixed point on the boundary of U for some $\lambda \in (0,1)$, if $f : \overline{U} \to E$ is a contractive mapping, where U is a bounded domain containing the origin, and E is Banach space. Indeed, this is a consequence of continuation results for k-set contractions, see for example [6], [14], [15], [18], [19]. More generally, we know that the property of having a fixed point is invariant by homotopy for contractions. In the first section of this paper, an elementary proof of this result is presented without using the notion of measure of noncompactness. Moreover, it is shown that there is a Lipschitzian curve of fixed points. This combines results of [8] and [10]. In addition, the homotopy invariance of the property of having a fixed point is also established for weakly contractive maps. However, this fact does not hold for nonexpansive mappings, as it is shown in section 2.

The topological degree theory for condensing multivalued mappings with nonempty compact, convex values is well developped, see for example [2], [12], [17]. Howeover, it does not permit to obtain homotopy invariance of the property of having a fixed point for contractive multivalued mappings when we assume the values to be only closed subsets of a complete metric space. This result due to Granas and myself [7] is presented in section 3. In section 4, a Nonlinear Alternative is given for multivalued nonexpansive mappings in a uniformly convex Banach space. All our results do not use the notion of measure of noncompactness.

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Let (X,d) be a metric space. By $B(x,r)$, we denote the open ball in X centered in X of radius r; and by $B(C,r)$, we denote $\cup_{x\in C}B(x,r)$, where C is a subset of X. For C and K two nonempty closed subsets of X, we define

$$D(C,K)=\inf\{\varepsilon>0\ :\ C\subset B(K,\varepsilon),\ K\subset B(C,\varepsilon)\}\in[0,\infty].$$

D is called the *generalised Hausdorff distance*.

Let C be a nonempty subset of X. A function $f:C\to X$ is said *contractive* if there exists a constant $\alpha<1$ such that $d(f(x),f(y))\leq\alpha d(x,y)$; and it is said *nonexpansive* if $d(f(x),f(y))\leq d(x,y)$.

Similarly, a multivalued mapping $F:C\to X$ with (nonempty) closed values is said *contractive* if there exists a constant $\alpha<1$ such that $D(F(x),F(y))\leq\alpha d(x,y)$; and it is said *nonexpansive* if $D(F(x),F(y))\leq d(x,y)$.

We mention that contractive mappings are called strictly contractive by some authors.

1. Contractive and weakly contractive mappings

The results presented in this section extend and unify those of [8] and [10]. First of all, we want to give an elementary proof of the fact that the property of having a fixed point is invariant by homotopy for contractions.

Let (X,d) be a complete metric space, and U a domain of X.

Definition 1.1. Let f and $g:\overline{U}\to X$ be two contractions. We say that they are *homotopic* if there exists $H:\overline{U}\times[0,1]\to X$ such that

(a) $H(\cdot,1)=f$, $H(\cdot,0)=g$;
(b) $H(x,t)\neq x$ for every $x\in\partial U$, and $t\in[0,1]$;
(c) there exists $\alpha<1$ such that $d(H(x,t),H(y,t))\leq\alpha d(x,y)$ for every x, $y\in\overline{U}$, and $t\in[0,1]$;
(d) there exists a constant $M\geq 0$ such that for every $x\in\overline{U}$, and t, $s\in[0,1]$, $d(H(x,t),H(x,s))\leq M|t-s|$.

Theorem 1.2. *Let f and $g:\overline{U}\to X$ be two homotopic contractive mappings. Then f has a fixed point if and only if g has a fixed point. Moreover, if f has a fixed point, there exists a Lipschitzian curve defined on $[0,1]$, $\lambda\mapsto x_\lambda$ such that $x_\lambda=H(x_\lambda,\lambda)$ where H is an homotopy between f and g.*

Proof. Let us denote

$$Q=\{\lambda\in[0,1]\ :\ H(\cdot,\lambda)\text{ has a fixed point}\},$$

where H is an homotopy between f and g. If g has a fixed point, $0\in Q$. First of all, we want to show that Q is open in $[0,1]$. Let $\lambda_0\in Q$, and x the fixed point of $H(\cdot,\lambda_0)$. Fix $r>0$ such that $B(x,r)\subset U$. Take $\delta>0$ such that for $|\lambda-\lambda_0|<\delta$, $M|\lambda-\lambda_0|<(1-\alpha)r$, (where M and α are given by the homotopy). Then, the function $H(\cdot,\lambda)$ maps the ball $\overline{B(x,r)}$ into itself. Indeed, for every $y\in\overline{B(x,r)}$,

$$\begin{aligned}d(x,H(y,\lambda)) &\leq d(H(x,\lambda_0),H(y,\lambda_0))+d(H(y,\lambda_0),H(y,\lambda))\\ &\leq \alpha d(x,y)+M|\lambda-\lambda_0|\\ &< \alpha r+(1-\alpha)r=r.\end{aligned}$$

By the Banach contraction principle, $H(\cdot,\lambda)$ admits a fixed point for each $\lambda\in B(\lambda_0,\delta)\cap[0,1]$.

Now, to show that Q is closed, take λ_1, $\lambda_2\in Q$, and x_1, x_2 the fixed points of $H(\cdot,\lambda_1)$ and $H(\cdot,\lambda_2)$ respectively. We get

$$\begin{aligned}d(x_1,x_2) &\leq d(H(x_1,\lambda_1),H(x_1,\lambda_2))+d(H(x_1,\lambda_2),H(x_2,\lambda_2))\\ &\leq M|\lambda_1-\lambda_2|+\alpha d(x_1,x_2),\end{aligned}$$

and hence

$$d(x_1,x_2)\leq\frac{M|\lambda_1-\lambda_2|}{1-\alpha}.$$

It follows that $Q=[0,1]$ and the curve of fixed point $\lambda\longmapsto x_\lambda$ is Lipschitzian. □

Remark 1.3. If in the definition 1.1, we replace the condition (d) by

(d') there exists a continuous function $\phi:[0,1]\to\mathbb{R}$ such that for every $x\in\overline{U}$, and t, $s\in[0,1]$, $d(H(x,t),H(x,s))\leq|\phi(t)-\phi(s)|$;

the previous result holds except that the curve of fixed points is no longer Lipschitzian but it is continuous.

In the particular case where X is a Banach space, we get as corollaries the following results.

Corollary 1.4. *Let E be a Banach space, U a domain of E containing the origin, and let $f:\overline{U}\to E$ be a contraction such that $f(\overline{U})$ is bounded. Then, there exist $\lambda^*\in(0,1]$ and a Lipschitzian curve of fixed points of λf, $\lambda\longmapsto x_\lambda$ defined on $[0,\lambda^*]$ such that $(\lambda^*,x_{\lambda^*})$ is on the boundary of $[0,1]\times U$.*

Corollary 1.5. *Let E be a Banach space, U a domain of E containing the origin, and let $f:\overline{U}\to E$ be a contraction such that $f(\overline{U})$ is bounded. Assume that for every $x\in\partial U$, one of the following conditions is satisfied:*

(i) $\|f(x)\|\leq\|x\|$; *E. Rothe*
(ii) $\|f(x)\|\leq\|x-f(x)\|$;
(iii) $\|f(x)\|\leq\left(\|x\|^2+\|x-f(x)\|^2\right)^{1/2}$; *M. Altman*
(iv) $\|f(x)\|\leq\max\{\|x\|,\|x-f(x)\|\}$;

(v) $-x \in \overline{U}$ *and* $f(x) = -f(-x)$.
Then f *has a unique fixed point.*

More generally, we can consider weakly contractive maps. Again (X, d) is a complete metric space and U is a domain of X.

Definition 1.6. A function $f : \overline{U} \to X$ is said *weakly contractive* if there exists $\psi : X \times X \to (0, \infty)$ *compactly positive* (i.e. $\inf\{\psi(x,y) : a \leq d(x,y) \leq b\} = \theta(a,b) > 0$ for every $0 < a \leq b$) such that

$$d(f(x), f(y)) \leq d(x,y) - \psi(x,y).$$

If ψ is a compactly positive function, we define for $0 < a \leq b$

$$\gamma(a,b) = \min\{a, \theta(a,b)\}.$$

The following lemma is due to Dugundji and Granas [3] and will be used later. We give the proof for sake of completeness.

Lemma 1.7. *Let* $x_0 \in X$, $r > 0$, *and* $f : \overline{B(x_0, r)} \to X$ *weakly contractive. If* $d(x_0, f(x_0)) < \gamma(r/2, r)$, *then* f *has a fixed point.*

Proof. Let $x_1 = f(x_0)$, since $d(x_0, f(x_0)) < \gamma(r/2, r) \leq r/2$, $x_1 \in \overline{B(x_0, r)}$. Define $x_2 = f(x_1)$. Then

$$d(x_2, x_1) \leq d(x_1, x_0) - \psi(x_1, x_0),$$

and

$$\begin{aligned} d(x_2, x_0) &\leq d(x_2, x_1) + d(x_1, x_0) \\ &\leq 2d(x_1, x_0) - \psi(x_1, x_0) \leq r. \end{aligned}$$

Again, take $x_3 = f(x_2)$. Then

$$d(x_3, x_2) \leq d(x_1, x_0) - \psi(x_1, x_0) - \psi(x_2, x_1),$$

and

$$\begin{aligned} d(x_3, x_0) &\leq d(x_3, x_1) + d(x_1, x_0) \\ &\leq d(x_2, x_0) - \psi(x_2, x_0) + d(x_1, x_0) \\ &\leq d(x_2, x_0) - \psi(x_2, x_0) + \gamma(r/2, r) \\ &\leq r. \end{aligned}$$

By repeating this argument, we get a sequence $\{x_n = f^n(x_0)\}$ in $\overline{B(x_0, r)}$ such that

$$d(x_n, x_{n+1}) \leq d(x_0, x_1) - \sum_{j=1}^{n} \psi(x_{j-1}, x_j). \tag{1.1}$$

Moreover, by the same argument, if for some $s > 0$

(1.2) $d(x_n, x_{n+1}) < \gamma(s/2, s)$, then $x_m \in \overline{B(x_n, s)} \cap \overline{B(x_0, r)}$ for all $m \geq n$.

The inequality (1.1) implies that the sequence $\{\sum_{j=1}^{\infty} \psi(x_{j-1}, x_j)\}$ is convergent. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that

$$\psi(x_n, x_{n+1}) < \theta(\gamma(\varepsilon/2, \varepsilon), d(x_0, x_1)) \quad \text{for all } n \geq N.$$

This, and the fact that $d(x_N, x_{N+1}) \leq d(x_0, x_1)$ implies that

$$d(x_N, x_{N+1}) < \gamma(\varepsilon/2, \varepsilon).$$

From (1.2), we get that $x_n \in \overline{B(x_N, \varepsilon)} \cap \overline{B(x_0, r)}$ for every $n \geq N$. So,

$$d(x_n, x_m) < 2\varepsilon \quad \text{for all } n,\ m \geq N.$$

Since ε is arbitrary, $\{x_n\}$ is a Cauchy sequence, and hence converges to $x \in \overline{B(x_0, r)}$. From the continuity of f, we get

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_{n-1}) = f(x),$$

and this completes the proof. $\square$

Definition 1.8. Let f and $g : \overline{U} \to X$ be two weakly contractive maps. We say that they are *homotopic* if there exists $H : \overline{U} \times [0, 1] \to X$ such that

(a) $H(\cdot, 1) = f$, $H(\cdot, 0) = g$;
(b) $H(x, t) \neq x$ for every $x \in \partial U$, and $t \in [0, 1]$;
(c) there exists a compactly positive function $\psi : X \times X \to (0, \infty)$ such that $d(H(x, t), H(y, t)) \leq d(x, y) - \psi(x, y)$ for every x, $y \in \overline{U}$, and $t \in [0, 1]$;
(d) there exists a continuous function $\phi : [0, 1] \to \mathbb{R}$ such that for every $x \in \overline{U}$, and t, $s \in [0, 1]$, $d(H(x, t), H(x, s)) \leq |\phi(t) - \phi(s)|$.

Theorem 1.9. *Let f and $g : \overline{U} \to X$ be two homotopic weakly contractive maps. Assume one of the following conditions is satisfied:*

(a) *$f(\overline{U})$ is bounded;*
(b) *there exists an homotopy H between f and g such that the positively compact function ψ associated to H satisfies $\inf\{\theta(a, b) : b \geq a\} > 0$ for all $a > 0$.*

Then f has a fixed point if and only if g has a fixed point.

Proof. Let H be an homotopy between f and g. Define

$$Q = \{\lambda \in [0,1] : H(\cdot, \lambda) \text{ has a fixed point}\}.$$

To show that Q is open, take $\lambda_0 \in Q$ and $x = H(x, \lambda_0)$. Let $r > 0$ be such that $B(x,r) \subset U$, and let $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$, $|\phi(\lambda) - \phi(\lambda_0)| < \gamma(r/2, r)$. Then

$$\begin{aligned} d(x, H(x,\lambda)) &\leq d(x, H(x,\lambda_0)) + d(H(x,\lambda_0), H(x,\lambda)) \\ &\leq |\phi(\lambda) - \phi(\lambda_0)| < \gamma(r/2, r). \end{aligned}$$

By Lemma 1.7, $H(\cdot, \lambda)$ has a fixed point for every λ such that $|\lambda - \lambda_0| < \delta$.

To show that Q is closed, take $\{\lambda_n\}$ in Q such that $\lambda_n \to \lambda$. Let $x_n = H(x_n, \lambda_n)$. Fix $\varepsilon > 0$. If assumption (a) is satisfied, there exists $M > 0$ such that $H(\overline{U} \times [0,1]) \subset B(0, M)$. So, $d(x_n, x_m) < 2M$ for all n, $m \in \mathbb{N}$. Define $\mu = \theta(\varepsilon, 2M)$. If assumption (b) is satisfied, define $\mu = \inf\{\theta(\varepsilon, b) : b \geq \varepsilon\}$.

Let $N \in \mathbb{N}$ be such that for all n, $m \geq N$, $|\phi(\lambda_n) - \phi(\lambda_m)| < \mu$. Then $d(x_n, x_m) < \varepsilon$ for every n, $m \geq N$. Indeed, otherwise,

$$\begin{aligned} d(x_n, x_m) &\leq d(H(x_n,\lambda_n), H(x_n,\lambda_m)) + d(H(x_n,\lambda_m), H(x_m,\lambda_m)) \\ &\leq |\phi(\lambda_n) - \phi(\lambda_m)| + d(x_n, x_m) - \psi(x_n, x_m) \\ &< \mu + d(x_n, x_m) - \psi(x_n, x_m) \leq d(x_n, x_m), \end{aligned}$$

which is a contradiction. So Q is closed.

Therefore, if f has a fixed point, $Q = [0,1]$, so g has a fixed point. □

2. Nonexpansive mappings

It is natural to ask whether the results of the previous section hold for nonexpansive mappings. The following result was obtained by Z. E. A. Guennoun.

Theorem 2.1. *Let E be a uniformly convex Banach space, U a bounded, convex domain of E containing the origin, and let $f : \overline{U} \to E$ be a nonexpansive map. Then, there exist $\lambda^* \in (0,1]$ and a curve of fixed points of λf, $\lambda \mapsto x_\lambda$ defined on $[0, \lambda^*]$ and locally Lipschitzian on $[0,\lambda^*] \cap [0,1)$, such that $(\lambda^*, x_{\lambda^*})$ is on the boundary of $[0,1] \times U$.*

Proof. By Corollary 1.4, for all $t \in (0,1)$, there exist $s \in (0,t]$ and a Lipschitzian curve $\lambda \mapsto x_\lambda$ of fixed points of λf defined on $[0,s]$ such that $(s, x_s) \in \partial([0,t] \times U)$. If $x_s \notin \partial U$ then $s = t$ and the curve can be extended beyond t. Let $\lambda^* = \sup\{t : \text{the curve is defined on } [0,t]\}$. If $\lambda^* < 1$ then we have the conclusion. If $\lambda^* = 1$, let $\{\lambda_n\}$ be an increasing sequence converging to 1, and $x_n = \lambda_n f(x_n)$. The sequence $\{x_n\}$ has a subsequence converging weakly to x, and the sequence $\{x_n - f(x_n)\}$ converges strongly to 0. The demi-closedness of $(I - f)$ implies that $x = f(x)$ (see the following lemma), and the proof is completed. □

Lemma 2.2. *Let E be a uniformly convex Banach space, C a closed, bounded, convex subset of E, and $f : C \to E$ a nonexpansive map. If $\{x_n\}$ converges weakly to x in C, and if $(I - f)(x_n) \to y$, then $(I - f)(x) = y$.*

Theorem 2.1 is only a partial answer to the following question: does the analogous result of Theorem 1.2 hold for nonexpansive mappings. The answer to this question is no as it is shown in the following example.

Example 2.3. Take $X = l^2$, $U = B(0,1)$, and $H : \overline{U} \times [0,1] - X$ defined by

$$H(x,t) = (ta, x_1, x_2, \dots),$$

where $a \in \mathbb{R}\backslash\{0\}$, and $x = (x_n)_{n\in\mathbb{N}}$. It is clear that for every t, $H(\cdot,t)$ is nonexpansive. Also,

$$\|H(x,t) - H(x,s)\| \leq |a|\,|t-s|,$$

for every $x \in \overline{U}$ and every s and t. On the other hand,

$$\begin{aligned} x = H(x,t) \quad & \text{if and only if} \quad ta = x_1 = x_2 = \cdots \\ & \text{if and only if} \quad t = 0 \text{ and } x = 0. \end{aligned}$$

Thus, $H(\cdot,t)$ has a fixed point if and only if $t = 0$.

The next example shows that Theorem 1.2 does not hold also for homotopy H such that $\|H(x,t) - H(y,t)\| < \|x - y\|$.

Example 2.4. Take $X = l^2$, $U = B(0,1)$, and $H : \overline{U} \times [0,1] - X$ defined by

$$H(x,t) = (ta, \frac{1}{\sqrt{2}}x_1, \frac{\sqrt{2}}{\sqrt{3}}x_2, \frac{\sqrt{3}}{\sqrt{4}}x_3 \dots),$$

where $a \in \mathbb{R}\backslash\{0\}$, and $x = (x_n)_{n\in\mathbb{N}}$. As in the previous example,

$$\|H(x,t) - H(x,s)\| \leq |a|\,|t-s|,$$

for every $x \in \overline{U}$ and every s and t. Also, it is clear that for every $t \in [0,1]$, and $x, y \in \overline{U}$,

$$\|H(x,t) - H(y,t)\| < \|x - y\|.$$

Moreover,

$$\begin{aligned} x = H(x,t) \quad & \text{if and only if} \quad ta = x_1, x_2 = \frac{ta}{\sqrt{2}}, x_3 = \frac{ta}{\sqrt{3}}, \cdots \\ & \text{if and only if} \quad t = 0 \text{ and } x = 0. \end{aligned}$$

Thus, $H(\cdot,t)$ has a fixed point if and only if $t = 0$. Observe that, in this example, H is not condensing.

Definition 4.1. Let $\{x_n\}$ be a bounded sequence in E. The *asymptotic radius of* $\{x_n\}$ *in* K is the number defined by

$$r(K,\{x_n\}) = \inf_{x\in K} \limsup_{n\to\infty} \|x - x_n\|.$$

Definition 4.2. Let $\{x_n\}$ be a bounded sequence in E. The *asymptotic center of* $\{x_n\}$ *in* K is the (possibly empty) set defined by

$$A(K,\{x_n\}) = \{x \in K : \limsup_{n\to\infty} \|x - x_n\| \leq r(K,\{x_n\})\}.$$

Definition 4.3. A bounded sequence $\{x_n\}$ is said to be *regular relative to* K if $r(K,\{x_n\}) = r(K,\{x_{n_k}\})$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Lemma 4.4. *Each bounded sequence has a subsequence regular relative to* K.

Lemma 4.5. *Let E be a uniformly convex Banach space, K a nonempty, bounded, closed, convex subset of E, and $\{x_n\}$ a bounded sequence in E. Then $A(K,\{x_n\})$ is a singleton.*

Now, we can give the main theorem of this section.

Theorem 4.6. *Let E be a uniformly convex Banach space, U a nonempty, bounded, convex domain of E. Let $F : \overline{U} \to E$ be a multivalued nonexpansive mapping with compact values. Assume there exists $H : \overline{U} \to E$ with closed values such that*

(a) $H(\cdot,1) = F$;
(b) $H(\cdot,0)$ *has a fixed point;*
(c) *for every $t \in [0,1)$, there exists $\alpha < 1$ such that for all x, $y \in \overline{U}$, and all $s \in [0,t]$, $D(H(x,s),H(y,s)) \leq \alpha\|x-y\|$;*
(d) *there exists a continuous function $\phi : [0,1] \to \mathbb{R}$ such that for every $x \in \overline{U}$ and every t, $s \in [0,1]$, $D(H(x,t),H(x,s)) \leq |\phi(t) - \phi(s)|$.*

Then one of the following statement holds:

(i) *F has a fixed point;*
(ii) *there exist $t \in [0,1)$ and $x \in \partial U$ such that $x \in H(x,t)$.*

Proof. Assume (ii) does not hold. By Theorem 3.3, for every $t \in [0,1)$, $H(\cdot,t)$ has a fixed point in U. Thus, we can take sequences $\{t_n\}$ and $\{x_n\}$ such that $x_n \in H(x_n,t_n)$, and $t_n \to 1$. Since F has compact values, the sequence $\{x_n\}$ is bounded and we can assume that it is regular relative to $\overline{U}$ by Lemma 4.4.

Let $\{x\} = A(\overline{U},\{x_n\})$, and $r = r(\overline{U},\{x_n\})$. Choose $y_n \in F(x)$ such that

$$\|x_n - y_n\| \leq D(H(x_n,t_n),F(x)).$$

Since $F(x)$ is compact, $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ converging to $y \in F(x)$. Therefore,

$$\begin{aligned}\|x_{n_k} - y\| &\le \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - y\| \\ &\le |\phi(t_{n_k}) - \phi(1)| + \|x_{n_k} - x\| + \|y_{n_k} - y\|.\end{aligned}$$

So,

$$\limsup_{k\to\infty} \|x_{n_k} - y\| \le \limsup_{k\to\infty} \|x_{n_k} - x\| = r.$$

By Lemma 4.5, this implies that $y = x$. Thus, $x \in F(x)$. $\square$

Corollary 4.7. (Nonlinear Alternative) *Let E be a uniformly convex Banach space, U a nonempty, bounded, convex domain of E. Let $F : \overline{U} \to E$ be a multivalued nonexpansive mapping with compact values such that $F(\overline{U})$ is bounded. Then one of the following statements holds:*

(i) *F has a fixed point;*
(ii) *there exist $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda Fx$.*

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