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12. Fixed Point Results for Multivalued Contractions on Gauge Spaces*

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Abstract: In this chapter, we present a fixed point result for set-valued contractions on complete gauge spaces which generalizes fixed point theorems of Nadler, and of Cain and Nashed. Also, we consider contractions $F: X \to E$ defined on an arbitrary closed subset of a gauge space. We show that the property of having a fixed point is invariant by suitable homotopies. From that, we deduce a fixed point theorem of Leray-Schauder type. Then we present an application of those results to first order differential inclusions on the half line.

1. INTRODUCTION AND PRELIMINARIES

In 1969, S. Nadler [14] established a generalization of the well known Banach Contraction Principle for multivalued contractions with closed, bounded, nonempty values, defined on a complete metric space. One year later, Covitz and Nadler [5] proved that the values don't need to be bounded. On the other hand, in 1971, Cain and Nashed [3] extended the notion of singlevalued contraction to Hausdorff locally convex linear spaces. They showed that on sequentially complete subset, the Banach Contraction Principle is still valid.

In this chapter, we present a generalization of those two results to multivalued contractions defined on a complete gauge space E.

In Section 3, we consider multivalued contractions defined on a closed subset of E. We introduce a notion of homotopy for such contractions. Then we show that the property of having a fixed point is invariant by homotopy. Our result generalizes results of Granas and myself [12,13]. It is worthwhile to mention that even in the singlevalued case, our result can not be obtained from the index theory for condensing operators (see [1,9,15]). Indeed, there is no vectorial structure on E; moreover, the set X can have empty interior. The reader is referred to [4,8,10,11] and the references therein for other results on contractions.

Finally, we present an application to first order differential inclusions on the half line.

In what follows, $E = (E, \{d_\alpha\}_{\alpha\in\Lambda})$ denotes a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$, see [7] for definitions.

For $r = \{r_\alpha\}_{\alpha\in\Lambda} \in]0,\infty[^\Lambda$, and $x \in E$, we define the *pseudo-ball* centered in x of radius r by

$$B(x,r) = \{y \in E : d_\alpha(x,y) \le r_\alpha \text{ for all } \alpha \in \Lambda\}.$$

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For all subset X of E, we denote $\bar{X}$ and ∂X respectively the closure and the boundary of X in E. Also, we denote

$$X_{k,M} = \left\{ x \in X : B(x,r) \not\subset X \ \forall r \in]0,\infty[^\Lambda \text{ with } \inf_{\alpha \in \Lambda} \frac{r_\alpha(1-k_\alpha)}{M_\alpha} > 0 \right\}.$$

for $k \in [0,1[^\Lambda$ and $M \in [0,\infty[^\Lambda$.

Remark: If $X = \bar{U}$ for U an open subset of E, then $X_{k,M} = \partial U$ for every $k \in [0,1[^\Lambda$ and $M \in [0,\infty[^\Lambda$.

We denote by D_α, the generalized Hausdorff pseudometric induced by d_α; that is, for $X, Y \subset E$,

$$D_\alpha(X,Y) = \inf\{\varepsilon > 0 : \forall x \in X, \forall y \in Y, \exists \hat{x} \in X, \exists \hat{y} \in Y \text{ such that } d_\alpha(x,\hat{y}) < \varepsilon, d_\alpha(\hat{x},y) < \varepsilon\},$$

with the convention that $\inf(\emptyset) = \infty$. In the particular case where E is a complete locally convex space, we say that a subset $X \subset E$ is *bounded* if $D_\alpha(\{0\}, X) < \infty$ for every $\alpha \in \Lambda$.

2. FIXED POINT RESULT

In what follows, a multivalued map $F : X \to E$ is a map with closed, nonempty values $F(x) \subset E$.

Definition 2.1: A multivalued map $F : X \to E$ is called an *admissible contraction* with constant $k = \{k_\alpha\}_{\alpha \in \Lambda} \in [0,1[^\Lambda$ if

(i) for every $\alpha \in \Lambda$, $D_\alpha(F(x), F(y)) \leq k_\alpha d_\alpha(x,y)$ for every $x, y \in X$;
(ii) for every $x \in X$ and every $\varepsilon \in]0,\infty[^\Lambda$, there exists $y \in F(x)$ such that $d_\alpha(x,y) \leq d_\alpha(x, F(x)) + \varepsilon_\alpha$ for every $\alpha \in \Lambda$.

Observe that if $\Lambda = \mathbb{N}$, a multivalued map F can be a contraction in the sense of the previous definition without being a contraction in the usual sense when X is endowed with the metric $d(x,y) = \sum_{n \in \mathbb{N}} d_n(x,y)/2^n(1 + d_n(x,y))$.

First of all, we establish a fixed point result for a multivalued contractive map defined on a pseudo-ball, and which does not move its center too far away. It generalizes a result of [12].

Proposition 2.2: *Let E be a complete gauge space, $r \in]0,\infty[^\Lambda$, $x_0 \in E$, and $F : B(x_0, r) \to E$ be an admissible multivalued contraction with constant $k \in]0,1[^\Lambda$ such that $d_\alpha(x_0, F(x_0)) < (1-k_\alpha)r_\alpha$ for every $\alpha \in \Lambda$. Then F has a fixed point.*

Proof: By assumptions and Definition 2.1(ii), we can choose $x_1 \in F(x_0)$ such that

$$d_\alpha(x_1, x_0) < (1-k_\alpha)r_\alpha \quad \text{for every } \alpha \in \Lambda.$$

Then, choose $x_2 \in F(x_1)$ such that for every $\alpha \in \Lambda$,

$$\begin{aligned} d_\alpha(x_1, x_2) &< d_\alpha(x_1, F(x_1)) + k_\alpha((1-k_\alpha)r_\alpha - d_\alpha(x_0, x_1)) \\ &\le D_\alpha(F(x_0), F(x_1)) + k_\alpha((1-k_\alpha)r_\alpha - d_\alpha(x_0, x_1)) \\ &\le k_\alpha d_\alpha(x_0, x_1) + k_\alpha((1-k_\alpha)r_\alpha - d_\alpha(x_0, x_1)) \\ &= k_\alpha(1-k_\alpha)r_\alpha. \end{aligned}$$

In repeating this process, we obtain a sequence $\{x_m\}$ such that

$$d_\alpha(x_m, x_{m+1}) < k_\alpha^m(1-k_\alpha)r_\alpha \quad \text{for every } \alpha \in \Lambda.$$

Therefore, $\{x_m\}$ is a Cauchy sequence and hence converges to $x \in B(x_0, r)$. The continuity of F implies that $x \in F(x)$. □

As a direct consequence of the previous result, we obtain a generalization of Covitz and Nadler's fixed point Theorem [5] and of Cain and Nashed's result [3].

Theorem 2.3: *Let E be a complete gauge space, and let $F: E \to E$ be an admissible multivalued contraction. Then F has a fixed point.*

Proof: Let $k \in [0,1[^\Lambda$ be a constant of contraction of F. Fix $x_0 \in E$. For every $\alpha \in \Lambda$, choose $r_\alpha > 0$ such that $d_\alpha(x_0, F(x_0)) < (1-k_\alpha)r_\alpha$. The conclusion follows from the previous proposition.

3. LERAY-SCHAUDER TYPE RESULTS

We introduce the notion of homotopy of contractions.

Definition 3.1: Let X be a closed subset of E. An *homotopy of admissible contractions* is a multivalued map $H: X \times [0,1] \to E$ such that

(a) there exists $k \in [0,1[^\Lambda$ such that for every $t \in [0,1]$, $H(\cdot, t)$ is an admissible contraction with constant k;
(b) there exits $M \in [0,\infty[^\Lambda$ such that $D_\alpha(H(x,t), H(x,s)) \le M_\alpha|t-s|$ for every s, $t \in [0,1]$, every $x \in X$, and every $\alpha \in \Lambda$;
(c) $x \notin H(x,t)$ for every $t \in [0,1]$, and every $x \in X_{k,M}$.

Remark: If $X = \bar{U}$ for U an open subset of E, since $X_{k,M} = \partial U$ for every $k \in [0,1[^\Lambda$ and $M \in [0,\infty[^\Lambda$, condition (c) becomes: $x \notin H(x,t)$ for every $t \in [0,1]$, and every $x \in \partial U$. Also, it is easy to see that (b) can be generalized. We stated this condition for sake of simplicity.

Definition 3.2: Let X be a closed subset of E. We say that the multivalued admissible contractions $F, G: X \to E$ are *homotopic* if there exists an homotopy of contractions $H: X \times [0,1] \to E$ such that $F = H(\cdot, 1)$ and $G = H(\cdot, 0)$.

We obtain the invariance by homotopy of the property of having a fixed point. It is worthwhile to mention that X can have empty interior. The proof is in fact a slight modification of the proof of [12, Th. 4.3]. We present it for sake of completness.

Theorem 3.3: *Let X be a closed subset of a complete gauge space E, and let $F,G: X \to E$ be two homotopic admissible contractions. Then F has a fixed point if and only if G has a fixed point.*

Proof: Let $H: X \times [01] \to E$ be an homotopy of admissible contractions between F and G. Consider

$$Q = \{(x,t) \in X \times [0,1] : x \in H(x,t)\}.$$

We define on Q the partial order

$$(x,t) \leq (y,s) \quad \text{iff} \quad t \leq s \quad \text{and} \quad d_\alpha(x,y) \leq \frac{2M_\alpha(s-t)}{1-k_\alpha} \quad \text{for every } \alpha \in \Lambda,$$

where k and M are given in the definition of H.

Let $P \subset Q$ be a totally ordered set. Set $t^* = \sup\{t : (x,t) \in P\}$. Let $\{(x_m, t_m)\}$ be a sequence in P such that $(x_m, t_m) \leq (x_{m+1}, t_{m+1})$ and $t_m \to t^*$. The order on Q yealds

$$d_\alpha(x_m, x_l) \leq \frac{2M_\alpha(t_m - t_l)}{1-k_\alpha} \quad \text{for every } m > l \text{ and every } \alpha \in \Lambda.$$

Thus, $\{x_m\}$ is a Cauchy sequence and hence converges to $x^* \in X$. The continuity of H implies that $x^* \in H(x^*, t^*)$, so $(x^*, t^*) \in Q$. It is easy to see the (x^*, t^*) is an upper bound of P.

From Zorn's Lemma, Q has a maximal element $(x_0, t_0) \in Q$, so, $x_0 \in H(x_0, t_0)$.

To conclude, we need to show that $t_0 = 1$. If this is false, since $x_0 \notin X_{k,M}$ by Definition 3.1(c), there exist $r \in [0,\infty[^\Lambda$ and $t_1 \in (t_0, 1]$ such that

$$B(x_0, r) \subset X \quad \text{and} \quad \frac{2M_\alpha(t_1 - t_0)}{1-k_\alpha} = r_\alpha \quad \text{for every } \alpha \in \Lambda.$$

On the other hand, for every $\alpha \in \Lambda$,

$$\begin{aligned} d_\alpha(x_0, H(x_0,t_1)) &\leq d_\alpha(x_0, H(x_0,t_0)) + D_\alpha(H(x_0,t_0), H(x_0,t_1)) \\ &\leq M_\alpha(t_1 - t_0) < (1-k_\alpha)r_\alpha. \end{aligned}$$

From Proposition 2.2, there exists $x_1 \in B(x_0, r)$ a fixed point of $H(\cdot, t_1)$. So, $(x_1, t_1) \in Q$ and $(x_0, t_0) < (x_1, t_1)$, which is a contradiction. □

In the particular case where E is a complete locally convex space, we deduce the following corollary.

Corollary 3.4 (Nonlinear Alternative): *Let X be a closed subset of a complete locally convex space E, and let $F: X \to E$ be an admissible multivalued contraction with constant k. Assume that F is bounded and $0 \in X \backslash X_{k,M}$, where $M \in [0,\infty[^\Lambda$ with $M_\alpha = D_\alpha(\{0\}, F(X))$. Then one of the following statements holds:*

(1) *F has a fixed point;*
(2) *there exist $t \in]0,1[$ and $x \in X_{k,M}$ such that $x \in tF(x)$.*

This corollary can be stated more simply for contractions defined on the closure of an open set.

Corollary 3.5: *Let U be an open neighborhood of the origin in a complete locally convex space space E, and let $F : \bar{U} \to E$ be an admissible multivalued contraction. Assume that F is bounded. Then one of the following statements holds:*

(1) *F has a fixed point;*
(2) *there exist $t \in]0,1[$ and $x \in \partial U$ such that $x \in tF(x)$.*

4. APPLICATION

Let us consider the first order differential inclusion

$$\begin{aligned} &x'(t) \in f(t, x(t)) \quad \text{a.e. } t \in [0, \infty[, \\ &x(0) = 0 \in H, \end{aligned} \tag{4.1}$$

where H is an Hilbert space, and $f : [0, \infty[\times H \to H$ is a locally Carathéodory multivalued map with closed, bounded, nonempty values; i.e., satisfying:

(i) $t \mapsto f(t, x)$ is measurable for all $x \in H$ ($\{t : f(t, x) \cap C \neq \emptyset\}$ is measurable for every closed subset C);
(ii) $x \mapsto f(t, x)$ is continous for almost every $t \in [0, \infty[$ (here 2^H is endowed with the Hausdorff metric D);
(iii) for all $R > 0$, there exists a function $h_R \in L^1_{\text{loc}}[0, \infty[$ such that for almost every $t \in [0, \infty[$ and for every $x \in H$ with $\|x\| \leq R$, we have $D(\{0\}, f(t, x)) \leq h_R(t)$.

Theorem 4.1: *Let $(H, \|\cdot\|)$ be an Hilbert space, and $f : [0, \infty[\times H \to H$ a locally Carathéodory multivalued map with closed, bounded, nonempty values. Assume that*

(a) *for every $R > 0$, there exists $l_R \in L^1_{\text{loc}}[0, \infty[$ such that for almost every $t \in [0, \infty[$, and every $x, y \in H$ satisfying $\|x\|, \|y\| \leq R$, we have*

$$D(f(t, x), f(t, y)) \leq l_R(t)\|x - y\|;$$

(b) *there exist $\theta \in L^1_{\text{loc}}[0, \infty[$ and $\psi : [0, \infty[\to]0, \infty[$ a Borel measurable function such that $D(\{0\}, f(t, x)) \leq \theta(t)\psi(\|x\|)$ a.e. $t \in [0, \infty[$, and all $x \in H$, with $1/\psi \in L^1_{\text{loc}}[0, \infty[$; and*

$$\int_0^\infty \frac{dz}{\psi(z)} > \|\theta\|_{L^1[0,r]} \quad \textit{for all } r > 0.$$

Then the problem (4.1) has a solution in $W^{1,1}_{\text{loc}}([0, \infty[, H)$.

Proof: Let $M : [0, \infty[\to [0, \infty[$ be a continuous nondecreasing function such that

$$\int_0^{M(t)} \frac{ds}{\psi(s)} \geq \|\theta\|_{L^1[0,t]}.$$

Let us define $\hat{f} : [0, \infty[\times H \to H$ by

$$\hat{f}(t,x) = \begin{cases} f(t,x), & \text{if } \|x\| \le M(t), \\ f(t, \frac{M(t)x}{\|x\|}), & \text{if } \|x\| > M(t); \end{cases}$$

and define $F : C([0,\infty[, H) \to C([0,\infty[, H)$ by

$$F(x)(t) = \int_0^t \hat{f}(s, x(s))ds.$$

Using assumption (b) and the definition of M, one shows that if x is a fixed point of F then x is a solution of (4.1).

Set $l(t) = l_{M(n)}(t)$ pour $t \in]n-1, n]$, $n \in \mathbb{N}$, where $l_{M(n)}$ is given in (a). For each $n \in \mathbb{N}$, we define on $C([0,\infty[, H)$ the semi-norm

$$|x|_n = \sup\{e^{-\int_0^t l(s)ds} \|x(t)\| : t \in [0,n]\}.$$

It follows directly from assumption (a) and the theory of multivalued maps (see [2]) that F is an admissible contraction. Theorem 2.3 gives the existence of solution of (4.1).

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