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# HEAT EQUATIONS WITH DISCONTINUOUS NONLINEARITIES ON CONVEX AND NONCONVEX CONSTRAINTS

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## INTRODUCTION

IN THIS paper we use a variational method for solving some semilinear parabolic equations with a discontinuous nonlinearity, possibly on either some convex or nonconvex constraints. The approach is based on the fact that the solutions of the above-mentioned problems can be viewed as "steepest descent curves", in a suitable sense to be specified, for some lower semicontinuous functionals, possibly restricted to suitable constraints. This abstract framework, which is presented in Section 1, seems interesting to us in that it provides a unifying tool for treating various kinds of constrained problems, including cases in which the constraint is not convex (see Section 4); moreover the existence theorem that we get holds under reasonably weak assumptions. All these ideas in great part originated from the paper [10], where a general framework for variational evolution was proposed, and are also related to the theory of maximal monotone operators (see [4]) and some of its extensions (see [5, 6, 9, 11-15, 17, 19-24] for some applications), the main difference being in the fact that, using compactness, we find existence theorems without uniqueness.

The applications presented can be described, roughly speaking, as follows: given an open set $\Omega \subset \mathbf{R}^N$ and $g: \Omega \times \mathbf{R} \to \mathbf{R}$, possibly discontinuous, we search for $\mathcal{U}: [0, T[ \to L^2(\Omega)$ which solve

$$\left.\begin{aligned} &\mathcal{U}(t) \in H_0^1(\Omega) \qquad \forall\, t \text{ in } I \text{ and a.e. in } I: \\ &\mathcal{U}'(t) = \Delta\mathcal{U}(t) + g(\mathcal{U}(t)); \end{aligned}\right\} \tag{P.1}$$

$$\left.\begin{aligned} &\mathcal{U}(t) \in H_0^1(\Omega), \mathcal{U}(t) \geq \varphi && \text{a.e. in } \Omega\ \forall\, t \text{ in } I \text{ and a.e. in } I: \\ &\mathcal{U}'(t) = \Delta\mathcal{U}(t) + g(\mathcal{U}(t)) && \text{a.e. in } \{x \mid \mathcal{U}(t)(x) > \varphi(x)\}, \\ &\mathcal{U}'(t) = [\Delta\mathcal{U}(t) + g(\mathcal{U}(t))]^+ && \text{a.e. in } \{x \mid \mathcal{U}(t)(x) = \varphi(x)\}, \end{aligned}\right\} \tag{P.2}$$

(here the convex constraint $K = \{u \geq \varphi\}$ is involved);

$$\left.\begin{array}{ll} \mathcal{U}(t) \in H_0^1(\Omega),\ \mathcal{U}(t) \geq \varphi & \text{a.e. in } \Omega,\ \displaystyle\int_\Omega \mathcal{U}(t)^2\,dx = \rho^2\ \forall\, t \text{ in } I \\ \text{and there exists } \Lambda\colon I \to \mathbf{R} \text{ such that a.e. in } I\colon & \\ \mathcal{U}'(t) = \Delta\mathcal{U}(t) + g(\mathcal{U}(t)) + \Lambda(t)\mathcal{U}(t) & \text{a.e. in } \{x \mid \mathcal{U}(t)(x) > \varphi(x)\}, \\ \mathcal{U}'(t) = [\Delta\mathcal{U}(t) + g(\mathcal{U}(t)) + \Lambda(t)\mathcal{U}(t)]^+ & \text{a.e. in } \{x \mid \mathcal{U}(t)(x) = \varphi(x)\} \end{array}\right\} \quad \text{(P.3)}$$

(here the additional nonconvex constraint $S_\rho = \{\int_\Omega u^2\,dx = \rho^2\}$ is considered).

In Sections 2, 3 and 4 the precise meanings of the above problems are given and some existence theorems are proved (see theorems 2.13, 3.7 and 4.7) by finding the "curves of maximal relaxed slope" (see definition 1.3) associated with the functional

$$f(u) = \tfrac{1}{2}\int_\Omega |Du(x)|^2\,dx + \int_\Omega \int_0^{u(x)} g(x, s)\,ds$$

with no constraints [for (P.1)], subjected to the condition $u \in K$ [for (P.2)] or to the condition $u \in K \cap S_\rho$ [for (P.3)].

Problems (P.1) and (P.2) were already treated by Shi Shuzhong in [26], with techniques of differential inclusion (see [2, 25]). We are mainly interested in solving precise equations (so we have the assumption (g.2)); moreover the variational approach allows more general growth conditions for the nonlinearity (see assumption (g.1)). The results presented in Section 4, which were suggested by [7], are new as far as we know.

## 1. THE CURVES OF MAXIMAL RELAXED SLOPE AND GENERALIZED EVOLUTION EQUATIONS

The concepts defined in this section are set in a Hilbert space structure; a lot of them could be as well considered just in a metric space, as in [20].

Let $H$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let $f\colon H \to \mathbf{R} \cup \{+\infty\}$ be a given function. We set $\mathfrak{D}(f) = \{u \in H \mid f(u) < +\infty\}$.

If $u, v \in \mathfrak{D}(f)$, we consider the "graph distance" between $v$ and $u$ as $d^*(v, u) = \|v - u\| + |f(v) - f(u)|$ and denote by $\mathfrak{D}(f)^*$ the metric space $\mathfrak{D}(f)$ endowed with the metric $d^*$. We also frequently use the following notations: if $R > 0$ we set:

$$B(u, R) = \{v \in H \mid \|v - u\| \leq R\} \qquad \text{for } u \text{ in } H,$$

$$B^*(u, R) = B(u, R) \cap \{v \in \mathfrak{D}(f) \mid f(v) \leq f(u)\} \qquad \text{for } u \text{ in } \mathfrak{D}(f).$$

We recall the definition of slope (see [20]).

*Definition* 1.1. Let $u \in \mathfrak{D}(f)$. The "slope of $f$ at $u$", denoted by $|\nabla f|(u)$, is defined as:

$$|\nabla f|(u) = -\liminf_{v \to u}\left(\frac{f(v) - f(u)}{\|v - u\|} \wedge 0\right).$$

*Definition* 1.2. Let $u \in \mathfrak{D}(f)$. The "relaxed slope of $f$ at $u$", denoted by $\overline{|\nabla f|}(u)$, is defined as:

$$\overline{|\nabla f|}(u) = \sup\{\varphi(u) \mid \varphi\colon \mathfrak{D}(f)^* \to \mathbf{R} \text{ is continuous, } \varphi \leq |\nabla f|\}.$$

It is clear that $\overline{|\nabla f|}\colon \mathfrak{D}(f)^* \to \mathbf{R} \cup \{+\infty\}$ is a lower semicontinuous function such that $\overline{|\nabla f|} \leq |\nabla f|$.

*Definition* 1.3. Let $I$ be an interval with nonempty interior and $\mathcal{U}: I \to H$ be a curve. We say that $\mathcal{U}$ is a "curve of maximal relaxed slope almost everywhere for $f$", if there exists a negligible subset $E$ of $I$ such that:

(a) $\mathcal{U}$ is continuous on $I$;

(b) $f \circ \mathcal{U}(t) < +\infty \ \forall\, t \in I \backslash E$, $f \circ \mathcal{U}(t) \leq f \circ \mathcal{U}(\min I) \ \forall\, t \in I \backslash E$ if $I$ has a minimum;

(c) $\|\mathcal{U}(t_2) - \mathcal{U}(t_1)\| \leq \int_{t_1}^{t_2} \overline{|\nabla f|}(\mathcal{U}(t))\, dt \ \forall\, t_1, t_2 \in I$ with $t_1 \leq t_2$;

(d) $f \circ \mathcal{U}(t_2) - f \circ \mathcal{U}(t_1) \leq -\int_{t_1}^{t_2} (\overline{|\nabla f|}(\mathcal{U}(t)))^2\, dt \ \forall\, t_1, t_2 \in I \backslash E$ with $t_1 \leq t_2$.

If $E = \emptyset$, then we say that $\mathcal{U}$ is a curve of maximal relaxed slope for $f$.

*Remark* 1.4. It is straightforward to see, as in [20], that, if $\mathcal{U}$ is a curve of maximal relaxed slope almost everywhere for $f$, then

(a) $\mathcal{U}$ is absolutely continuous on any compact subintervals of $I \backslash \{\inf I\}$ (of $I$ if $I$ has a minimum and $f \circ \mathcal{U}(\min I) < +\infty$) and

$$\|\mathcal{U}'(t)\| \leq \overline{|\nabla f|}(\mathcal{U}(t)) \qquad \text{a.e. in } I;$$

(b) there exists a nonincreasing function $m: I \to \mathbf{R} \cup \{+\infty\}$, which is almost everywhere equal to $f \circ \mathcal{U}$, such that:

$$m'(t) \leq -(\overline{|\nabla f|}(\mathcal{U}(t)))^2 \qquad \text{a.e. in } I.$$

*Definition* 1.5. Let $W \subset H$. We say that $f$ is "coercive on $W$", if the set $\{v \in H \mid f(v) \leq C\} \cap W$ is compact for every $C$ in $\mathbf{R}$.

Let $u \in \mathcal{D}(f)$. We say that $f$ is "coercive at $u$", if there exists $R > 0$ such that $f$ is coercive on the set $B^*(u, R)$.

We say that $f$ is locally coercive, if $f$ is coercive at every $u$ in $\mathcal{D}(f)$.

*Definition* 1.6. We say that $f$ is $\nabla$-continuous, if for every $u$ in $\mathcal{D}(f)$, for every sequence $(u_k)_k$ converging to $u$ such that $\sup_k f(u_k) < +\infty$ and $\sup_k |\nabla f|(u_k) < +\infty$, one has:

$$\lim_{k \to \infty} f(u_k) = f(u).$$

*Definition* 1.7. We say that $f$ is $d\overline{\nabla}$-continuous, if for all $u$ in $\mathcal{D}(f)$ and all sequences $(u_k)_k$ converging to $u$ with $\sup_k f(u_k) < +\infty$ and $\lim_{k \to \infty} \overline{|\nabla f|}(u_k)\|u_k - u\| = 0$, one has:

$$\lim_{k \to \infty} f(u_k) = f(u).$$

The proof of the following two theorems are essentially contained in [20].

THEOREM 1.8. (Existence.) Let $u_0 \in \mathcal{D}(f)$ and suppose that:

(a) $f$ is coercive at $u_0$;

(b) $f$ is $\nabla$-continuous.

Then there exist $T > 0$ and $\mathcal{U}: [0, T[ \to H$, a curve of maximal relaxed slope almost everywhere for $f$, such that $\mathcal{U}(0) = u_0$, $f \circ \mathcal{U}(t) \leq f(u_0) \ \forall\, t$ and $f \circ \mathcal{U}$ is lower semicontinuous.

*Proof.* This statement is precisely the first part of [20, theorem 4.10]: for the proof simply observe that $\nabla$-continuity implies the assumption (b) of that theorem.

THEOREM 1.9. Suppose that $f$ is $d\bar{\nabla}$-continuous and let $\mathcal{U}: I \to H$ be a curve of maximal relaxed slope almost everywhere for $f$.

Then $f \circ \mathcal{U}$ is continuous and nonincreasing.

*Proof.* This is a "relaxed version" of [20, lemma (3.10)]. For the proof it suffices to repeat all the arguments carried on in the proofs of [20, (3.9) (b) and (3.10)], just replacing the slope with the relaxed slope and to remark that $d\bar{\nabla}$-continuity provides the relaxed version of the condition of [20, (3.11)].

The following proposition individuates a class of functions to which the previous theorems apply.

PROPOSITION 1.10. Assume that $f$ satisfies the following inequality:

$$\begin{gathered} f(v) \geq f(u) - \Phi(u, v, |f(u)|, |f(v)|, |\nabla f|(u))\|v - u\| \\ \forall\, u, v \in \mathcal{D}(f) \text{ with } |\nabla f|(u) < +\infty \end{gathered} \tag{1.1}$$

where $\Phi: \mathcal{D}(f)^2 \times \mathbf{R}^3 \to \mathbf{R}$ is a function which is bounded on bounded subsets.

Then $f$ is $\nabla$-continuous.

Furthermore, if $\Phi(u, v, p_1, p_2, p) = \Phi_0(u, v, p_1, p_2)(1 + p)$, with $\Phi_0$ bounded on bounded subsets, then $f$ is $d\bar{\nabla}$-continuous.

So, if in addition $f$ is coercive, we can apply the previous theorems to get existence and regularity of a curve of maximal relaxed slope for $f$.

*Proof.* The fact that (1.1) implies $\nabla$-continuity of $f$ is trivial. For the second part just observe that from (1.1) we can deduce:

$$f(v) \geq f(u) - \tilde{\Phi}_0(u, v, |f(u)|, |f(v)|)(1 + \overline{|\nabla f|}(u))\|v - u\|$$

for another suitable $\tilde{\Phi}_0$ bounded on bounded subsets, which implies $d\bar{\nabla}$-continuity.

PROPOSITION 1.11. Suppose that $f = f_0 + \mathcal{G}$, where $f_0: H \to \mathbf{R} \cup \{+\infty\}$ is a convex lower semicontinuous function and $\mathcal{G}: \mathcal{D}(f_0) \to \mathbf{R}$ satisfies the following inequalities:

$$\left.\begin{array}{ll} |\mathcal{G}(u)| \leq K(u) + \mu |f_0(u)| & \forall\, u \in \mathcal{D}(f) \\ |\mathcal{G}(v) - \mathcal{G}(u)| \leq L(u, v, |f_0(u)|, |f_0(v)|)\|v - u\| & \forall\, u, v \in \mathcal{D}(f) = \mathcal{D}(f_0) \end{array}\right\} \tag{1.2}$$

where $K: \mathcal{D}(f) \to \mathbf{R}$, $L: \mathcal{D}(f)^2 \times (\mathbf{R}^+)^2 \to \mathbf{R}$ are continuous functions and $\mu \in [0, 1[$. For instance $\mathcal{G}$ could be a locally Lipschitzian function.

Then $f$ is $d\bar{\nabla}$-continuous.

*Proof.* Fix $\mu_0$ in $\mathcal{D}(f)$; by (1.2) and the lower semicontinuity of $f_0$, it is easy to prove that

$$\begin{cases} f_0(u) \leq \dfrac{1}{1 - \mu} f(v) + C, \\ |\nabla f_0|(u) \leq |\nabla f|(u) + L(u, u, |f_0(u)|, |f_0(u)|) \end{cases} \qquad \forall\, u \in B(u_0, R)$$

for suitable $R, C > 0$. Since $f_0$ is convex, it verifies (1.1) with $\Phi(u, v, p_1, p_2, p) = p$; it follows, from the second inequality in (1.2) and the two previous ones, that $f$ satisfies (1.1) in $B(u_0, R)$ with $\Phi(u, v, p_1, p_2, p) = p + 2L(u, v, p_1/(1 - \mu) + C, p_2/(1 + \mu) + C)$.

Now we are going to show that, under suitable assumptions, a curve of maximal relaxed slope for $f$ solves an equation of the type:

$$\mathcal{U}' = -\operatorname{grad} f \circ \mathcal{U}.$$

To this aim the following lemma is a crucial point.

LEMMA 1.12. Let $\mathcal{U}: I \to H$ be a curve of maximal relaxed slope almost everywhere for $f$ and set $I' = \{t \in I \mid \overline{|\nabla f|}(\mathcal{U}(t)) < +\infty\}$. Let $A: I' \to H$ be an operator which satisfies the following properties:

(a) $\|A(t)\| \le \overline{|\nabla f|}(\mathcal{U}(t))$ a.e. in $I$;

(b) $D_+(f \circ \mathcal{U})(t) \ge \langle A(t), \mathcal{U}'(t)\rangle$ a.e. in $I$

($D_+$ denotes the right-lower derivative). Then for almost every $t$ in $I$ we have:

$$\begin{cases} \mathcal{U}'(t) = -A(t), \\ m'(t) = -\|A(t)\|^2 \end{cases}$$

where $m: I \to \mathbf{R} \cup \{+\infty\}$ is a nondecreasing function equivalent to $f \circ \mathcal{U}$ as in (1.4).

*Proof.* We recall that $\operatorname{meas}(I \setminus I') = 0$. From (a) and (b) of remark (1.4) we get:

$$\|\mathcal{U}'(t)\| \le \overline{|\nabla f|}(\mathcal{U}(t)) \qquad \text{a.e. in } I,$$
$$D^+ m(t) \le -(\overline{|\nabla f|}(\mathcal{U}(t)))^2 \qquad \text{a.e. in } I.$$

Then, for almost every $t$ in $I'$, we have:

$$\begin{aligned} -(\overline{|\nabla f|}(\mathcal{U}(t)))^2 &\ge D^+ m(t) \ge D_+(f \circ \mathcal{U})(t) \ge \langle A(t), \mathcal{U}'(t)\rangle \\ &\ge -\|A(t)\|\,\|\mathcal{U}(t)\| \ge -(\overline{|\nabla f|}(\mathcal{U}(t)))^2. \end{aligned}$$

It follows:

$$\langle \mathcal{U}'(t), A(t)\rangle = -\|A(t)\|\,\|\mathcal{U}(t)\| = -(\overline{|\nabla f|}(\mathcal{U}(t)))^2.$$

Then we have $\|\mathcal{U}'(t)\|^2 = (\overline{|\nabla f|}(\mathcal{U}(t)))^2 = \|A(t)\|^2 = -m'(t)$ and hence

$$\mathcal{U}'(t) = -A(t).$$

The remainder of the section is devoted to finding "good candidates" for the operator $A$. In several situations (e.g. in the $\varphi$-convex context: see [14]) the concept of subgradient is well fitted to this aim. We recall the definition (see [10]).

*Definition* 1.13. If $u \in \mathcal{D}(f)$, we call "subdifferential of $f$ at $u$" the set:

$$\partial^- f(u) = \left\{\alpha \in H \,\middle|\, \liminf_{v \to u} \frac{f(v) - f(u) - \langle \alpha, v - u\rangle}{\|v - u\|} \ge 0\right\}.$$

As well known $\partial^- f(u)$ is a closed and convex subset of $H$ (possibly empty). If $\partial^- f(u) \ne \emptyset$, then we can define the "subgradient of $f$ at $u$" as the element $\operatorname{grad}^- f(u)$ in $\partial^- f(u)$ which has minimal norm.

It is quite simple to check that $A(\mathcal{U}(t)) = \operatorname{grad}^- f \circ \mathcal{U}$ satisfies (b) of lemma 1.12; in general (and in the cases that we want to treat) it does not satisfy (a); for this reason we are led to introduce a larger set than $\partial^- f(u)$.

*Definition* 1.14. We define the multivalued map $\mathcal{A}(f)\colon \mathcal{D}(f) \to 2^H$ by:

$$\alpha \in \mathcal{A}(f)(u) \Leftrightarrow \begin{cases} \text{there exist a sequence } (u_k)_k \text{ in } \mathcal{D}(f), \text{ such that} \\ \lim\limits_{k\to\infty} u_k = u, \qquad \lim\limits_{k\to\infty} f(u_k) = f(u) \\ \text{and a sequence } (\alpha_k)_k \text{ in } H \text{ such that} \\ \alpha_k \in \partial^- f(u_k)\ \forall\, k, \qquad \alpha_k \to \alpha \text{ weakly in } H. \end{cases}$$

We have the following result.

PROPOSITION 1.15. Let $f$ be locally coercive. Then for all $u$ in $\mathcal{D}(f)$ with $\overline{|\nabla f|}(u) < +\infty$, one has:
(a) there exists $\alpha$ in $\mathcal{A}(f)(u)$ such that $\|\alpha\| \le \overline{|\nabla f|}(u)$;
(b) $\overline{|\nabla f|}(u) = \liminf\limits_{\substack{v\to u \\ \partial^- f(v)\neq\varnothing}} \|\mathrm{grad}^- f(v)\|$.

*Proof.* Let $(u_k)_k$ be a sequence such that $u_k \to u$, $f(u_k) \to f(u)$ and

$$|\nabla f|(u_k) \to \overline{|\nabla f|}(u).$$

Fix $k$ integer, then arguing as in the proof of [20, lemma 5.5, part (c)], we can find $u_k'$ such that

$$\|u_k' - u_k\| \le \frac{1}{k}, \qquad f(u_k') \le f(u_k)$$

$$\partial^- f(u_k') \neq \varnothing, \qquad \|\mathrm{grad}^- f(u_k')\| \le \left(1 + \frac{1}{k}\right) |\nabla f|(u_k).$$

It follows that $u_k' \to u$, $f(u_k') \to f(u)$ and $\mathrm{grad}^- f(u_k') \to \alpha \in H$ weakly (passing to a subsequence). Then $\alpha \in \mathcal{A}(f)(u)$ and by the weak lower semicontinuity of the norm, we have:

$$\|\alpha\| \le \liminf_{k\to\infty} \|\mathrm{grad}^- f(u_k')\| \le \overline{|\nabla f|}(u).$$

But, since $\|\mathrm{grad}^- f(u_k')\| \ge |\nabla f|(u_k')$, we also get (b).

To state the main theorem we need another definition.

*Definition* 1.16. Let $\mathcal{A}\colon \mathcal{D}(f) \to 2^H$ be a multivalued map. We say that $\mathcal{A}$ is a "subdifferential along curves" for $f$ if:

$$\begin{cases} \text{for all absolute continuous curves } \mathcal{U}\colon I \to X \text{ such that} \\ \sup\limits_{t\in I} f\circ\mathcal{U}(t) < +\infty, \qquad \mathcal{A}(\mathcal{U}(t)) \neq \varnothing \text{ a.e. in } I \\ \text{one has for almost all } t \text{ in } I \\ D_+(f\circ\mathcal{U})(t) \ge \langle \alpha, \mathcal{U}'(t)\rangle \qquad \forall\, \alpha \text{ in } \mathcal{A}(\mathcal{U}(t)). \end{cases}$$

In the next theorem we essentially require that $\mathcal{A}(f)$ is a subdifferential along curves for $f$; however, since the explicit determination of $\mathcal{A}(f)$ may be nontrivial, the use of a larger (and easier to compute) $\mathcal{A}$ may be useful.

THEOREM 1.17. Suppose that $f$ is locally coercive and let $\mathcal{A}: \mathcal{D}(f) \to 2^H$ be a multivalued map such that:

$$\begin{cases} \mathcal{A}(f)(u) \subset \mathcal{A}(u) & \forall\, u \in \mathcal{D}(f); \\ \mathcal{A} \text{ is a subdifferential along curves for } f. \end{cases}$$

Then, if $\mathcal{U}: I \to H$ is a curve of maximal relaxed slope almost everywhere for $f$, one has for almost every $t$ in $I$:

(a) $\mathcal{A}(\mathcal{U}(t)) \neq \emptyset$;

(b) $\mathcal{A}(\mathcal{U}(t))$ has a unique minimal section (namely a unique element of minimal norm), which we denote by $A(\mathcal{U}(t))$; moreover $\mathcal{A}(\mathcal{U}(t))$ belongs to $\mathcal{A}(f)(\mathcal{U}(t))$ and is therefore the unique minimal section of $\mathcal{A}(f)(\mathcal{U}(t))$;

(c) the equations

$$\begin{cases} \mathcal{U}'(t) = -A(\mathcal{U}(t)), \\ m'(t) = -\|A(\mathcal{U}(t))\|^2 \end{cases}$$

hold ($m$ is as in lemma 1.12).

*Proof.* Since $\overline{|\nabla f|}(\mathcal{U}(t)) < +\infty$ a.e. in $I$, we have (a). By proposition 1.15 we can define $A(\mathcal{U}(t))$ almost everywhere in $I$ as an element with norm less or equal to $\overline{|\nabla f|}(\mathcal{U}(t))$ (there exists at least one of such elements). Since $\mathcal{A}$ is a subdifferential along curves, then $A \circ \mathcal{U}$ satisfies the assumptions of lemma 1.12, hence (c) holds. Moreover, this states that, for a.e. $t$, $A(\mathcal{U}(t))$ is uniquely determined; to see that $A(\mathcal{U}(t)) \in \mathcal{A}(f)(\mathcal{U}(t))$ it suffices to replace $\mathcal{A}$ with $\mathcal{A}(f)$ in the previous arguments.

Now we want to study functionals restricted to some constraints. For this we consider a smooth surface $M$ defined as

$$M = \{v \in W \mid \gamma(v) = 0,\ \mathrm{d}\gamma(v) \neq 0\},$$

where $W \subset H$ is an open set and $\gamma: W \to \mathbf{R}$ is a $C^1$ function. For $u \in M$ we denote by $N(u)$ the normal space to $M$ at $u$ (which has dimension one). Furthermore we consider the function $I_M: H \to \mathbf{R} \cup \{+\infty\}$ defined by

$$I_M(u) = \begin{cases} 0 & \text{if } u \in M, \\ +\infty & \text{if } u \in H \setminus M. \end{cases}$$

It should be as well possible to treat a wider class of constraints, but this is not necessary to our present aims.

The main result in constrained problems is the following lemma.

LEMMA 1.18. Let $f: H \to \mathbf{R} \cup \{+\infty\}$ be locally coercive and $M$, $\gamma$ as above.

Let $u_0 \in \mathcal{D}(f)$, $R_0 > 0$ and suppose that there exist two constants $C_f, C_\psi \geq 0$ and a map $\psi: B^*(u_0, R_0) \to \mathcal{D}(f) \cap M$ such that:

$$\left.\begin{aligned} & f(v) \geq f(\psi(v)) - C_f \|\psi(v) - v\|, \\ & \|\psi(v) - v\| \leq C_\psi \operatorname{dist}(v, M) \end{aligned} \quad \forall\, v \in B^*(u_0, R_0). \right\} \tag{M.f}$$

Then there exist $R \leq R_0$ and constant $K$ (depending on $u_0, C_f, C_\psi$) such that, for all $u$ in $B^*(u_0, R) \cap M$, for all $\alpha \in \partial^-(f + I_M)(u)$:

$$\begin{cases} \text{there exist a sequence } (u_k)_k \text{ in } \mathfrak{D}(f) \text{ such that:} \\ \lim\limits_{k \to \infty} u_k = u, \qquad \lim\limits_{k \to \infty} f(u_k) = f(u), \\ \text{a sequence } (\alpha'_k)_k \text{ in } H \text{ and } \nu \in N(u) \text{ such that:} \\ \alpha'_k \in \partial^- f(u_k)\ \forall\, k, \qquad \lim\limits_{k \to \infty} \alpha'_k = \alpha + \nu; \\ \text{moreover we have the inequality:} \\ \|\alpha'_k\| \leq K(1 + \|\alpha\|)\ \forall\, k. \end{cases}$$

*Proof.* Let $R \leq R_0$, $u \in B^*(u_0, R)$ and take $\rho > 0$ such that $B(u, \rho) \subset B(u_0, R)$ and $f$ is bounded below in $B^*(u, \rho)$ ($f$ is coercive at $u$). Let $\alpha \in \partial^-(f + I_M)(u)$, it is easy to see that, possibly reducing $\rho$, there exists a function $\omega\colon [0, 2\rho] \to [0, 2\rho]$ of class $C^1$ such that $\omega(0) = \omega'(0) = 0$, $\omega$ is 1-Lipschitz continuous and the function $h$ defined by

$$h(v) = f(v) - \langle \alpha, v - u\rangle + \omega(\|v - u\|)$$

restricted on $M \cap B(u, \rho)$ has a unique strict minimum at $v = u$.

Let $\lambda \geq 0$, since $f$ is locally coercive, then there exists $u_\lambda$, a minimizer in $B(u, \rho)$ for the function $v \mapsto h(v) + \lambda\gamma^2(v)$. Then:

$$h(u_\lambda) + \lambda\gamma^2(u_\lambda) \leq h(v) \qquad \forall\, v \in M \cap B(u, \rho). \tag{1.3}$$

It can be easily deduced, by coerciveness, that there exists $u' = \lim\limits_{\lambda \to +\infty} u_\lambda$, and going to the limit in (1.3), we get $\gamma(u') = 0 \Leftrightarrow u' \in M$. From (1.3) again we get that $u' = u$ (because $u$ is the unique strict minimum in $M \cap B(u, \rho)$) and $\lim\limits_{\lambda \to +\infty} f(u_\lambda) = f(u)$.

Then, for $\lambda$ large, $u_\lambda \in \operatorname{int}(B(u, \rho))$, which implies:

$$\alpha'_\lambda = \alpha - \lambda \operatorname{grad} \gamma^2(u_\lambda) + \omega'(\|u_\lambda - u\|)\frac{u_\lambda - u}{\|u_\lambda - u\|} \in \partial^- f(u_\lambda).$$

Now we want to estimate $\|\lambda \operatorname{grad} \gamma^2(u_\lambda)\|$: we have from (1.3) and (M.f)

$$\begin{aligned} \lambda\gamma^2(u_\lambda) &\leq f(\psi(u_\lambda)) - f(u_\lambda) - \langle \alpha, \psi(u_\lambda) - u_\lambda\rangle + \omega(\psi(u_\lambda)) - \omega(u_\lambda) \\ &\leq (C_f + \|\alpha\| + 1)C_\psi \operatorname{dist}(u_\lambda, M). \end{aligned}$$

Since $\gamma$ is $C^1$ and $d\gamma(u_0) \neq 0$, we get that, if $R$ is taken small enough, then there exist $\varepsilon > 0$, $C' \geq 0$ such that

$$\gamma(v) \geq \varepsilon \operatorname{dist}(v, M), \qquad \|\operatorname{grad} \gamma(v)\| \leq C' \Rightarrow \|\operatorname{grad} \gamma(v)\| \leq \frac{C'}{\varepsilon}\frac{\gamma(v)}{\operatorname{dist}(v, M)}$$

for all $v$ in $B(u_0, R)$. It follows:

$$\lambda\|\operatorname{grad} \gamma^2(u_\lambda)\| = 2\lambda|\gamma(u_\lambda)|\,\|\operatorname{grad} \gamma(u_\lambda)\| \leq \frac{2C'}{\varepsilon} C_\psi(C_f + \|\alpha\| + 1).$$

Then, we conclude that, for a sequence $(\lambda_k)_k$ going to $+\infty$:

$$\lambda_k \operatorname{grad} \gamma^2(u_{\lambda_k}) \to \nu \in N(u)$$

and the remaining part is trivial, since $\alpha'_\lambda \to \alpha + \nu$.

COROLLARY 1.19. If $f$, $M$ and $u_0$ are as in the assumptions of lemma 1.18, then there exists $R > 0$ such that

$$\mathcal{Q}(f + I_M)(u) \subset \mathcal{Q}(f)(u) + N(u) \qquad \forall\, u \in B^*(u_0, R).$$

*Proof*. Use the definition of $\mathcal{Q}(f + I_M)(u)$ and apply a standard diagonalization technique to the result of lemma 1.18.

THEOREM 1.20. Let $M$ be a $C^1$-surface (namely a manifold of codimension 1), $f_0 : H \to \mathbf{R} \cup \{+\infty\}$ be a convex lower semicontinuous function and $\mathcal{G}$: $\mathfrak{D}(f_0) \to \mathbf{R}$ a function satisfying (1.2). Set $f = f_0 + \mathcal{G}$ and suppose that $f$ is locally coercive.

Let $u_0 \in \mathfrak{D}(f) \cap M$ be a point such that

$$\mathfrak{D}(f) \text{ and } M \text{ are not tangent at } u_0$$

in the sense that the tangent plane to $M$ at $u_0$ is not tangent to $\mathfrak{D}(f)$ (which is a convex set). Then:

(a) there exists $R > 0$ such that, for all $u$ in $B^*(u_0, R)$

$$\mathcal{Q}(f + I_M)(u) \subset \mathcal{Q}(f)(u) + N(u);$$

(b) if $\mathcal{Q}$ is a multivalued map satisfying the assumptions of theorem 1.17 with respect to $f$, then the multivalued map $\mathcal{Q} + N$ satisfies the same assumptions with respect to $f + I_M$ in $B^*(u_0, R)$, for a suitable $R > 0$.

*Proof*. The proof of (b) is an immediate consequence of (a): just observe that subdifferentiability along curves behaves well with respect to the restriction to a constraint, since, if $\mathcal{U}$ lies in $M$, then $\mathcal{U}'$ is tangent to $M$.

Let us prove (a). Take $u_0 \in \mathfrak{D}(f)$, arguing as in proposition 1.11 we can find $R, L > 0$ such that $\mathcal{G}$ is $L$-Lipschitzian in $B^*(u_0, R)$. We define $\psi$: $B^*(u_0, R) \to \mathfrak{D}(f) \cap M$ satisfying (M.f) of lemma 1.18. Since $M$ and $\mathfrak{D}(f)$ are not tangent at $u_0$ (hence at any $u$ of $B(u_0, R)$, if $R$ is small), then we can find $u^+$ and $u^-$ in $\mathfrak{D}(f)$ such that, possibly reducing $R$

$$\gamma(u^+) > 0, \qquad \gamma(u^-) < 0,$$

$$d\gamma(u)(u^+ - u) \geq \varepsilon, \qquad d\gamma(u)(u^- - u) \leq -\varepsilon \qquad \forall\, u \in B^*(u_0, R)$$

for a suitable $\varepsilon > 0$ ($\gamma$ is as in lemma 1.12).

If $v \in B^*(u_0, R)$, we define $\lambda(v)$ by

$$\begin{cases} \gamma(v + \lambda(v)(u^+ - v)) = 0 & \text{if } \gamma(v) \leq 0, \\ \gamma(v + \lambda(v)(u^- - v)) = 0 & \text{if } \gamma(v) \geq 0. \end{cases}$$

It is not difficult to see, by computations, that $\lambda$ is well defined, and

$$|\lambda(v)| \leq C \operatorname{dist}(v, M) \qquad \forall\, v \in B^*(u_0, R)$$

for a suitable constant $C$. Then, setting

$$\psi(v) = \begin{cases} v + \lambda(v)(u^+ - v) & \text{if } \gamma(v) \le 0, \\ v + \lambda(v)(u^- - v) & \text{if } \gamma(v) \ge 0, \end{cases}$$

it follows that the second condition in (M.f) is satisfied with $C_\psi = C$. For the second one observe that, if for instance $\gamma(v) \le 0$, we have, for a suitable $C_1 > 0$:

$$f_0(v) - f_0(\psi(v)) \ge \lambda(v)(f_0(v) - f_0(u^+)) \ge C_1 \operatorname{dist}(v, M);$$

the first inequality holds by the convexity of $f_0$, the second one is true, taking $R$ small, since $f_0$ is lower semicontinuous.

Since $\mathcal{G}$ is $L$-Lipschitzian in $B^*(u_0, R)$, then the first inequality in (M.f) follows with $C_f = C_1 + L$.

Now (a) follows immediately from lemma 1.18.

## 2. THE UNCONSTRAINED CASE

In this section we study a semilinear parabolic equation with a discontinuous nonlinearity. The existence theorem (see theorem 2.13) that we prove provides a generalization to the one of [26] in the superlinear case. The analysis carried out in this section will be also used in the constrained cases.

Let $N \ge 2$, $\Omega$ be a bounded open subset of $\mathbf{R}^N$ and $g\colon \Omega \times \mathbf{R} \to \mathbf{R}$ a measurable function. We introduce the functions $\underline{g}, \bar{g}\colon \Omega \times \mathbf{R} \to \mathbf{R}$ defined by:

$$\bar{g}(x, s) = \inf\{\varphi(s) \mid \varphi\colon \mathbf{R} \to \mathbf{R} \text{ is continuous, } \varphi(\sigma) \ge g(x, \sigma) \text{ for a.e. } \sigma \text{ in } \mathbf{R}\},$$

$$\underline{g}(x, s) = \sup\{\varphi(s) \mid \varphi\colon \mathbf{R} \to \mathbf{R} \text{ is continuous, } \varphi(\sigma) \le g(x, \sigma) \text{ for a.e. } \sigma \text{ in } \mathbf{R}\}.$$

We shall denote by $2^*$ the number $(2N)/N - 2$, if $N \ge 3$ and $+\infty$ if $N = 2$.

In the following we denote by $\|\cdot\|_q$ the standard norm in the space $L^q(\Omega)$ and by $B_q(u, R)$ the ball $\{v \mid \|v - u\|_q \le R\}$. We also consider in the Hilbert space $L^2(\Omega)$ the standard inner product $\langle u, v\rangle = \int_\Omega uv\,dx$:

We introduce the following assumptions on $g$.

$$\left.\begin{array}{l} \text{there exist } a \in L^2(\Omega),\ b \in \mathbf{R},\ p \in \left[1, \dfrac{2^*}{2}\right[ \cap \left[1, 1 + \dfrac{4}{N}\right[ \text{ such that} \\ |g(x, s)| \le a(x) + b|s|^p \qquad \forall\, x \in \Omega,\ \forall\, s \in \mathbf{R}; \end{array}\right\} \tag{g.1}$$

$$\left.\begin{array}{l} \text{for every measurable function } u\colon \Omega \to \mathbf{R}, \text{ the functions:} \\ x \mapsto \underline{g}(x, u(x)), \qquad x \mapsto \bar{g}(x, u(x)) \\ \text{are measurable;} \end{array}\right\} \tag{g.2}$$

$$\left.\begin{array}{l} \text{there exists } E \subset \mathbf{R} \text{ such that } \operatorname{meas}(E) = 0 \text{ and for all } x \text{ in } \Omega \text{ the function:} \\ s \mapsto g(x, s) \\ \text{is continuous on } \mathbf{R}\backslash E. \end{array}\right\} \tag{g.3}$$

If (g.1) holds, we can define $G\colon \Omega \times \mathbf{R} \to \mathbf{R}$, by $G(x, s) = \int_0^s g(x, \sigma)\,d\sigma$.

Really assumption (g.1) could be weakened to allow the "natural" growth condition $p < 2^*$, using the techniques of [24]; we do not do this for sake of simplicity.

The proofs of the following two remarks are quite standard, so we have omitted them.

*Remark* 2.1. Under the assumption (g.1), $G$ is a Caratheodory function and there exist $a_1 \in L^1(\Omega)$, $b_1 \in \mathbf{R}$ such that:

$$|G(x,s)| \le a_1(x) + b_1|s|^{p+1} \qquad \forall\, x \in \Omega,\ \forall\, s \in \mathbf{R}, \tag{2.1}$$

$$|G(x,s_2) - G(x,s_1)| \le (a(x) + b(|s_2|^p + |s_1|^p))|s_2 - s_1| \qquad \forall\, x \in \Omega,\ \forall\, s_2, s_2 \in \mathbf{R}. \tag{2.2}$$

*Remark* 2.2. The following facts hold:

(a) if $g$ does not depend on $x$, then (g.2) is automatically fulfilled;

(b) the function $s \mapsto \bar{g}(x,s)$ ($s \mapsto \underline{g}(x,s)$) is upper (lower) semicontinuous, $\forall\, x \in \Omega$;

(c) for every $x$ in $\Omega$ we have:

$$\underline{g}(x,s) \le \bar{g}(x,s) \qquad \forall\, s \in \mathbf{R};$$

if $s \mapsto g(x,s)$ is continuous at $s_0$, then $\underline{g}(x,s_0) = g(x,s_0) = \bar{g}(x,s_0)$;

(d) under the assumption (g.1), we have:

$$\underline{g}(x,s) \le D_- G(x,s), \qquad \bar{g}(x,s) \ge D^+ G(x,s) \qquad \forall\, s \in \mathbf{R};$$

$$|\bar{g}(x,s)|,\ |\underline{g}(x,s)| \le a(x) + b|s|^p \qquad \forall\, x \in \Omega,\ \forall\, s \in \mathbf{R};$$

(e) if (g.1) and (g.2) hold, then for all $u \in H_0^1(\Omega)\bar{g}(\cdot, u)$, $\underline{g}(\cdot, u) \in L^2(\Omega)$.

*Definition* 2.3. If (g.1) holds, we define $f_1 : L^2(\Omega) \to \mathbf{R} \cup \{+\infty\}$, by

$$f_1(u) = \begin{cases} \frac{1}{2}\displaystyle\int_\Omega |Du|^2\,\mathrm{d}x + \int_\Omega G(x,u)\,\mathrm{d}x & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

PROPOSITION 2.4. Under the assumption (g.1), the following facts hold:

(a) $f_1$ satisfies the assumptions of proposition 1.11, with $f_0(u) = \frac{1}{2}\int_\Omega |Du|^2\,\mathrm{d}x$ and $\mathcal{G}(u) = \int_\Omega G(x,u)\,\mathrm{d}x$; then, by proposition 1.11 $f_1$ is $\mathrm{d}\bar{\nabla}$-continuous;

(b) for every $C_1, C_2$ in $\mathbf{R}$, there exists $M > 0$ such that:

$$\|u\|_2 \le C_1, f_1(u) \le C_2 \Rightarrow \|Du\|_2 \le M;$$

(c) if $(u_k)_k$ is a sequence in $H_0^1(\Omega)$, $u \in L^2(\Omega)$, and

$$u_k \to u \text{ in } L^2(\Omega), \qquad \sup_k f_1(u_k) < +\infty,$$

then $u \in H_0^1(\Omega)$, $u_k \to u$ weakly in $H_0^1(\Omega)$ and $u_k \to u$ in $L^q(\Omega)$ for all $q < 2^*$;

(d) $f_1$ is lower semicontinuous;

(e) for every $u_0$ in $H_0^1(\Omega)$, $C$ in $\mathbf{R}$, $f_1$ is coercive on $B_2(u_0, R)$.

*Proof.* We prove (a). By (2.1) we get:

$$\mathcal{G}(u) \geq -\|a_1\|_1 - b_1\|u\|_{p+1}^{p+1} \qquad \forall\, u \in H_0^1(\Omega).$$

By interpolation, since $p < 1 + 4/N$, there exist $q \in ]p + 1, 2^*[$, $\theta < 2$ and $C, C_1 > 0$ such that, for all $u$ in $H_0^1(\Omega)$:

$$\|u\|_{p+1}^{p+1} \leq \|u\|_2^{p+1-\theta}\|u\|_q^\theta \leq C\|u\|_2^{p+1-\theta}\|Du\|_2^\theta \leq C_1\|u\|_2^{(p+1-\theta)/(2-\theta)} + \tfrac{1}{4}\|Du\|_2^2$$

($C$ is related to the imbedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$), which implies the first inequality in (1.2). Notice that we have proved:

$$f_1(u) \geq \tfrac{1}{4}\|Du\|_2^2 - C_1\|u\|_2^{(p+1-\theta)/(2-\theta)} - \|a_1\|_1 \qquad \forall\, u \in H_0^1(\Omega). \tag{2.3}$$

To prove the second inequality in (1.2) we use (2.2), estimating $\|u\|_p^p$, $\|v\|_p^p$ in terms of $f_1(u), f_1(v)$, as above.

The proof of (b) is a direct consequence of (2.3).

To prove (c), (d) and (e), just use (b) and the compact imbedding of $H_0^1(\Omega)$ in $L^q(\Omega)$, for $q < 2^*$.

PROPOSITION 2.5. Under the assumptions (g.1) and (g.2), we have:

(a)
$$-|\nabla f_1|(u)\|v - u\|_2 \leq \int_\Omega DuD(v - u)\,dx + \int_\Omega \bar{g}(x, u)(v - u)^+\,dx - \int_\Omega \underline{g}(x, u)(v - u)^-\,dx \qquad \forall\, u, v \in H_0^1(\Omega);$$

(b) $u \in H_0^1(\Omega)$, $|\nabla f_1|(u) < +\infty \Rightarrow \Delta u \in L^2(\Omega)$ and $\|\Delta u\|_2 \leq |\nabla f_1|(u) + \|a\|_2 + b\|u\|_{2p}^p$;

(c) if $u \in H_0^1(\Omega)$, $\alpha \in \partial^- f_1(u)$, then:

$$\int_\Omega \alpha(v - u)\,dx \leq \int_\Omega DuD(v - u)\,dx + \int_\Omega \bar{g}(x, u)(v - u)^+\,dx - \int_\Omega \underline{g}(x, u)(v - u)^-\,dx \qquad \forall\, u, v \in H_0^1(\Omega).$$

*Proof.* Let $u, v \in H_0^1(\Omega) = \mathcal{D}(f_1)$ and $(t_k)_k$ be a sequence such that $t_k \to 0^+$; then $u + t_k(v - u) \in H_0^1(\Omega)$ and we have

$$\begin{aligned}-|\nabla f_1|(u)\|v - u\|_2 &\leq \liminf_{k\to\infty} \frac{f_1(u + t_k(v - u)) - f_1(u)}{t_k} \\ &\leq \int_\Omega DuD(v - u)\,dx + \limsup_{k\to\infty} \int_\Omega \frac{G(x, u + t_k(v - u)) - G(x, u)}{t_k}\,dx.\end{aligned}$$

By (2.2), we get, for $k$ large:

$$\left|\frac{G(\cdot, u + t_k(v - u)) - G(\cdot, u)}{t_k}\right| \leq a + b((2^p + 1)|u|^p + 2^p|v - u|^p)|v - u| \in L^1(\Omega).$$

Applying Fatou's lemma and remark 2.2, we obtain the conclusion, since, in an a.e. sense:

$$\limsup_{k\to\infty} \frac{G(\cdot, u + t_k(v-u)) - G(\cdot, u)}{t_k} \le D^+G(\cdot, u)(v-u)^+ - D_-G(\cdot, u)(v-u)^-$$

$$\le \bar{g}(\cdot, u)(v-u)^+ - \underline{g}(\cdot, u)(v-u)^-.$$

We prove (b). Let $w \in C_0^\infty(\Omega)$; applying (a) with $v = u + w$, we get:

$$-\int_\Omega DuDw\,dx \le |\nabla f_1|(u)\|w\|_2 + \int_\Omega |\bar{g}(x,u)|w^+\,dx + \int_\Omega |\underline{g}(x,u)|w^-\,dx$$

$$\le |\nabla f_1|(u)\|w\|_2 + \int_\Omega (a + b|u|^p)|w|\,dx$$

$$\le (|\nabla f_1|(u) + \|a\|_2 + b\|u\|_{2p}^p)\|w\|_2,$$

which implies (b).

To prove (c) proceed as in the proof of (a), noting that, if $\alpha \in \partial^- f_1(u)$, then

$$\int_\Omega \alpha(v-u)\,dx \le \liminf_{k\to\infty} \frac{f_1(u + t_k(v-u)) - f_1(u)}{t_k}.$$

*Remark* 2.6. Let $u \in H_0^1(\Omega)$, $\Delta u \in L^2(\Omega)$. Then $|\nabla f_1|(u) < +\infty$.

*Proof.* It can be deduced by (a) of proposition 2.4 and by the inequality:

$$f_1(v) - f_1(u) \ge -(\|\Delta u\|_2 + \|a\|_2 + b(\|u\|_{2p}^p + \|v\|_{2p}^p))\|v-u\|_2.$$

*Definition* 2.7. Assume (g.1) and (g.2). We define the multivalued map $\mathfrak{A}_1 : H_0^1(\Omega) \to 2^{L^2(\Omega)}$ by:

$$\alpha \in \mathfrak{A}_1(u) \Leftrightarrow \begin{cases} \displaystyle\int_\Omega \alpha(v-u)\,dx \le \int_\Omega DuD(v-u)\,dx + \int_\Omega \bar{g}(x,u)(v-u)^+\,dx \\ \displaystyle -\int_\Omega \underline{g}(x,u)(v-u)^-\,dx \qquad \forall\, v \in H_0^1(\Omega). \end{cases}$$

Proposition 2.8. Under the assumptions (g.1) and (g.2), we have:

(a) $\mathfrak{A}_1(u)$ is closed and convex for every $u$ in $H_0^1(\Omega)$;

(b) $\mathfrak{A}(f_1)(u) \subset \mathfrak{A}_1(u)$ for every $u$ in $H_0^1(\Omega)$;

(c) there exists a continuous function $\Psi_1 : (\mathbf{R}^+)^4 \to \mathbf{R}^+$, increasing in all its arguments, such that:

$$\begin{cases} f_1(v) \ge f_1(u) - \Psi_1(\|u\|_2, \|v\|_2, |f_1(u)|, |f_1(v)|)(1 + \|\alpha\|_2)\|v-u\|_2 \\ \forall\, u, v \in H_0^1(\Omega)\ \forall\, \alpha \in \mathfrak{A}_1(u); \end{cases} \tag{2.4}$$

(d) if $\mathfrak{A}_1(u) \ne \emptyset$, then $|\nabla f_1|(u) < +\infty$ $(\Rightarrow \Delta u \in L^2(\Omega))$, moreover for every $\alpha$ in $\mathfrak{A}_1(u)$ there exists $\beta$ in $L^2(\Omega)$ such that:

$$\alpha = -\Delta u + \beta,\ \underline{g}(x, u(x)) \le \beta(x) \le \bar{g}(x, u(x)) \qquad \text{a.e. in } \Omega:$$

if $A_1(u)$ denotes the minimal section of $\mathfrak{A}_1(u)$, then:

$$A_1(u) = (-\Delta u + \underline{g}(\cdot, u)) \vee 0 + (-\Delta u + \bar{g}(\cdot, u)) \wedge 0.$$

*Proof.* Part (a) is obvious. To prove (b) first observe that, by (c) of proposition 2.5, it follows

$$\partial^- f_1(u) \subset \mathcal{A}_1(u) \qquad \forall\, u \in H_0^1(\Omega).$$

Now let $(u_k)_k$, $u$ be in $H_0^1(\Omega)$, $(\alpha_k)_k$, $\alpha$ be in $L^2(\Omega)$, such that $u_k \to u$ in $L^2(\Omega)$, $f_1(u_k) \to f_1(u)$, $\alpha_k \in \mathcal{A}_1(u_k)$ $\forall\, k$ and $\alpha_k \to \alpha$ weakly in $L^2(\Omega)$. By (c) of proposition 2.4 we obtain that $Du_k \to Du$ weakly in $L^2(\Omega)$ and $u_k \to u$ in $L^{2p}(\Omega)$ $(2p < 2^*)$; we can also suppose that $u_k \to u$ almost everywhere in $\Omega$. From the lower semicontinuity of $\|\cdot\|_2$ with respect to weak convergence, we get:

$$\limsup_{k\to\infty} -\|Du_k\|_2^2 \le -\|Du\|_2^2.$$

If $v \in H_0^1(\Omega)$, then

$$\int_\Omega \alpha_k(v - u_k)\,dx \le \int_\Omega Du_k D(v - u_k)\,dx + \int_\Omega \bar{g}(x, u_k)(v - u_k)^+\,dx - \int_\Omega \underline{g}(x, u_k)(v - u_k)^-\,dx.$$

Passing to the limit, applying Fatou's lemma and the semicontinuity of $\underline{g}, \bar{g}$, we obtain $\alpha \in \mathcal{A}_1(u)$. So $\mathcal{A}_1$ turns out to be closed with respect to this sort of "weak-strong convergence": in particular, since it contains $\partial^- f_1$, then it contains $\mathcal{A}(f_1)$.

We prove (c). Let $u, v \in H_0^1(\Omega)$, $\alpha \in \mathcal{A}_1(u)$, then

$$\begin{aligned} f(v) - f(u) &\ge \int_\Omega DuD(v - u)\,dx + \int_\Omega G(x, v)\,dx - \int_\Omega G(x, u)\,dx \\ &\ge \int_\Omega \alpha(v - u)\,dx + \int_\Omega G(x, v)\,dx - \int_\Omega G(x, u)\,dx \\ &\quad - \int_\Omega \bar{g}(x, u)(v - u)^+\,dx + \int_\Omega \underline{g}(x, u)(v - u)^-\,dx \\ &\ge -K(1 + \|\alpha\|_2)(1 + \|u\|_2^\gamma + \|v\|_2^\gamma + |f_1(u)|^p + |f_1(v)|^p)\|v - u\|_2 \end{aligned}$$

where $K$, $\gamma$ are suitable constants, which can be obtained by (2.3).

We prove (d). It is easy to deduce from (c) and from (a) of proposition 2.5 that $\mathcal{A}_1(u) \ne \varnothing \Rightarrow |\nabla f_1|(u) < +\infty \Rightarrow \Delta u \in L^2(\Omega)$. Then, if $\alpha \in \mathcal{A}_1(u)$

$$\int_\Omega \alpha w\,dx \le -\int_\Omega \Delta u w\,dx + \int_\Omega \bar{g}(x, u) w\,dx \qquad \forall\, w \in H_0^1(\Omega) \text{ with } w \ge 0,$$

$$\int_\Omega \alpha w\,dx \ge -\int_\Omega \Delta u w\,dx + \int_\Omega \underline{g}(x, u) w\,dx \qquad \forall\, w \in H_0^1(\Omega) \text{ with } w \ge 0,$$

which is the weak formulation of (d).

PROPOSITION 2.9. Assume that (g.1), (g.2) and (g.3) hold. Then $\mathcal{A}_1$ is a subdifferential along curves (see definition 1.16).

For the proof of proposition 2.9 we need some lemmas.

LEMMA 2.10. Let $I$ be an interval, $\mathcal{U} \in L^1(I, L^1(\Omega))$ be absolutely continuous and such that $\mathcal{U}' \in L^1(I, L^1(\Omega))$. Let $N \subset \mathbf{R}$ be such that $\operatorname{meas}(N) = 0$. Then for almost every $t$ in $I$ one has:

$$\operatorname{meas}(\{x \in \Omega \mid \mathcal{U}(t)(x) \in N, \mathcal{U}'(t)(x) \neq 0\}) = 0.$$

The proof is contained in the Appendix.

*Remark* 2.11. The assumption (g.3) implies that for every $x$ in $\Omega$ and $s$ in $\mathbf{R}\backslash E$ $G'(x, s)$ (the derivative with respect to $s$) exists and $G'(x, s) = \underline{g}(x, s) = \bar{g}(x, s)$.

LEMMA 2.12. Assume that (g.1) and (g.2) hold. Let $\mathcal{U} \in L^2(I, L^2(\Omega))$ be an absolutely continuous curve such that $\sup\{f_1 \circ \mathcal{U}(t) \mid t \in I\} < +\infty$. Then the function $t \mapsto \mathcal{G}(\mathcal{U}(t)) = \int_\Omega G(x, \mathcal{U}(t))\, dx$ is absolutely continuous and for a.e. $t$ in $I$:

$$\begin{cases} (\mathcal{G} \circ \mathcal{U})'(t) = \displaystyle\int_\Omega g(x, \mathcal{U}(t))\mathcal{U}'(t)\, dx \\ \mathcal{U}'(t) = 0 \qquad \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) \in E\} \end{cases} \tag{2.5}$$

[we recall that $E$ was defined in (g.3)].

*Proof.* By (a) of proposition 2.4 $\mathcal{G}$ is locally Lipschitzian in the sublevels of $f_1$, so it is clear that $\mathcal{G} \circ \mathcal{U}$ is absolutely continuous. Let $t \in I$ be such that $\mathcal{U}'(t)$ and $(\mathcal{G} \circ \mathcal{U})'(t)$ exist and $\operatorname{meas}(\{x \in \Omega \mid \mathcal{U}(t)(x) \in E, \mathcal{U}'(t)(x) \neq 0\}) = 0$. Then we can find $(h_k)_k$ converging to $0^+$, such that $(\mathcal{U}(t + h_k) - \mathcal{U}(t))/(h_k) \to \mathcal{U}'(t)$ almost everywhere in $\Omega$. We claim that, for a.e. $x$ in $\Omega$:

$$\lim_{k \to \infty} \frac{G(x, \mathcal{U}(t + h_k)(x)) - G(x, \mathcal{U}(t)(x))}{h_k} = \begin{cases} g(x, \mathcal{U}(t)(x))\mathcal{U}'(t)(x) & \text{if } \mathcal{U}(t)(x) \notin E, \\ 0 & \text{if } \mathcal{U}(t)(x) \in E, \end{cases} \tag{2.6}$$

which can be expressed as $g(x, \mathcal{U}(t)(x))\mathcal{U}'(t)(x)$, with the obvious convention. To prove this we first note that:

$$\begin{aligned} \left|\frac{G(\cdot, \mathcal{U}(t + h_k)) - G(\cdot, \mathcal{U}(t))}{h_k}\right| &\leq a + b(|\mathcal{U}(t + h_k)|^p + |\mathcal{U}(t)|^p)\left|\frac{\mathcal{U}(t + h_k) - \mathcal{U}(t)}{h_k}\right| \\ &\leq c\left|\frac{\mathcal{U}(t + h_k) - \mathcal{U}(t)}{h_k}\right|, \end{aligned} \tag{2.7}$$

where $c \in L^2(\Omega)$, by (a) of proposition 2.4. Then, for almost every $x$ in $\Omega$ such that $U'(t)(x) = 0$, the right-hand side of (2.6) is zero, namely (2.6) holds. In the other case $\mathcal{U}(t)(x) \notin E$ for a.e. $x$ and

$$\begin{aligned} &\lim_{k \to \infty} \frac{G(x, \mathcal{U}(t + h_k)(x)) - G(x, \mathcal{U}(t)(x))}{h_k} \\ &\quad = \lim_{k \to \infty} \frac{G(x, \mathcal{U}(t + h_k)(x)) - G(x, \mathcal{U}(t)(x))}{\mathcal{U}(t + h_k)(x) - \mathcal{U}(t)(x)} \frac{\mathcal{U}(t + h_k)(x) - \mathcal{U}(t)(x)}{h_k} \\ &\quad = G'(x, \mathcal{U}(t)(x))\mathcal{U}'(t)(x) = g(x, \mathcal{U}(t)(x))\mathcal{U}'(t)(x). \end{aligned}$$

From (2.6) and (2.7), applying Lebesgue's theorem, we get the conclusion.

*Proof of proposition* 2.9. Let $t$ be such that $\mathcal{A}_1(\mathcal{U}(t)) \neq \emptyset$, $\mathcal{U}'(t)$, $(\mathcal{G} \circ \mathcal{U})'(t)$ exist and (2.5) holds. If $t' > t$, $\alpha \in \mathcal{A}_1(\mathcal{U}(t))$, we have:

$$\begin{aligned}\frac{f_1(\mathcal{U}(t')) - f_1(\mathcal{U}(t))}{t' - t} \geq & \int_\Omega \alpha \frac{\mathcal{U}(t') - \mathcal{U}(t)}{t' - t}\,dx + \int_\Omega \frac{G(x, \mathcal{U}(t')) - G(x, \mathcal{U}(t))}{t' - t}\,dx \\ & - \int_\Omega \bar{g}(x, \mathcal{U}(t)) \frac{(\mathcal{U}(t') - \mathcal{U}(t))^+}{t' - t}\,dx \\ & + \int_\Omega \underline{g}(x, \mathcal{U}(t)) \frac{(\mathcal{U}(t') - \mathcal{U}(t))^-}{t' - t}\,dx.\end{aligned}$$

Going to the limit, as $t' \to t^+$, we get:

$$\begin{aligned}D_+(f \circ \mathcal{U})(t) \geq \langle \alpha, \mathcal{U}'(t)\rangle + \int_\Omega g(x, \mathcal{U}(t))\mathcal{U}'(t)\,dx - \int_\Omega \bar{g}(x, \mathcal{U}(t))(\mathcal{U}'(t))^+\,dx \\ + \int_\Omega \underline{g}(x, \mathcal{U}(t))(\mathcal{U}'(t))^-\,dx.\end{aligned}$$

Using lemma 2.10 and remark 2.11 we get the conclusion.

We can prove now the main theorem of this section.

THEOREM 2.13. Let $g: \Omega \times \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying (g.1), (g.2) and (g.3). Then for every $u_0$ in $H_0^1(\Omega)$ there exist $T > 0$ and $\mathcal{U}: [0, T[ \to L^2(\Omega)$, an absolutely continuous curve, such that $\mathcal{U}(0) = u_0$, $\mathcal{U}(t) \in H_0^1(\Omega)$ for every $t$ in $[0, T[$ and for almost all $t$ in $[0, T[$: [we recall that $E$ was introduced in (g.3)]

$$\Delta\mathcal{U}(t) \in L^2(\Omega);$$

$$\mathcal{U}'(t) = \begin{cases} \Delta\mathcal{U}(t) - g(\cdot, \mathcal{U}(t)) & \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) \notin E\}, \\ 0 & \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) \in E\}; \end{cases}$$

$$\Delta\mathcal{U}(t) = 0 \in [\underline{g}(\cdot, \mathcal{U}(t)), \bar{g}(\cdot, \mathcal{U}(t))] \qquad \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) \in E\}.$$

Moreover the function:

$$t \mapsto m(t) = \tfrac{1}{2} \int_\Omega |D\mathcal{U}(t)|^2\,dx + \int_\Omega G(x, \mathcal{U}(t))\,dx$$

is continuous, nonincreasing and $m'(t) = \|\mathcal{U}'(t)\|_2^2$ a.e. in $[0, T[$.

*Proof.* By propositions 2.4 (a) and 1.11 we can apply theorem 1.8 and prove the existence of $\mathcal{U}$ a curve of maximal relaxed slope almost everywhere for $f_1$, such that $\mathcal{U}(0) = u_0$. Such a curve has the property that $f_1 \circ \mathcal{U}$ is continuous, by theorem 1.9 and proposition 2.4 (a). Moreover $\mathcal{A}_1$ satisfies the assumptions of theorem 1.17, as was proved in propositions 2.8 (b) and 2.9. Then, applying theorem 1.17, we obtain the conclusion by the characterization of the minimal section of $\mathcal{A}$, given in proposition 2.8 (d).

## 3. THE PROBLEM WITH AN OBSTACLE

In this section we study the evolution of the functional $f_1$ on the convex constraint produced by the presence of an obstacle. The result is an existence theorem (see theorem 3.7) for a problem of parabolic variational inequalities (see also [26]).

Let $\Omega$, $g$ be as in Section 2 and let $\varphi: \Omega \to \mathbf{R}$ be a measurable function (the obstacle). We denote by $K$ the closed and convex set:

$$K = \{u \in L^2(\Omega) \mid u(x) \geq \varphi(x) \text{ a.e. in } \Omega\}$$

and for $u$ in $K$ we introduce the "contact set"

$$C(u) = \{x \in \Omega \mid u(x) = \varphi(x)\}.$$

We define $f_2: L^2(\Omega) \to \mathbf{R} \cup \{+\infty\}$ by $f_2 = f_1 + I_K$, where, for a generic subset $V$ of $L^2(\Omega)$,

$$I_V(u) = \begin{cases} 0 & \text{if } u \in V, \\ +\infty & \text{if } u \notin V. \end{cases}$$

PROPOSITION 3.1. Suppose that (g.1) holds. Then:

(a) $\mathfrak{D}(f_2) = H_0^1(\Omega) \cap K$; if $\varphi \in W^{1,2}(\Omega)$, then $\mathfrak{D}(f_2) \neq \emptyset \Leftrightarrow \varphi^+ \in H_0^1(\Omega)$;

(b) $f_2 = \bar{f}_0 + \mathcal{G}$, where $\bar{f}_0 = f_0 + I_K$ is a convex lower semicontinuous function and $\mathcal{G}$ satisfies the inequalities (1.2); therefore $f_2$ is $d\bar{\nabla}$-continuous, by proposition (1.11);

(c) for every $u_0 \in H_0^1(\Omega) \cap K$, $R > 0$, $f_2$ is coercive on $B_2(u_0, R)$;

(d) $f_2$ is lower semicontinuous.

The proof is straightforward.

PROPOSITION 3.2. Suppose that (g.1) holds and that $\varphi \in W^{1,2}(\Omega)$, $\Delta\varphi \in L^2(\Omega)$. Then for every $u_0$ in $H_0^1(\Omega) \cap K$, $R > 0$ there exists a constant $C$, such that:

$$|\nabla f_2|(u) \leq |\nabla f_1|(u) \leq C + |\nabla f_2|(u) \qquad \forall\, u \in B_2(u, R) \text{ with } f_2(u) \leq f_2(u_0).$$

In particular $|\nabla f_2|(u) < +\infty \Leftrightarrow |\nabla f_1|(u) < +\infty (\Leftrightarrow \Delta u \in L^2(\Omega))$.

*Proof.* Let $u_0 \in H_0^1(\Omega) \cap K$, $R > 0$: we construct a map $\pi: B_2(u_0, R) \cap \{f_1(v) \leq f_1(u)\} \to H_0^1(\Omega) \cap K$ such that for all $u \in B_2(u_0, R)$ with $f_1(u) \leq f_1(u_0)$:

$$\begin{cases} \|\pi(v) - u\|_2 \leq \|v - u\|_2, \\ f_1(v) \geq f_1(\pi(v)) - C\|v - u\|, \end{cases} \qquad \forall\, v \in B_2(u_0, R) \text{ with } f_1(v) \leq f_1(u_0)$$

for a suitable $C$. If this can be done, the conclusion follows from [7, lemma 3.4], applied in the metric space $X = B_2(u_0, R) \cap \{f_1(v) \leq f_1(u)\}$. The function $\pi$ is simply defined by:

$$\pi(v) = v \vee \varphi$$

($\pi(v) \in H_0^1(\Omega)$ because $\varphi^+ \in H_0^1(\Omega)$ by (3.1) (a) and $v \in H_0^1(\Omega)$). It is clear that $\|\pi(v) - u\|_2 \leq \|v - u\|_2$. Moreover

$$\begin{aligned} f_1(v) - f_1(\pi(v)) &\geq \int_\Omega D\pi(v)D(v - \pi(v))\,dx + \int_\Omega G(x, v) - G(x, \pi(v))\,dx \\ &= \int_\Omega D\varphi D(v - \pi(v))\,dx + \int_\Omega G(x, v) - G(x, \pi(v))\,dx \\ &\geq -\int_\Omega \Delta\varphi(v - \pi(v))\,dx - \int_\Omega a + b(|v|^p + |\pi(v)|^p)|v - \pi(v)|\,dx \\ &\geq -(\|\Delta\varphi\|_2 + \|a\|_2 + b(\|v\|_{2p}^p + \|\pi(v)\|_{2p}^p))\|v - \pi(v)\|_2 \\ &\geq -C\|v - \pi(v)\|_2 \end{aligned}$$

(since $\|v\|_2$ and $f_1(v)$ are bounded, then $\|v\|_{2p}$ and $\|v\|_{2p}$ are bounded, by (a) of proposition 2.4). This concludes the proof.

PROPOSITION 3.3. Suppose that (g.1) and (g.2) hold. Then:

(a) the following inequality holds:

$$-|\nabla f_2|(u)\|v - u\|_2 \leq \int_\Omega DuD(v - u)\,dx + \int_\Omega \bar{g}(x, u)(v - u)^+\,dx - \int_\Omega \underline{g}(x, u)(v - u)^-\,dx \qquad \forall\, v \in H_0^1(\Omega) \cap K;$$

(b) if $u \in H_0^1(\Omega) \cap K$, $\alpha \in \partial^- f_2(u)$, then:

$$\int_\Omega \alpha(v - u)\,dx \leq \int_\Omega DuD(v - u)\,dx + \int_\Omega \bar{g}(x, u)(v - u)^+\,dx - \int_\Omega \underline{g}(x, u)(v - u)^-\,dx \qquad \forall\, v \in H_0^1(\Omega) \cap K.$$

*Proof.* The proofs are similar to those of (a) and (c) of proposition 2.5.

*Definition* 3.4. Under the assumptions (g.1) and (g.2), we introduce the multivalued map $\mathfrak{A}_2 : H_0^1(\Omega) \cap K \to 2^{L^2(\Omega)}$ by

$$\alpha \in \mathfrak{A}_2(u) \Leftrightarrow \begin{cases} \displaystyle\int_\Omega \alpha(v - u)\,dx \leq \int_\Omega DuD(v - u)\,dx + \int_\Omega \bar{g}(x, u)(v - u)^+\,dx \\ \displaystyle\qquad - \int_\Omega \underline{g}(x, u)(v - u)^-\,dx \qquad \forall\, v \in H_0^1(\Omega) \cap K. \end{cases}$$

PROPOSITION 3.5. If (g.1) and (g.2) hold, then the following facts are true:

(a) $\mathfrak{A}_2(u)$ is closed and convex for all $u$ in $H_0^1(\Omega) \cap K$;

(b) $\mathfrak{A}(f_2)(u) \subset \mathfrak{A}_2(u)$ for all $u$ in $H_0^1(\Omega) \cap K$;

(c) there exists a continuous function $\Psi_2 : (\mathbf{R}^+)^4 \to \mathbf{R}^+$ increasing in all its arguments, such that:

$$\begin{cases} f_2(v) \geq f_2(u) - \Psi_2(\|u\|_2, \|v\|_2, |f_2(u)|, |f_2(v)|)(1 + \|\alpha\|_2)\|v - u\|_2 \\ \forall\, u, v \in H_0^1(\Omega) \cap K\ \forall\, \alpha \in \mathfrak{A}_2(u); \end{cases}$$

(d) let $\varphi \in W^{1,2}(\Omega)$, $\Delta\varphi \in L^2(\Omega)$; we have that, if $\mathfrak{A}_2(u) \neq \emptyset$ then $|\nabla f_2|(u) < +\infty$ and for every $u$ in $H_0^1(\Omega) \cap K$, $\alpha$ in $\mathfrak{A}_2(u)$ there exists $\beta$ in $L^2(\Omega)$ such that $\alpha = -\Delta u + \beta$ and

$$\underline{g}(x, u(x)) \leq \beta(x) \leq \bar{g}(x, u(x)) \qquad \text{for a.e. } x \text{ in } \Omega \setminus C(u),$$

$$\beta(x) \leq \bar{g}(x, u(x)) \qquad \text{for a.e. } x \text{ in } C(u);$$

furthermore, if $A_2(u)$ denotes the minimal section of $\mathfrak{A}_2(u)$, then:

$$A_2(u) = \begin{cases} (-\Delta u + \underline{g}(\cdot, u)) \vee 0 + (-\Delta u + \bar{g}(\cdot, u)) \wedge 0 & \text{a.e. in } \Omega \setminus C(u), \\ (-\Delta u + \bar{g}(\cdot, u)) \wedge 0 & \text{a.e. in } C(u). \end{cases}$$

*Proof*. The proofs of (a), (b) and (c) are quite similar to the corresponding in proposition 2.8. We prove (d). If $\mathfrak{A}_2(u) \neq \emptyset$, then, by (c), $|\nabla f_2|(u) < +\infty$ and by proposition 3.2 we get $|\nabla f_1|(u) < +\infty$, namely $\Delta u \in L^2(\Omega)$. Let $\alpha \in \mathfrak{A}_2(u)$, we claim that:

$$\int_\Omega \alpha w \,\mathrm{d}x \leq -\int_\Omega \Delta u w \,\mathrm{d}x + \int_\Omega \bar{g}(x, u) w^+ \,\mathrm{d}x - \int_\Omega \underline{g}(x, u) w^- \,\mathrm{d}x \tag{3.1}$$

$$\forall\, w \in H_0^1(\Omega) \text{ with } w \geq 0 \text{ a.e. in } C(u).$$

Let $t > 0$ and $w \in H_0^1(\Omega)$ such that $w \geq 0$ a.e. in $C(u)$. We set:

$$v_t = u + tw, \qquad w_t = v_t \vee \varphi,$$

then $w_t \in H_0^1(\Omega) \cap K$. Moreover $|(w_t - u)/t| \leq |w|$ a.e., because

$$\frac{w_t(x) - u(x)}{t} = \begin{cases} w(x) & \text{if } v_t(x) \geq \varphi(x), \\ \dfrac{\varphi(x) - u(x)}{t} \in [w(x), 0] & \text{if } v_t(x) < \varphi(x). \end{cases}$$

Moreover $(w_t - u)/t \to w$ a.e. in $\Omega$, since $w \geq 0$ in $C(u)$. From the relation

$$\frac{1}{t}\int_\Omega \alpha(w_t - u)\,\mathrm{d}x \leq \frac{1}{t}\left(-\int_\Omega \Delta u(w_t - u)\,\mathrm{d}x + \int_\Omega \bar{g}(x, u)(w_t - u)^+\,\mathrm{d}x - \int_\Omega \underline{g}(x, u)(w_t - u)^-\,\mathrm{d}x\right)$$

going to the limit as $t \to 0^+$, we get (3.1), by means of Lebesgue's theorem. This inequality can be extended to all $w$ in $L^2(\Omega)$ such that $w \geq 0$ in $C(u)$; so it follows:

$$\int_{\Omega \setminus C(u)} \alpha w \,\mathrm{d}x \geq -\int_{\Omega \setminus C(u)} \Delta u w \,\mathrm{d}x + \int_{\Omega \setminus C(u)} \underline{g}(x, u) w \,\mathrm{d}x \qquad \forall\, w \in L^2(\Omega \setminus C(u)),\ w \geq 0,$$

$$\int_\Omega \alpha w \,\mathrm{d}x \leq -\int_\Omega \Delta u w \,\mathrm{d}x + \int_\Omega \bar{g}(x, u) w \,\mathrm{d}x \qquad \forall\, w \in L^2(\Omega),\ w \geq 0,$$

which gives the conclusion.

PROPOSITION 3.6. Suppose that (g.1), (g.2) and (g.3) hold. Then $\mathcal{A}_2$ is a subdifferential along curves for $f_2$.

The proof goes in the same way as the corresponding proof of proposition 2.9.

Arguing as in the proof of theorem 2.13, we can prove the following theorem. Note that, even if we did not assume that $\varphi \in W^{1,2}(\Omega)$, $\Delta\varphi \in L^2(\Omega)$, an existence theorem would still hold, in terms of variational inequalities; this can be easily seen looking at the proofs and at the definition of $\mathcal{A}_2$ (the regularity of $\varphi$ is used only for the "regularization result" stated in proposition 3.5 (d).

THEOREM 3.7. Assume that $g: \Omega \times \mathbf{R} \to \mathbf{R}$ is a measurable function satisfying (g.1), (g.2) and (g.3), $\varphi \in W^{1,2}(\Omega)$ with $\Delta\varphi \in L^2(\Omega)$.

Then for every $u_0$ in $H_0^1(\Omega) \cap K$ there exist $T > 0$ and $\mathcal{U}: [0, T[ \to L^2(\Omega)$, an absolutely continuous curve, such that $\mathcal{U}(0) = u_0$, $\mathcal{U}(t) \in H_0^1(\Omega) \cap K$ for every $t$ in $[0, T[$ and for almost all $t$ in $[0, T[$ [we recall that $E$ was introduced in (g.3)]:

$$\Delta\mathcal{U}(t) \in L^2(\Omega);$$

$$\mathcal{U}'(t) = \begin{cases} \Delta\mathcal{U}(t) - g(\cdot, \mathcal{U}(t)) & \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) > \varphi(x),\ \mathcal{U}(t)(x) \notin E\}, \\ [\Delta\mathcal{U}(t) - g(\cdot, \mathcal{U}(t))] \vee 0 & \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) = \varphi(x),\ \mathcal{U}(t)(x) \notin E\}, \\ 0 & \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) \in E\}; \end{cases}$$

$$\Delta\mathcal{U}(t) = 0 \in \begin{cases} [\underline{g}(\cdot, \mathcal{U}(t)), \bar{g}(\cdot, \mathcal{U}(t))] & \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) > \varphi(x),\ \mathcal{U}(t)(x) \in E\}, \\ ]-\infty, \bar{g}(\cdot, \mathcal{U}(t))] & \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) = \varphi(x),\ \mathcal{U}(t)(x) \in E\}; \end{cases}$$

moreover the function:

$$t \mapsto m(t) = \tfrac{1}{2}\int_\Omega |D\mathcal{U}(t)|^2\,dx + \int_\Omega G(x, \mathcal{U}(t))\,dx$$

is continuous, nonincreasing and $m'(t) = \|\mathcal{U}'(t)\|_2^2$ a.e. in $[0, T[$.

## 4. THE PROBLEM WITH A NONCONVEX CONSTRAINT

Let $\Omega$, $g$, $\varphi$, $K$ be as in the previous sections and let $\rho > 0$. We set:

$$S_\rho = \left\{u \in L^2(\Omega) \,\middle|\, \int_\Omega u^2\,dx = \rho^2\right\}.$$

If $u_0 \in K \cap S_\rho$, we consider the following assumption:

$$\left.\begin{aligned} &\int_\Omega u_0^2\,dx > \int_\Omega (\varphi^+)^2\,dx, \\ &\operatorname{meas}(\{x \in \Omega \mid \varphi(x) < u_0(x) < 0\} \cup \{x \in \Omega \mid 0 < u_0(x)\}) > 0. \end{aligned}\right\} \qquad (\text{N.T.}u_0)$$

In [7, (3.12)] the following result is proved.

PROPOSITION 4.1. Let $u_0 \in K \cap S_p$, then (N.T.$u_0$) holds if and only if $K$ and $S_p$ are not tangent at $u_0$.

*Definition* 4.2. If (g.1) holds, we define $f_3 : L^2(\Omega) \to \mathbf{R} \cup \{+\infty\}$ by $f_3 = f_1 + I_{K \cap S_p} = f_2 + I_{S_p}$.

The following proposition is straightforward.

PROPOSITION 4.3. We have:
(a) $\mathfrak{D}(f_3) = H_0^1(\Omega) \cap K \cap S_p$;
(b) for every $u \in H_0^1(\Omega) \cap K \cap S_p$ $f_3$ is coercive at $u$;
(c) $f_3$ is lower semicontinuous.

PROPOSITION 4.4. Assume that (g.1) holds and let $u_0 \in H_0^1(\Omega) \cap K \cap S_p$ satisfy (N.T.$u_0$). Then there exist $R, C_1, C_2 > 0$ such that:

$$|\nabla f_3|(u) \leq |\nabla f_2|(u) \leq C_1 + C_2 |\nabla f_3|(u) \qquad \forall\, u \in B_2(u_0, R) \text{ with } f_3(u) \leq f_3(u_0).$$

In particular, for every $u$ in $B_2(u_0, R)$ $|\nabla f_2|(u) < +\infty \Leftrightarrow |\nabla f_3|(u) < +\infty$.

*Proof.* As in the proof of proposition 3.2 we construct a map $\pi' : B_2(u_0, R) \cap \{f_2(v) \leq f_2(u_0)\} \to H_0^1(\Omega) \cap K \cap S_p$ such that for all $u$ in $B_2(u_0, R)$ with $f_2(u) \leq f_2(u_0)$:

$$\begin{cases} \|\pi'(v) - u\|_2 \leq C_1 \|v - u\|_2, \\ f_2(v) \geq f_2(\pi'(v)) - C_2 \|v - u\|_2, \end{cases} \qquad \forall\, v \in B_2(u_0, R) \text{ with } f_2(v) \leq f_2(u_0),$$

for suitable $R, C_1, C_2$. This can be done observing that $f_2 = \bar{f}_0 + \mathcal{G}$, where $\bar{f}_0 = f_0 + I_K$ is a convex lower semicontinuous function and $\mathcal{G}$ satisfies the inequality (1.2): then we can define $\pi'$ as the function $\psi$ of theorem (1.20), with $M = S_p$. It is trivial to see that the required inequalities are verified, so the proof is over.

*Definition* 4.5. Under the assumptions (g.1) and (g.2), we define the multivalued map $\mathfrak{A}_3 : H_0^1(\Omega) \cap K \cap S_p \to 2^{L^2(\Omega)}$ by:

$$\mathfrak{A}_3(u) = \mathfrak{A}_2(u) + \{\lambda u \mid \lambda \in \mathbf{R}\}.$$

Notice that $\{\lambda u \mid \lambda \in \mathbf{R}\}$ is the set of normal vectors to $S_p$ at $u$.

PROPOSITION 4.6. If (g.1), (g.2) hold and $u_0$ in $H_0^1(\Omega) \cap K \cap S_p$ satisfies (N.T.$u_0$), then there exists $R > 0$ such that:
(a) $\mathfrak{A}(f_3)(u) \subset \mathfrak{A}_3(u)$ for all $u$ in $B_2(u_0, R)$ with $f_3(u) \leq f_3(u_0)$;
(b) if, in addition, $\varphi \in W^{1,2}(\Omega)$, $\Delta\varphi \in L^2(\Omega)$, then $\mathfrak{A}_3(u) \neq \emptyset \Rightarrow \Delta u \in L^2(\Omega)$ and, if $\mathfrak{A}_3(u)$ has a minimal section $\alpha$, then there exist $\lambda \in \mathbf{R}$ and $\beta$ in $L^2(\Omega)$ such that $\alpha = -\Delta u + \beta + \lambda u$ and:

$$\underline{g}(x, u(x)) \leq \beta(x) \leq \bar{g}(x, u(x)) \qquad \text{for a.e. } x \text{ in } \Omega \setminus C(u),$$

$$\beta(x) \leq \bar{g}(x, u(x)) \qquad \text{for a.e. } x \text{ in } C(u);$$

(one could prove that $\mathfrak{A}_3(u)$ actually has a minimal section for all $u$, but this is not really necessary, since it turns out to be true at least almost everywhere on a curve of maximal relaxed slope for $f_3$, as was proved in theorem 1.17;
(c) if, in addition, (g.3) holds, then $\mathfrak{A}_3$ is a subdifferential along curves for $f_3$.

*Proof.* Parts (a) and (c) follow from theorem 1.20. (b) follows from the definition of $\mathcal{A}_3$ and the characterization of the minimal section of $\mathcal{A}_2$.

We can finally state the following theorem.

THEOREM 4.7. Assume that $g: \Omega \times \mathbf{R} \to \mathbf{R}$ is a measurable function satisfying (g.1), (g.2) and (g.3), $\varphi \in W^{1,2}(\Omega)$ with $\Delta\varphi \in L^2(\Omega)$.

Then for every $u_0$ in $H_0^1(\Omega) \cap K \cap S_\rho$ such that (N.T.$u_0$) holds at $u_0$ there exist $T > 0$, $\mathcal{U}: [0, T[ \to L^2(\Omega)$, an absolutely continuous curve and $\Lambda: [0, T[ \to \mathbf{R}$ such that $\mathcal{U}(0) = u_0$, $\mathcal{U}(t) \in H_0^1(\Omega) \cap K \cap S_\rho$ for every $t$ in $[0, T[$ and for almost all $t$ in $[0, T[$ [$E$ was introduced in (g.3)]:

$$\Delta\mathcal{U}(t) \in L^2(\Omega);$$

$$\mathcal{U}'(t) = \begin{cases} \Delta\mathcal{U}(t) - g(\cdot, \mathcal{U}(t)) + \Lambda(t)\mathcal{U}(t) \\ \qquad \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) > \varphi(x), \mathcal{U}(t)(x) \notin E\}, \\ [\Delta\mathcal{U}(t) - g(\cdot, \mathcal{U}(t)) + \Lambda(t)\mathcal{U}(t)] \vee 0 \\ \qquad \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) = \varphi(x), \mathcal{U}(t)(x) \notin E\}, \\ 0 \qquad \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) \in E\}, \end{cases}$$

$$\Delta\mathcal{U}(t) = 0 \in \begin{cases} [\underline{g}(\cdot, \mathcal{U}(t)) + \Lambda(t)\mathcal{U}(t), \bar{g}(\cdot, \mathcal{U}(t)) + \Lambda(t)\mathcal{U}(t)] \\ \qquad \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) > \varphi(x), \mathcal{U}(t)(x) \in E\}, \\ ]-\infty, \bar{g}(\cdot, \mathcal{U}(t)) + \Lambda(t)\mathcal{U}(t)] \\ \qquad \text{a.e. in } \{x \in \Omega \mid \mathcal{U}(t)(x) = \varphi(x), \mathcal{U}(t)(x) \in E\}; \end{cases}$$

moreover the function:

$$t \mapsto m(t) = \tfrac{1}{2}\int_\Omega |D\mathcal{U}(t)|^2 \, dx + \int_\Omega G(x, \mathcal{U}(t)) \, dx$$

is continuous, nonincreasing and $m'(t) = \|\mathcal{U}'(t)\|_2^2$ a.e. in $[0, T[$.

*Proof.* From proposition 4.4 and from the fact that $f_2$ is $d\bar{\nabla}$-continuous, we deduce that $f_3$ is $d\bar{\nabla}$-continuous, so there exists $\mathcal{U}$, a curve of maximal relaxed slope for $f_3$ such that $\mathcal{U}(0) = u_0, f_3 \circ \mathcal{U}$ is continuous. The remainder of the proof consists in applying theorem 1.20, using proposition 4.6.

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## APPENDIX

We want to prove lemma 2.10. We can prove it in a more general form.

Lemma A.1. Let $I$ be an interval, $\mathcal{U} \in L^1(I, L^1(\Omega))$, $\mathcal{U}$ be absolutely continuous and $\mathcal{U}' \in L^1(I, L^1(\Omega))$. Suppose that $N$ is a negligible subset of $\mathbf{R}$.

Then for a.e. $t$ in $I$:

$$\text{meas}(\{x \in \Omega \mid \mathcal{U}(t)(x) \in N, \mathcal{U}'(t)(x) \neq 0\}) = 0.$$

For the proof we need some preliminary results.

Lemma A.2. Let $u \in L^1_{\text{loc}}(\Omega)$ and suppose that the $i$th distributional derivative of $u$, denoted by $\partial u/\partial x_i$, is an element of $L^1_{\text{loc}}(\Omega)$. Then there exists a function $\tilde{u}$, which is almost everywhere equal to $u$, such that, for a.e. $(n-1)$-tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, the function $\xi \mapsto \tilde{u}(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)$ is absolutely continuous and

$$\left[\frac{\partial \tilde{u}(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)}{\partial x_i}\right] = \left[\frac{\partial u(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)}{\partial x_i}\right] \qquad \text{for a.e. } \xi$$

where $[\partial \tilde{u}/\partial x_i]$ denotes the classical pointwise derivative.

*Proof.* The proof is given in [18, (5.6.3)].

LEMMA A.3. If $q \geq 1$, $\mathcal{U} \in L^q(I, L^q(\Omega))$, then there exists $\tilde{\mathcal{U}} \in L^q(I \times \Omega)$ such that, for a.e. $t$ in $I$:

$$\tilde{\mathcal{U}}(t, x) = \mathcal{U}(t)(x) \qquad \text{a.e. in } \Omega.$$

*Proof.* See [18, (2.20.9)].

LEMMA A.4. Let $h: \mathbf{R} \to \mathbf{R}$ be an absolutely continuous function and $N$ be a negligible subset of $\mathbf{R}$. Then:

$$\operatorname{meas}(\{t \in \mathbf{R} \mid h(t) \in N, h'(t) \text{ exists and } h'(t) \neq 0\}) = 0.$$

*Proof.* See [16, (4.14)].

*Proof of lemma* A.1. By lemma A.3 we can find $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ in $L^1(I \times \Omega)$ such that, for a.e. $t$ in $I$:

$$\tilde{\mathcal{U}}(t, x) = \mathcal{U}(t)(x), \qquad \tilde{\mathcal{V}} = \mathcal{U}'(t)(x) \qquad \text{a.e. in } \Omega.$$

We claim that $\tilde{\mathcal{V}}$ is the distributional $t$-derivative of $\tilde{\mathcal{U}}$. To see this, take $\tilde{\varphi}$ in $C_0^\infty(I \times \Omega)$; applying Fubini's theorem we get:

$$\begin{aligned}
\int_{I \times \Omega} \tilde{\mathcal{U}}(t, x) \frac{\partial}{\partial t} \tilde{\varphi}(t, x) \, \mathrm{d}x \, \mathrm{d}t &= \int_I \mathrm{d}t \int_\Omega \tilde{\mathcal{U}}(t, x) \frac{\partial}{\partial t} \tilde{\varphi}(t, x) \, \mathrm{d}x \\
&= \int_I \langle \mathcal{U}(t), \varphi'(t) \rangle \, \mathrm{d}t = -\int_I \langle \mathcal{U}'(t), \varphi(t) \rangle \, \mathrm{d}t \\
&= -\int_I \mathrm{d}t \int_\Omega \tilde{\mathcal{V}}(t, x) \tilde{\varphi}(t, x) \, \mathrm{d}x = -\int_{I \times \Omega} \tilde{\mathcal{V}}(t, x) \tilde{\varphi}(t, x) \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

where we denote by $\varphi$ the map $\varphi \in C_0^\infty(I, L^1(\Omega))$ defined by $\varphi(t)(x) = \tilde{\varphi}(t, x)$.

Now let $F = \{(t, x) \in I \times \Omega \mid \tilde{\mathcal{U}}(t, x) \in N, \tilde{\mathcal{V}}(t, x) \neq 0\}$. To prove lemma A.1 it suffices to prove that $F$ is measurable and that its measure is zero. For this we take a Borel set $N_1$ such that $N \subset N_1$ and $N_1$ is still negligible and we set: $F_1 = \{(t, x) \in I \times \Omega \mid \tilde{\mathcal{U}}(t, x) \in N_1, \tilde{\mathcal{V}}(t, x) \neq 0\}$. Clearly $F_1$ is measurable and contains $F$. By Fubini's theorem:

$$\operatorname{meas}(F_1) = \int_\Omega \operatorname{meas}_I(\{t \in I \mid \tilde{\mathcal{U}}(t, x) \in N_1, \tilde{\mathcal{V}}(t, x) \neq 0\}) \, \mathrm{d}x.$$

By lemma A.2, we can suppose that, for a.e. $x$ in $\Omega$, the function $t \mapsto \tilde{\mathcal{U}}(t, x)$ is absolutely continuous and its pointwise derivative is equal to $\tilde{\mathcal{V}}(t, x)$ for a.e. $t$ in $I$. By lemma A.4 we get:

$$\operatorname{meas}_I(\{t \in I \mid \tilde{\mathcal{U}}(t, x) \in N_1, \tilde{\mathcal{V}}(t, x) \neq 0\}) = 0 \qquad \text{a.e. in } \Omega.$$

So $\operatorname{meas}(F_1) = 0 \Rightarrow \operatorname{meas}(F) = 0$, and the lemma is proved.

*Remark* A.5. Finally we wish to point out a finite dimensional result which follows easily by the previous arguments. Let $\Omega \subset \mathbf{R}^N$ be open and $G: \Omega \to \mathbf{R}$ be Lipschitz continuous. It is not difficult to see that, if $G$ has almost everywhere continuous partial derivatives [compare with (g.3)], then $\mathcal{A}(G)$ is a subdifferential along curves for $G$ (see definitions 1.14 and 1.16). Then, using the results of Section 1, it can be proved that the problem

$$\begin{cases} -\mathcal{U}' \in \mathcal{A}(G)(\mathcal{U}) \\ \mathcal{U}(0) = u_0 \end{cases}$$

has a solution. This seems interesting to us, since, in general, the map $\mathcal{A}(G)$ does not have convex values.