Differential Integral Equations **8**, (1995), 1789–1804.

BOUNDARY AND PERIODIC VALUE PROBLEMS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS UNDER BERNSTEIN-NAGUMO GROWTH CONDITION

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Abstract. In this paper, we establish some existence results for boundary and periodic value problems for systems of nonlinear differential equations with right-hand side satisfying a Berntein-Nagumo growth condition. Hartman's condition ($|f| \leq 2k(\langle x, f\rangle + |x'|^2) + K$) is not assumed. This assumption is replaced by one which is automatically satisfied in the scalar case.

1. Introduction

In this paper, we consider the boundary and periodic value problem for systems of nonlinear differential equations

$$(\star)\quad \begin{cases} x''(t) = f(t, x(t), x'(t)) \text{ a.e. } t \in [0,1] \\ \quad x \in BC \end{cases}$$

where $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a Carathéodory function and BC denotes a boundary condition such as non-homogeneous Dirichlet, Neumann, Sturm-Liouville conditions, or the periodic condition that we write

$$(SL)\quad \begin{cases} A_0x(0) - \beta_0 x'(0) = r_0, \\ A_1x(1) + \beta_1 x'(1) = r_1; \end{cases}$$

$$(P)\quad \begin{cases} x(0) = x(1), \\ x'(0) = x'(1); \end{cases}$$

where A_i is a $n \times n$ matrix (possibly nonsymmetric) for which there exists $\alpha_i \geq 0$ such that $\langle x, A_i x\rangle \geq \alpha_i \|x\|^2$ for all x in $\mathbb{R}^n$; $\beta_i = 0, 1$; $\alpha_i + \beta_i > 0$; $i = 0, 1$.

The literature on this problem is voluminous, and we refer to [1,2,4–6,9–14] and the references therein. Among those results, let us mention the following well known result of Hartman [11], and a result in the scalar case ($n = 1$).

1991 *Mathematics Subject Classification.* 34B15, 34A34.

Reaserch supported in part by NSREC-Canada and FCAR-Québec.

Theorem 1.1. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a continuous function. Assume*

(1.1) *there exists a constant $M > 0$ such that $\langle x, f(t,x,p)\rangle + \|p\|^2 \geq 0$ for $\|x\| = M$ and $\langle x,p\rangle = 0$;*

(1.2) *there exist k, $K \geq 0$ such that $\|f(t,x,p)\| \leq 2k(\langle x, f(t,x,p)\rangle + \|p\|^2) + K$ for $\|x\| \leq M$;*

(1.3) *there exists a continuous function $\psi : [0,\infty) \to (0,\infty)$ such that $\|f(t,x,p)\| \leq \psi(\|p\|)$ for $\|x\| \leq M$, and $\int^\infty s\,ds/\psi(s) = \infty$.*

Let $\|r_0\|$, $\|r_1\| \leq M$, then the problem

$$\begin{aligned} &x''(t) = f(t, x(t), x'(t)) \text{ a.e. } t \in [0,1] \\ &x(0) = r_0, \; x(1) = r_1 \end{aligned} \tag{1.4}$$

has a solution.

Theorem 1.2. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume*

(1.5) *there exist $\alpha \leq \beta \in C^2([0,1],\mathbb{R})$ respectively lower and upper solutions of $(\star)$;*

(1.6) *there exists a continuous function $\psi : [0,\infty) \to (0,\infty)$ such that $|f(t,x,p)| \leq \psi(|p|)$ for $\alpha(t) \leq x \leq \beta(t)$, and $\int^\infty s\,ds/\psi(s) = \infty$.*

Then the problem $(\star)$ has a solution.

Observe that in the scalar case, if (1.1) and (1.3) are satisfied and $|r_0|$, $|r_1| \leq M$, Theorem 1.2 gives the existence of a solution to (1.4) while Theorem 1.1 could not be applied since conditon (1.2) is not necessarily satisfied.

Also, Theorem 1.1 does not give the existence of a solution to very simple problems such as

$$\begin{aligned} &x''(t) = \|x'(t)\|\, x'(t) - c \\ &x(0) = x(1) = (0, \cdots, 0) \end{aligned} \tag{1.7}$$

where $c \in \mathbb{R}^n$ with $\|c\| = 1$.

In this paper, we give theorems of existence of solution to $(\star)$ containing, as a particular case, Theorem 1.2 (see Theorem 4.1), and which could be applied to problems such as (1.7). Our existence results are obtained under an assumption of existence of solution-tube. This notion generalizes in a natural way conditions (1.1) and (1.5) and is slightly more general than the notion of Nagumo pair (see [6,9]). Also, condition (1.2) is not assumed. This condition is replaced by one ((H3) or (H5)) which is automatically satisfied in the scalar case. Let us mention that our condition (H3) generalizes a condition of Gaprindashvili [9, condition (1.4)] while his Nagumo growth condition is weaker than ours.

This paper is divided in five sections. Section 2 contains notations, definitions and results which will be used throughout this paper. In section 3, theorems of existence are established under a Bernstein-type growth condition, while in section 4, results are obtained under a Nagumo-type growth condition. In section 5, very simple examples of the previous results are given. Proofs are obtained via the theory of topological transversality for continuous, compact operators in §3, and for upper semi-continuous, compact, multivalued operators in §4.

2. Preliminaries

In this section, we establish notations, definitions, and results which are used throughout this paper. We denote $\langle\,,\rangle$ the scalar product, and $\|\cdot\|$ the Euclidian norm in $\mathbb{R}^n$. The Banach space of k-times continuously differentiable functions x is denoted by $C^k([0,1],\mathbb{R}^n)$ with the norm: $\|x\|_k = \max\{\|x\|_0, \|x'\|_0, \dots, \|x^{(k)}\|_0\}$, where $\|x\|_0 = \max\{\|x(t)\| : t \in [0,1]\}$. The Sobolev space of functions in $C^1([0,1],\mathbb{R}^n)$ with the derivative being absolutely continuous is denoted by $W^{2,1}([0,1],\mathbb{R}^n)$. We define $C_0([0,1],\mathbb{R}^n) = \{x \in C([0,1],\mathbb{R}^n) : x(0) = 0\}$, and $C^k_B([0,1],\mathbb{R}^n)$, (resp. $W^{2,1}_B([0,1],\mathbb{R}^n)$) the set of functions $x \in C^k([0,1],\mathbb{R}^n)$ (resp. $W^{2,1}([0,1],\mathbb{R}^n)$) satisfying the boundary condition $x \in BC$. Let $L^1([0,1],\mathbb{R}^n)$ denote the space of integrable functions, with the usual norm $\|\cdot\|_{L^1}$.

We say that a function $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ (resp. $G : [0,1] \times \mathbb{R}^{2n} \to 2^{\mathbb{R}^n}$ a multivalued function with non-empty, closed, convex values) is a *Carathéodory function* if, (i) for every (x,p) in $\mathbb{R}^{2n}$, the function $t \mapsto f(t,x,p)$ (resp. $t \mapsto G(t,x,p)$) is measurable; (ii) the function $(x,p) \mapsto f(t,x,p)$ (resp. $(x,p) \mapsto G(t,x,p)$) is continuous (resp. upper semi-continuous) for almost every t in $[0,1]$; (iii) for every $k > 0$, there exists a function h_k in $L^1([0,1],[0,\infty))$ such that $\|f(t,x,p)\| \le h_k(t)$ (resp. $\|G(t,x,p)\| \le h_k(t)$ i.e. $\|v\| \le h_k(t)$ for all $v \in G(t,x,p)$) a.e. $t \in [0,1]$, and for all $\|x\| \le k$ and $\|p\| \le k$. Observe that if $G(t,x,p) = \{f(t,x,p)\}$ then G is Carathéodory if and only if f is Carathéodory. A function $F : C^1([0,1],\mathbb{R}^n) \to L^1([0,1],\mathbb{R}^n)$ (resp. $\mathcal{G} : C^1([0,1],\mathbb{R}^n) \to 2^{L^1([0,1],\mathbb{R}^n)}$) is said *integrably bounded on bounded* if for every bounded set $B \subset C^1([0,1],\mathbb{R}^n)$, there exists an integrable function h_B in $L^1([0,1],[0,\infty))$ such that for every $x \in B$, $\|F(x)(t)\| \le h_B(t)$ a.e. $t \in [0,1]$ (resp. $\|u(t)\| \le h_B(t)$ a.e. $t \in [0,1]$ and for all $u \in \mathcal{G}(x)$). We associate to F (resp. $\mathcal{G}$) an operator $N_F : C^1([0,1],\mathbb{R}^n) \to C_0([0,1],\mathbb{R}^n)$ (resp. $N_{\mathcal{G}} : C^1([0,1],\mathbb{R}^n) \to 2^{C_0([0,1],\mathbb{R}^n)}$) defined by

$$N_F(x)(t) = \int_0^t F(x)(s)ds$$

$$(\text{resp.} \quad N_{\mathcal{G}}(x) = \{w(t) = \int_0^t u(s)ds : u \in \mathcal{G}(x)\,\}\,).$$

We recall the following result (see for example [7,10]).

Lemma 2.1.

(i) *Let $G : [0,1] \times \mathbb{R}^{2n} \to 2^{\mathbb{R}^n}$ be a Carathéodory multivalued function with non-empty, closed, convex values then the operator $\mathcal{G} : C^1([0,1],\mathbb{R}^n) \to 2^{L^1([0,1],\mathbb{R}^n)}$ defined by $\mathcal{G}(x) = \{u : u(t) \in G(t,x(t),x'(t))$ a.e. $t \in [0,1]\}$ is upper semi-continuous, integrably bounded on bounded, with non-empty, closed, convex values.*

(ii) *Let $\mathcal{F} : C^1([0,1],\mathbb{R}^n) \to 2^{L^1([0,1],\mathbb{R}^n)}$ be an upper semi-continuous multivalued function, integrably bounded on bounded, with non-empty, closed, convex values, then the associated operator $N_{\mathcal{F}}$ is upper semi-continuous and completely continuous, with non-empty, compact, convex values.*

Let us give some notions of the theory of topological transversality; for more details and generality, see [3].

Let U be a bounded open set in $C^1_B([0,1],\mathbb{R}^n)$. By $\mathcal{K}_{\partial U}(\overline{U}, 2^{C^1_B([0,1],\mathbb{R}^n)})$, we denote the set of upper semi-continuous and compact operators with non-empty, compact, convex values $T : \overline{U} \to 2^{C^1_B([0,1],\mathbb{R}^n)}$ fixed point free on ∂U. We say that $T \in \mathcal{K}_{\partial U}(\overline{U}, 2^{C^1_B([0,1],\mathbb{R}^n)})$ is *essential* if for every $R \in \mathcal{K}_{\partial U}(\overline{U}, 2^{C^1_B([0,1],\mathbb{R}^n)})$

such that $T\big|_{\partial U} = R\big|_{\partial U}$, R has a fixed point. Let T, $R \in \mathcal{K}_{\partial U}(\overline{U}, 2^{C^1_B([0,1],\mathbb{R}^n)})$, T is *homotopic* to R ($T \approx R$) if there exists $H : [0,1] \times \overline{U} \to 2^{C^1_B([0,1],\mathbb{R}^n)}$ upper semi-continuous and compact with non-empty, compact, convex values such that $H(\lambda, \cdot) \in \mathcal{K}_{\partial U}(\overline{U}, 2^{C^1_B([0,1],\mathbb{R}^n)})$ for every $\lambda \in [0,1]$; $T = H(1,\cdot)$ and $R = H(0,\cdot)$. We have similar definitions for $\mathcal{K}_{\partial U}(\overline{U}, C^1_B([0,1],\mathbb{R}^n))$, the set of continuous, compact operators fixed point free on ∂U.

Theorem 2.2. (Topological Transversality). *Let T and R be homotopic operators in $\mathcal{K}_{\partial U}(\overline{U}, C^1_B([0,1],\mathbb{R}^n))$ (resp. $\mathcal{K}_{\partial U}(\overline{U}, 2^{C^1_B([0,1],\mathbb{R}^n)})$), then T is essential if and only if R is essential.*

Let us consider the problem $(\star)$ where BC denotes (P) or (SL). A *solution* to $(\star)$ is a function $x \in W^{2,1}_B([0,1],\mathbb{R}^n)$ satisfying $(\star)$.

Now, we give the definition of *solution-tube* to the problem $(\star)$ which was introduced in [6] and which is slightly more general than the definition of Nagumo pair given by Gaprindashvili [9]. This notion will play an essential role in our existence results.

Definition 2.3. A *solution-tube* to the problem $(\star)$ is a couple (v, M) where M is a non-negative function in $W^{2,1}([0,1],\mathbb{R})$, and $v \in W^{2,1}([0,1],\mathbb{R}^n)$ such that

(i) $\langle x - v(t), f(t,x,p) - v''(t)\rangle + \|p - v'(t)\|^2 \geq M(t)M''(t) + (M'(t))^2$ a.e. $t \in [0,1]$ and for all $(x,p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| = M(t)$, and $\langle x - v(t), p - v'(t)\rangle = M(t)M'(t)$;
and $v''(t) = f(t, v(t), v'(t))$ a.e. on $\{t \in [0,1] : M(t) = 0\}$;

(ii) if BC denotes (SL), $\|r_0 - (A_0 v(0) - \beta_0 v'(0))\| \leq \alpha_0 M(0) - \beta_0 M'(0)$, $\|r_1 - (A_1 v(1) + \beta_1 v'(1))\| \leq \alpha_1 M(1) + \beta_1 M'(1)$;
and if BC denotes (P), $v(0) = v(1)$, $\|v'(1) - v'(0)\| \leq M'(1) - M'(0)$, and $M(0) = M(1)$.

Observe that it is assumed in Theorem 1.1 that $(0, M)$ is a solution-tube to (1.4). In fact, many results were obtained under an assumption of existence of a solution-tube of the form $(0, M)$ with M being a positive constant, see for example [1,4,11,14].

Remark also that in the scalar case, the notion of upper and lower solutions is equivalent to the notion of solution-tube.

For sake of completeness, we state the following results which will be used later in this paper.

Lemma 2.4. *Let $u : [0,1] \to \mathbb{R}^n$ be an absolutely continuous function and let E be a negligeable set in $\mathbb{R}^n$, then $meas\{t \in [0,1] : u(t) \in E$ and $u'(t) \neq 0\} = 0$.*

Lemma 2.5. *Let $u \in W^{2,1}([0,1],\mathbb{R})$ and $\varepsilon \geq 0$. Assume one of the following properties is satisfied:*

(i) $u''(t) - \varepsilon u(t) \geq 0$ *a.e.* $t \in [0,1]$*;* $a_0 u(0) - b_0 u'(0) \leq 0$, $a_1 u(1) + b_1 u'(1) \leq 0$, *where* $a_i, b_i \geq 0$, $\max\{a_i, b_i\} > 0$, $\max\{a_0, a_1, \varepsilon\} > 0$*;*

(ii) $u''(t) - \varepsilon u(t) \geq 0$ *a.e.* $t \in [0,1]$*;* $\varepsilon > 0$, $u(0) = u(1)$, $u'(1) - u'(0) \leq 0$*;*

(iii) $u''(t) - \varepsilon u(t) \geq 0$ *a.e.* $t \in [0,t_1] \cup [t_2,1]$*;* $\varepsilon > 0$, $u(0) = u(1)$, $u'(1) - u'(0) \leq 0$, *and* $u(t) \leq 0$ *for* $t \in [t_1, t_2]$.

Then $u(t) \leq 0$ for all $t \in [0,1]$.

3. Bernstein-type growth condition

The following theorem gives the existence of solution when the function f satisfies a Bernstein-type growth condition.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume*

(H1) *there exists (v, M) a solution-tube to $(\star)$;*

(H2) *there exist $C, D > 0$ and a function $h \in L^1([0,1])$ such that $|\langle p, f(t,x,p)\rangle| \leq (C\|p\|^2 + D)(h(t) + \|p\|)$ a.e. $t \in [0,1]$ and for all $(x,p) \in \mathbb{R}^{2n}$ with $\|x - v(t)\| \leq M(t)$;*

(H3) *there exist $k, \theta, \gamma > 0$, $m \geq 0$, $h_1, h_2 \in L^1([0,1])$ such that for a.e. $t \in [0,1]$ and for all $(x,p) \in \mathbb{R}^{2n}$ with $\|x - v(t)\| \leq M(t)$ and $\|p\| \geq k$,*

(i) $$\frac{\langle x, f(t,x,p)\rangle + \|p\|^2}{\|p\|} - \frac{\langle p, f(t,x,p)\rangle \langle x, p\rangle}{\|p\|^3} \geq \theta\|p\| - m|\langle x, p\rangle| - h_1(t);$$

(ii) $$\|x\| \left(\frac{\langle x, f(t,x,p)\rangle + \|p\|^2}{\|p\|} - \frac{\langle p, f(t,x,p)\rangle\langle x, p\rangle}{\|p\|^3} \right) + \frac{\langle x, p\rangle^2}{\|x\|\,\|p\|} \geq \gamma|\langle x, p\rangle| - h_2(t).$$

Then the problem $(\star)$ has a solution such that $\|x(t) - v(t)\| \leq M(t)$ for all $t \in [0,1]$.

To prove this theorem, we need the following three lemmas.

Fix $\varepsilon \geq 0$ such that the operator $L_\varepsilon : C^1_B([0,1], \mathbb{R}^n) \to C_0([0,1], \mathbb{R}^n)$ defined by

$$L_\varepsilon(x)(t) = x'(t) - x'(0) - \varepsilon \int_0^t x(s)ds.$$

is invertible. In particular, if BC denotes (SL) with $\max\{\alpha_0, \alpha_1\} > 0$, we can take $\varepsilon = 0$ (see [8]).

Lemma 3.2. *Let (v, M) be a solution-tube to $(\star)$. If $x \in W^{2,1}_B([0,1], \mathbb{R}^n)$ satisfies*

$$\frac{\langle x(t) - v(t), x''(t) - v''(t)\rangle + \|x'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} - \varepsilon\|x(t) - v(t)\| \geq M''(t) - \varepsilon M(t)$$

a.e. on $\{t \in [0,1] : \|x(t) - v(t)\| > M(t)\}$. Then $\|x(t) - v(t)\| \leq M(t)$ for every $t \in [0,1]$.

Proof. Let $E_\delta = \{t \in [0,1] : \|x(t) - v(t)\| > M(t) + \delta\}$ with $\delta > 0$. If, $E_\delta \neq \emptyset$ for some $\delta > 0$, then, for every interval $(t_0, t_1) \subset E_\delta$ such that $\|x(t_0) - v(t_0)\| = M(t_0) + \delta$ or $t_0 = 0$, and $\|x(t_1) - v(t_1)\| = M(t_1) + \delta$ or $t_1 = 1$, the function $\|x(t) - v(t)\|$ belongs to the space $W^{2,1}([t_0, t_1], \mathbb{R})$ and we have

$$\|x(t) - v(t)\|' = \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle}{\|x(t) - v(t)\|}$$

which exists for all $t \in [t_0, t_1]$, and

$$\|x(t) - v(t)\|'' = \frac{\langle x(t) - v(t), x''(t) - v''(t)\rangle + \|x'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3}$$

a.e. $t \in [t_0, t_1]$.

Denote $w(t) = \|x(t)-v(t)\|-(M(t)+\delta)$. By assumption, we have a.e. on $[t_0, t_1]$,

$$\begin{aligned} w''(t) - \varepsilon w(t) &= \|x(t) - v(t)\|'' - \varepsilon\|x(t) - v(t)\| - M''(t) + \varepsilon(M(t) + \delta) \\ &\geq 0. \end{aligned}$$

In order to apply the maximum principle (Lemma 2.5), we need to verify some boundary conditions. If BC denotes (SL) then, either

$$w(t_0) \leq 0, \quad \text{or} \quad t_0 = 0 \;\text{ and }\; \alpha_0 w(0) - \beta_0 w'(0) \leq 0.$$

Indeed,

$$\begin{aligned} &\|x(0) - v(0)\| \,(\alpha_0\|x(0) - v(0)\| - \beta_0\|x(0) - v(0)\|') \\ &\qquad \leq \langle x(0) - v(0), A_0(x(0) - v(0)) - \beta_0(x'(0) - v'(0))\rangle \\ &\qquad \leq \|x(0) - v(0)\| \,\|r_0 - (A_0 v(0) - \beta_0 v'(0))\| \\ &\qquad \leq \|x(0) - v(0)\|(\alpha_0 M(0) - \beta_0 M'(0)) \\ &\qquad \leq \|x(0) - v(0)\|(\alpha_0(M(0) + \delta) - \beta_0(M + \delta)'(0)). \end{aligned}$$

Similarly, either

$$w(t_1) \leq 0, \quad \text{or} \quad t_1 = 1 \;\text{ and }\; \alpha_1 w(1) + \beta_1 w'(1) \leq 0.$$

On the other hand, when BC denotes the periodic boundary condition (P), if $w(t_0) \leq 0$ and $w(t_1) \leq 0$, we argue as in the previous case. Otherwise, $[t_0, t_1] = [0, 1]$, or $(t_0, t_1) \subset (0, t_2) \cup (t_3, 1) \subset E_\delta$ and $w(t_2) = w(t_3) = 0$, and we have

$$\|x(0) - v(0)\| = \|x(1) - v(1)\|, \quad M(0) = M(1), \quad \text{and} \quad w'(1) - w'(0) \leq 0.$$

Indeed,

$$\begin{aligned} \|x(1) - v(1)\|' - \|x(0) - v(0)\|' &= \frac{\langle x(0) - v(0), v'(0) - v'(1)\rangle}{\|x(0) - v(0)\|} \\ &\leq \|v'(1) - v'(0)\| \leq M'(1) - M'(0). \end{aligned}$$

By Lemma 2.5 applied to w, we deduce that $\|x(t) - v(t)\| \leq M(t) + \delta$. But this inequality holds for every $\delta > 0$; therefore, $\|x(t) - v(t)\| \leq M(t)$ for all $t \in [0, 1]$. This completes the proof. $\square$

Lemma 3.3. *Let $u \in W^{2,1}([0,1], \mathbb{R}^n)$, $l_1, l_2 \in L^1([0,1])$, $\theta_1, \gamma_1, k_1 > 0$, $m_1 \geq 0$. If $x \in W^{2,1}([0,1], \mathbb{R}^n)$ satisfies a.e. on $\{t \in [0,1] : \|x'(t) - u'(t)\| \geq k_1\}$,*

(i) $$\begin{aligned} &\frac{\langle x(t) - u(t), x''(t) - u''(t)\rangle + \|x'(t) - u'(t)\|^2}{\|x'(t) - u'(t)\|} \\ &\qquad - \frac{\langle x'(t) - u'(t), x''(t) - u''(t)\rangle\langle x(t) - u(t), x'(t) - u'(t)\rangle}{\|x'(t) - u'(t)\|^3} \\ &\quad \geq \theta_1\|x'(t) - u'(t)\| - m_1|\langle x(t) - u(t), x'(t) - u'(t)\rangle| - l_1(t); \end{aligned}$$

(ii) $$\begin{aligned} &\|x(t) - u(t)\| \left(\frac{\langle x(t) - u(t), x''(t) - u''(t)\rangle + \|x'(t) - u'(t)\|^2}{\|x'(t) - u'(t)\|}\right) \\ &\quad -\|x(t)-u(t)\| \left(\frac{\langle x'(t) - u'(t), x''(t) - u''(t)\rangle\langle x(t) - u(t), x'(t) - u'(t)\rangle}{\|x'(t) - u'(t)\|^3}\right) \\ &\quad + \frac{\langle x(t) - u(t), x'(t) - u'(t)\rangle^2}{\|x(t) - u(t)\| \,\|x'(t) - u'(t)\|} \\ &\geq \gamma_1|\langle x(t) - u(t), x'(t) - u'(t)\rangle| - l_2(t). \end{aligned}$$

Then there exists $K_1(\|x-u\|_0)$ such that for any interval $[a,b]$ on which $\|x'(t)-u'(t)\| \geq k_1$ we have $\|x'-u'\|_{L^1[a,b]} \leq K_1(\|x-u\|_0)$. Moreover, there exists $t \in [0,1]$ such that $\|x'(t)-u'(t)\| \leq \max\{k_1, K_1(\|x-u\|_0)\}$.

Proof. Assume $\|x'(t)-u'(t)\| \geq k_1$ on $[a,b]$. Then, by (ii),

$$\begin{aligned}
&\int_a^b |\langle x(t)-u(t), x'(t)-u'(t)\rangle|\, dt \\
&\qquad \leq \frac{1}{\gamma_1}\int_a^b l_2(t) + \frac{d}{dt}\frac{\|x(t)-u(t)\|\,\langle x(t)-u(t), x'(t)-u'(t)\rangle}{\|x'(t)-u'(t)\|}\, dt \\
&\qquad \leq \frac{1}{\gamma_1}\left(\|l_2\|_{L^1[0,1]} + 2(\|x-u\|_0)^2\right) = K_2(\|x-u\|_0).
\end{aligned}$$

Now, (i) gives

$$\begin{aligned}
&\int_a^b \|x'(t)-u'(t)\|\, dt \\
&\leq \frac{1}{\theta_1}\int_a^b l_1(t) + m_1|\langle x(t)-u(t), x'(t)-u'(t)\rangle| + \frac{d}{dt}\frac{\langle x(t)-u(t), x'(t)-u'(t)\rangle}{\|x'(t)-u'(t)\|}\, dt \\
&\leq \frac{1}{\theta_1}\left(\|l_1\|_{L^1[0,1]} + m_1 K_2(\|x-u\|_0) + 2\|x-u\|_0\right) = K_1(\|x-u\|_0).
\end{aligned}$$

Moreover, there exists $t \in [0,1]$ such that $\|x'(t)-u'(t)\| \leq \max\{k_1, K_1(\|x-u\|_0)\}$ since, either $\|x'(t)-u'(t)\| \leq k_1$ for some $t \in [0,1]$, or $\|x'-u'\|_{L^1[0,1]} \leq K_1(\|x-u\|_0)$. □

Lemma 3.4. *Let $k_0, K_0 \geq 0$, $l \in L^1([0,1])$ and $\psi : [0,\infty) \to (0,\infty)$ be a Borel measurable function such that*

$$\int_{k_0}^{\infty} \frac{s ds}{\psi(s)} > \|l\|_{L^1} + K_0.$$

If $x \in W^{2,1}([0,1],\mathbb{R}^n)$ satisfies

- (i) *there exists $t \in [0,1]$ such that $\|x'(t)\| \leq k_0$;*
- (ii) *$\|x'\|_{L^1[a,b]} \leq K_0$ if $\|x'(t)\| \geq k_0$ on $[a,b]$;*
- (iii) *$|\langle x'(t), x''(t)\rangle| \leq \psi(\|x'(t)\|)(l(t) + \|x'(t)\|)$ a.e. on $\{t : \|x'(t)\| \geq k_0\}$.*

Then there exists $K = K(\psi, l, k_0, K_0)$ such that $\|x'\|_0 < K$.

Proof. Fix $K > k_0$ such that

$$\int_{k_0}^{K} \frac{s ds}{\psi(s)} > \|l\|_{L^1} + K_0. \tag{3.1}$$

We claim that $\|x'(t)\| < K$ for all $t \in [0,1]$. If not, there exist $a, b \in [0,1]$ such that $\|x'(a)\| = k_0$, $\|x'(b)\| = K$, and $k_0 < \|x'(t)\| \leq K$ for all t between a and b. Without loss of generality, assume that $a < b$, then

$$\|x'(t)\|' = \frac{\langle x'(t), x''(t)\rangle}{\|x'(t)\|}$$

exists for all $t \in (a, b]$. Thus,

$$\|x'(t)\| \, \|x'(t)\|' \leq \psi(\|x'(t)\|)(l(t) + \|x'(t)\|)$$

a.e. $t \in (a, b)$. Dividing by $\psi(\|x'(t)\|)$, integrating from a to b, we obtain

$$\int_a^b \frac{\|x'(t)\| \, \|x'(t)\|'}{\psi(\|x'(t)\|)} \, dt \leq \|l\|_{L^1} + K_0.$$

By the inequality (3.1) and the change of variables formula (see [7]), we get a contradiction. $\square$

To prove Theorem 3.1, we will modify the function f. To this modified function, we will associate a problem for which we will deduce the existence of a solution. Finally, we will observe that this solution is in fact a solution to our original problem $(\star)$.

Let $\lambda \in [0, 1]$ and $\varepsilon \geq 0$ be as before. We define the function $f_\lambda^\varepsilon : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ by

$$f_\lambda^\varepsilon(t, x, p) = \begin{cases} \lambda \left(\frac{M(t)}{\|x - v(t)\|} f(t, \widetilde{x}, \widehat{p}) - \varepsilon \widetilde{x} \right) - \varepsilon(1 - \lambda) v(t) + \\ \quad \left(1 - \frac{\lambda M(t)}{\|x - v(t)\|} \right) \left(v''(t) + \frac{M''(t)}{\|x - v(t)\|} (x - v(t)) \right), & \text{if } \|x - v(t)\| > M(t), \\ \lambda \left(f(t, x, p) - \varepsilon x \right) - \varepsilon(1 - \lambda) v(t) + \\ \quad (1 - \lambda) \left(v''(t) + \frac{M''(t)}{M(t)} (x - v(t)) \right), & \text{otherwise;} \end{cases}$$

where (v, M) is the solution-tube to $(\star)$ given in (H1), $\widetilde{x} = \frac{M(t)}{\|x - v(t)\|}(x - v(t)) + v(t)$, $\widehat{p} = p + \left(M'(t) - \frac{\langle x - v(t), p - v'(t) \rangle}{\|x - v(t)\|} \right) \left(\frac{x - v(t)}{\|x - v(t)\|} \right)$, and where we mean $\frac{M''(t)}{M(t)}(x - v(t)) = 0$ on $\{t \in [0, 1] : M(t) = 0\}$.

To the function f_λ^ε, we associate the operator $F_\lambda^\varepsilon : C^1([0, 1], \mathbb{R}^n) \to L^1([0, 1], \mathbb{R}^n)$ defined by

$$F_\lambda^\varepsilon(x)(t) = f_\lambda^\varepsilon(t, x(t), x'(t)).$$

The function f_λ^ε is not necessarily a Carathéodory function, but we have the following result.

Proposition 3.5. *Let $f : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function and let (v, M) be a solution-tube to $(\star)$. Then the previously defined operator F_λ^ε is continuous and integrably bounded on bounded.*

Proof. Obviously, F_λ^ε is integrably bounded on bounded independently of $\lambda \in [0, 1]$. Therefore, it is sufficient to show that if $x_n \to x$ in $C^1([0, 1], \mathbb{R}^n)$, then

$$f_\lambda^\varepsilon(t, x_n(t), {x'}_n(t)) \to f_\lambda^\varepsilon(t, x(t), x'(t)) \quad \text{a.e. } t \in [0, 1]. \tag{3.2}$$

The conclusion follows from the Lebesgue Dominated Convergence Theorem.

Since f is a Carathéodory function, it is clear from the definition of f_λ^ε that the relation (3.2) holds almost everywhere on $\{t \in [0, 1] : \|x(t) - v(t)\| \neq M(t)\}$. On the other hand, it follows from Lemma 2.4 that $\langle x(t) - v(t), x'(t) - v'(t) \rangle = M(t) M'(t)$

a.e. on $\{t \in [0,1] : \|x(t) - v(t)\| = M(t) > 0\}$. Therefore, it is easy to verify that almost everywhere on that set,

$$\widehat{x_n'(t)} \to \widehat{x'(t)};$$

hence, the relation (3.2) is satisfied.

Finally, on $\{t \in [0,1] : \|x(t) - v(t)\| = 0 = M(t)\}$, $x(t) = v(t)$, $x'(t) = v'(t)$, $M'(t) = 0$, $M''(t) = 0$ a.e. So,

$$\begin{aligned} f_\lambda^\varepsilon(t, x(t), x'(t)) &= \lambda(f(t, x(t), x'(t)) - \varepsilon x(t)) + (1-\lambda)(v''(t) - \varepsilon v(t)) \\ &= \lambda(f(t, v(t), v'(t)) - \varepsilon x(t)) + (1-\lambda)(v''(t) - \varepsilon v(t)) \\ &= \lambda(v''(t) - \varepsilon x(t)) + (1-\lambda)(v''(t) - \varepsilon v(t)) \\ &= v''(t) - \lambda\varepsilon x(t) - (1-\lambda)\varepsilon v(t) \end{aligned}$$

a.e. on that set. This completes the proof. $\square$

Let us consider the associated problems

$$(\star)_\lambda^\varepsilon \quad \begin{cases} x''(t) - \varepsilon\, x(t) = f_\lambda^\varepsilon(t, x(t), x'(t)) & \text{a.e. } t \in [0,1] \\ x \in BC \end{cases}$$

Now, we can prove Theorem 3.1.

Proof of Theorem 3.1. We will show that the problem $(\star)_1^\varepsilon$ has a solution satisfying $\|x(t) - v(t)\| \le M(t)$. By the definition of f_1^ε, this solution will be a solution to our original problem $(\star)$.

Let x be a solution to $(\star)_\lambda^\varepsilon$. On $\{t \in [0,1] : \|x - v(t)\| > M(t)\}$, we have $\|\widetilde{x}(t) - v(t)\| = M(t)$, $\langle \widetilde{x}(t) - v(t), \widehat{x'(t)} - v'(t)\rangle = M(t)M'(t)$, and $\|\widehat{x'(t)} - v'(t)\|^2 = \|x'(t) - v'(t)\|^2 + (M'(t))^2 - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^2}$. Thus, by using (H1), we obtain

$$\begin{aligned} &\frac{\langle x(t) - v(t), x''(t) - v''(t)\rangle + \|x'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} \\ &\qquad - \varepsilon\|x(t) - v(t)\| \\ &= \frac{\lambda(\langle \widetilde{x} - v(t), f(t, \widetilde{x}(t), \widehat{x'(t)}) - v''(t)\rangle + \|\widehat{x'(t)} - v'(t)\|^2) - M'(t)^2}{\|x(t) - v(t)\|} \\ &\qquad + \frac{(1-\lambda)\|\widehat{x'(t)} - v'(t)\|^2}{\|x(t) - v(t)\|} + \left(1 - \frac{\lambda M(t)}{\|x(t) - v(t)\|}\right) M''(t) - \lambda\varepsilon M(t) \\ &\ge M''(t) - \varepsilon M(t) + \frac{(1-\lambda)(\|\widehat{x'(t)} - v'(t)\|^2 - M'(t)^2)}{\|x(t) - v(t)\|} \\ &\ge M''(t) - \varepsilon M(t) \end{aligned}$$

a.e. on $\{t \in [0,1] : \|x - v(t)\| > M(t)\}$. It follows from Lemma 3.2 that every solution to $(\star)_\lambda^\varepsilon$ satisfies $\|x(t) - v(t)\| \le M(t)$ for all $t \in [0,1]$.

On the other hand, (H3) implies the existence of $\theta_1, \gamma_1 > 0$, $l_1, l_2 \in L^1([0,1])$ such that for every solution x of $(\star)_\lambda^\varepsilon$ we have a.e. on $\{t \in [0,1] : \|x'(t)\| \ge k\}$,

$$\begin{aligned} &\frac{\langle x(t), x''(t)\rangle + \|x'(t)\|^2}{\|x'(t)\|} - \frac{\langle x'(t), x''(t)\rangle\langle x(t), x'(t)\rangle}{\|x'(t)\|^3} \\ &\qquad\qquad \ge \theta_1\|x'(t)\| - m|\langle x(t), x'(t)\rangle| - l_1(t); \end{aligned}$$

and

$$\|x(t)\|\left(\frac{\langle x(t),x''(t)\rangle+\|x'(t)\|^2}{\|x'(t)\|}-\frac{\langle x'(t),x''(t)\rangle\langle x(t),x'(t)\rangle}{\|x'(t)\|^3}\right)+\frac{\langle x(t),x'(t)\rangle^2}{\|x(t)\|\,\|x'(t)\|}$$
$$\geq \gamma_1|\langle x(t),x'(t)\rangle|-l_2(t).$$

Indeed,

$$\begin{aligned}
&\frac{\langle x(t),x''(t)\rangle+\|x'(t)\|^2}{\|x'(t)\|}-\frac{\langle x'(t),x''(t)\rangle\langle x(t),x'(t)\rangle}{\|x'(t)\|^3}\\
&=\lambda\left(\frac{\langle x(t),f(t,x(t),x'(t))\rangle+\|x'(t)\|^2}{\|x'(t)\|}-\frac{\langle x'(t),f(t,x(t),x'(t))\rangle\langle x(t),x'(t)\rangle}{\|x'(t)\|^3}\right)\\
&\quad+(1-\lambda)\|x'(t)\|+\frac{(1-\lambda)\langle x(t),v''(t)+(\varepsilon+M''(t)/M(t))(x(t)-v(t))\rangle}{\|x'(t)\|}\\
&\quad-\frac{(1-\lambda)\langle x'(t),v''(t)+(\varepsilon+M''(t)/M(t))(x(t)-v(t))\rangle\langle x(t),x'(t)\rangle}{\|x'(t)\|^3}\\
&\geq \lambda\theta\|x'(t)\|-m|\langle x(t),x'(t)\rangle|+(1-\lambda)\|x'(t)\|-l_1(t)\\
&\geq \theta_1\|x'(t)\|-m|\langle x(t),x'(t)\rangle|-l_1(t);
\end{aligned}$$

and

$$\|x(t)\|\left(\frac{\langle x(t),x''(t)\rangle+\|x'(t)\|^2}{\|x'(t)\|}-\frac{\langle x'(t),x''(t)\rangle\langle x(t),x'(t)\rangle}{\|x'(t)\|^3}\right)+\frac{\langle x(t),x'(t)\rangle^2}{\|x(t)\|\,\|x'(t)\|}$$
$$\begin{aligned}
&\geq (1-\lambda)\|x(t)\|\,\|x'(t)\|+\lambda\gamma|\langle x(t),x'(t)\rangle|-l_2(t)\\
&\geq \gamma_1|\langle x(t),x'(t)\rangle|-l_2(t).
\end{aligned}$$

Lemma 3.3 applied with $u=0$ gives the existence of $K_1=K_1(\|M\|_0+\|v\|_0)>0$ such that $\|x'\|_{L^1[a,b]}\leq K_1$ for any interval $[a,b]$ on which $\|x'(t)\|\geq k$; and $\|x'(t)\|\leq \max\{k,K_1\}$ for some $t\in[0,1]$.

By (H2) and the definition of $f^{\varepsilon}_{\lambda}$, there exist $C_1,D_1>0$ such that

$$\begin{aligned}
|\langle x'(t),x''(t)\rangle|&\leq (1-\lambda)\|x'(t)\|\,\|v''(t)+(\varepsilon+M''(t)/M(t))(x(t)-v(t))\|\\
&\quad+\lambda(C\|x'(t)\|^2+D)(h(t)+\|x'(t)\|)\\
&\leq \left(C_1\|x'(t)\|^2+D_1\right)(h(t)+\|x'(t)\|+\|\varepsilon M(t)+M''(t)\|+\|v''(t)\|)
\end{aligned}$$

a.e. $t\in[0,1]$. It follows from Lemma 3.4 that there exists K such that $\|x'\|_0<K$.

A solution to $(\star)^{\varepsilon}_{\lambda}$ is a fixed point of the operator $L_\varepsilon^{-1}\circ N_{F^{\varepsilon}_{\lambda}}:\overline{U}\to C^1_B([0,1],\mathbb{R}^n)$, where $U=\{x\in C^1_B([0,1],\mathbb{R}^n):\|x\|_0<\|v\|_0+\|M\|_0+1,\ \|x'\|_0<K\}$, and L_ε and $N_{F^{\varepsilon}_{\lambda}}$ are previously defined. From Lemma 2.1 and Proposition 3.5, we deduce the continuity and the compactness of this operator. We showed that $L_\varepsilon^{-1}\circ N_{F^{\varepsilon}_{\lambda}}$ has no fixed point on ∂U. Also, it is easy to show that $L_\varepsilon^{-1}\circ N_{F^{\varepsilon}_{0}}$ is essential. The topological transversality Theorem (Theorem 2.2) gives the existence of a fixed point to $L_\varepsilon^{-1}\circ N_{F^{\varepsilon}_{1}}$, and then a solution to $(\star)^{\varepsilon}_{1}$ satisfying $\|x(t)-v(t)\|\leq M(t)$. The conclusion follows from the definition of f^{ε}_{1}. □

Remark. The assumption (H2) can we weaken. In fact, we need:
there exist a Borel measurable function $\psi:[0,\infty)\to(0,\infty)$ and a function $h\in$

$L^1([0,1])$ such that $|\langle p, \lambda f(t,x,p) + (1-\lambda)(v'' + (\varepsilon + (M''(t))^+/M(t))(x-v(t))\rangle| \le \psi(\|p\|)(h(t) + \|p\|)$ a.e. t and for all $\lambda \in [0,1]$, $p \in \mathbb{R}^n$ and x with $\|x - v(t)\| \le M(t)$;

and $\displaystyle\int_{\max\{k,K_1\}}^{\infty} \frac{sds}{\psi(s)} > \|h\|_{L^1} + K_1$,

where K_1 is the constant given in the proof of Theorem 3.1.

Gaprindashvili [9] obtained a similar result to the next one for the problem with Dirichlet boundary condition. Here, assumption (H2) is stronger while (H1) is weaker than his assumptions.

Corollary 3.6. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1), (H2) *and*

(H4) *there exist $k, \theta > 0$, $h_1 \in L^1([0,1])$ such that for a.e. t and for all (x,p) with $\|x - v(t)\| \le M(t)$ and $\|p\| \ge k$,*

$$\frac{\langle x, f(t,x,p)\rangle + \|p\|^2}{\|p\|} - \frac{\langle p, f(t,x,p)\rangle\langle x, p\rangle}{\|p\|^3} \ge \theta\|p\| - h_1(t).$$

Then the problem $(\star)$ has a solution such that $\|x(t) - v(t)\| \le M(t)$ for all $t \in [0,1]$.

Remark. In the scalar case, (H4) is satisfied with $\theta = 1$, $h_1 \equiv 0$ and any $k > 0$.

The following theorem is similar to Theorem 3.1.

Theorem 3.7. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1), (H2) *and*

(H5) *there exist $k, \theta, \gamma > 0$, $m \ge 0$, $h_1, h_2 \in L^1([0,1])$ such that for a.e. t and for all (x,p) with $\|x - v(t)\| \le M(t)$ and $\|p - v'(t)\| \ge k$,*

(i)
$$\frac{\langle x - v(t), f(t,x,p) - v''(t)\rangle + \|p - v'(t)\|^2}{\|p - v'(t)\|} - \frac{\langle p - v'(t), f(t,x,p) - v''(t)\rangle\langle x - v(t), p - v'(t)\rangle}{\|p - v'(t)\|^3} \ge \theta\|p - v'(t)\| - m|\langle x - v(t), p - v'(t)\rangle| - h_1(t);$$

(ii)
$$\|x - v(t)\|\left(\frac{\langle x - v(t), f(t,x,p) - v''(t)\rangle + \|p - v'(t)\|^2}{\|p - v'(t)\|}\right) - \|x - v(t)\|\left(\frac{\langle p - v'(t), f(t,x,p) - v''(t)\rangle\langle x - v(t), p - v'(t)\rangle}{\|p - v'(t)\|^3}\right) + \frac{\langle x - v(t), p - v'(t)\rangle^2}{\|x - v(t)\|\,\|p - v'(t)\|} \ge \gamma|\langle x - v(t), p - v'(t)\rangle| - h_2(t).$$

Then the problem $(\star)$ has a solution such that $\|x(t) - v(t)\| \le M(t)$ for all $t \in [0,1]$.

4. Nagumo-type growth condition

Theorem 4.1. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1), (H5) *and*

(H6) *there exist a Borel measurable function $\phi : [0,\infty) \to (0,\infty)$ and a function $h \in L^1([0,1])$ such that $\|f(t,x,p)\| \le \phi(\|p\|)(h(t) + \|p\|)$ a.e. $t \in [0,1]$ and for all $(x,p) \in \mathbb{R}^{2n}$ with $\|x - v(t)\| \le M(t)$;*

and $\displaystyle\int^{\infty} \frac{ds}{\phi(s)} = \infty.$

Then the problem $(\star)$ *has a solution such that* $\|x(t)-v(t)\| \leq M(t)$ *for all* $t \in [0,1]$.

First of all, observe that ϕ is not necessarily continuous, and v'' and M'' could be not essentially bounded. To prove this theorem, we can not use the problems $(\star)^\varepsilon_\lambda$ as we did for Theorem 3.1, since an assumption like (H6) is not necessarily satisfied by f^ε_λ. In fact, to prove this theorem, we will use the theory of differential inclusions. We will construct a multivalued mapping and we will deduce the existence of a solution of the differential inclusion associated. Finally, we will observe that this solution is in fact a solution to our original problem $(\star)$.

For $\varepsilon \geq 0$, $\lambda \in [0,1]$, we define the multivalued fonction $T^\varepsilon_\lambda : [0,1] \times \mathbb{R}^{2n} \to 2^{\mathbb{R}^n}$ by $T^\varepsilon_\lambda(t,x,p) = \widehat{f}^\varepsilon_\lambda(t,x,p) + G_\lambda(t,x,p)$ where $\widehat{f}^\varepsilon_\lambda$ is the function defined by

$$\widehat{f}^\varepsilon_\lambda(t,x,p) =$$
$$\begin{cases} \lambda\left(\frac{M(t)}{\|x-v(t)\|} f(t,\widetilde{x},\widehat{p}) - \varepsilon\widetilde{x}\right) - \varepsilon(1-\lambda)v(t), & \text{if } \|x-v(t)\| > M(t) > 0, \\ \lambda\left(f(t,x,p) - \varepsilon x\right) - \varepsilon(1-\lambda)v(t), & \text{if } \|x-v(t)\| \leq M(t),\ M(t) > 0, \\ v''(t) - \varepsilon v(t), & \text{if } M(t) = 0; \end{cases}$$

and G_λ is the multivalued function defined by

$$G_\lambda(t,x,p) =$$
$$\begin{cases} \Bigg(\left(1 - \frac{\lambda M(t)}{\|x-v(t)\|}\right)\left(M''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\right) + \\ \qquad (1-\lambda)\left(\frac{M'(t)^2 - \|\widehat{p}-v'(t)\|^2}{\|x-v(t)\|}\right)\Bigg)^+ \frac{(x-v(t))}{\|x-v(t)\|}, & \text{if } \|x-v(t)\| > M(t) > 0, \\ [0,(1-\lambda)]\Bigg(M''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|} + \\ \qquad \frac{M'(t)^2 - \|\widehat{p}-v'(t)\|^2}{\|x-v(t)\|}\Bigg)^+ \frac{(x-v(t))}{\|x-v(t)\|}, & \text{if } \|x-v(t)\| = M(t) > 0, \\ 0, & \text{if } \|x-v(t)\| < M(t) \\ & \quad \text{or } M(t) = 0; \end{cases}$$

where, as before, (v,M) is the solution-tube to $(\star)$ given in (H1), $\widetilde{x} = v(t) + \frac{M(t)}{\|x-v(t)\|}(x-v(t))$ and $\widehat{p} = p + \left(M'(t) - \frac{\langle x-v(t), p-v'(t)\rangle}{\|x-v(t)\|}\right)\left(\frac{x-v(t)}{\|x-v(t)\|}\right)$.

To the function T^ε_λ, we associate the operator $\mathcal{T}^\varepsilon_\lambda = \widehat{F}^\varepsilon_\lambda + \mathcal{G}_\lambda : C^1([0,1],\mathbb{R}^n) \to 2^{L^1([0,1],\mathbb{R}^n)}$, where $\widehat{F}^\varepsilon_\lambda$ and $\mathcal{G}_\lambda$ are respectively defined by

$$\widehat{F}^\varepsilon_\lambda(x)(t) = \widehat{f}^\varepsilon_\lambda(t,x(t),x'(t)),$$
$$\mathcal{G}_\lambda(x) = \{u \in L^1([0,1],\mathbb{R}^n) : u(t) \in G_\lambda(t,x(t),x'(t)) \text{ a.e. } t \in [0,1]\}.$$

Proposition 4.2. *Let* $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ *be a Carathéodory function and let* (v,M) *be a solution-tube to* $(\star)$. *Then the previously defined operator* $\mathcal{T}^\varepsilon_\lambda$ *is upper semi-continuous, and integrably bounded on bounded, with non-empty, closed, convex values.*

Proof. To show that $\widehat{F}^\varepsilon_\lambda$ is continuous and integrably bounded on bounded independently of $\lambda \in [0,1]$, we argue as in Proposition 3.5.

On the other hand, it is clear that $G_\lambda(t,x,p)$ has non-empty, closed, convex values, $t \mapsto G_\lambda(t,x,p)$ is measurable for all $(x,p) \in \mathbb{R}^{2n}$, and $(x,p) \mapsto G_\lambda(t,x,p)$ is upper semi-continuous for almost all $t \in [0,1]$. Observe that if $\|x-v(t)\| \geq M(t) > 0$ and

$$\Big(\big(1 - \frac{\lambda M(t)}{\|x-v(t)\|}\big)\big(M''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\big) \\ + (1-\lambda)\big(\frac{M'(t)^2 - \|\widehat{p}-v'(t)\|^2}{\|x-v(t)\|}\big)\Big)^+ > 0,$$

then

$$\begin{aligned} &\Big\| \Big(\big(1 - \frac{\lambda M(t)}{\|x-v(t)\|}\big)\big(M''(t) + \frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|}\big) \\ &\qquad + (1-\lambda)\big(\frac{M'(t)^2 - \|\widehat{p}-v'(t)\|^2}{\|x-v(t)\|}\big)\Big)^+ \frac{(x-v(t))}{\|x-v(t)\|} \Big\| \\ &\quad = (1-\lambda)\big(\frac{\frac{\langle M(t)(x-v(t)), v''(t)\rangle}{\|x-v(t)\|} + M''(t)M(t) + M'(t)^2 - \|\widehat{p}-v'(t)\|^2}{\|x-v(t)\|}\big) \\ &\qquad + \big(1 - \frac{M(t)}{\|x-v(t)\|}\big)\big(\frac{\langle x-v(t), v''(t)\rangle}{\|x-v(t)\|} + M''(t)\big) \\ &\quad \leq (1-\lambda)\frac{\langle M(t)(x-v(t)), f(t,\widetilde{x},\widehat{p})\rangle}{\|x-v(t)\|^2} + \|v''(t)\| + |M''(t)| \\ &\quad \leq \|f(t,\widetilde{x},\widehat{p})\| + \|v''(t)\| + |M''(t)|. \end{aligned}$$

Hence, G_λ is Carathéodory. It follows from Lemma 2.1 that $\mathcal{G}_\lambda$ is upper semi-continuous with non-empty, closed, convex values, and integrably bounded on bounded. This completes the proof. □

Let us consider the associated problems

$$(\star\star)^\varepsilon_\lambda \quad \begin{cases} x''(t) - \varepsilon\, x(t) \in T^\varepsilon_\lambda(t, x(t), x'(t)) & \text{a.e. } t \in [0,1] \\ x \in BC \end{cases}$$

Proof of Theorem 4.1. We will show that the problem $(\star\star)^\varepsilon_1$ has a solution satisfying $\|x(t) - v(t)\| \leq M(t)$. By the definition of T^ε_1, this solution will be a solution to our original problem $(\star)$.

Let x be a solution to $(\star\star)^\varepsilon_\lambda$. By Lemma 2.4, $M''(t) = 0$ a.e. on $\{t \in [0,1] : M(t) = 0\}$. This and (H1) imply that

$$\begin{aligned} &\frac{\langle x(t)-v(t), x''(t)-v''(t)\rangle + \|x'(t)-v'(t)\|^2}{\|x(t)-v(t)\|} - \frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle^2}{\|x(t)-v(t)\|^3} \\ &\qquad - \varepsilon\|x(t)-v(t)\| \\ &\quad \geq M''(t) - \varepsilon M(t) \end{aligned}$$

a.e. on $\{t \in [0,1] : \|x - v(t)\| > M(t)\}$. It follows from Lemma 3.2 that every solution to $(\star\star)^\varepsilon_\lambda$ satisfies $\|x(t) - v(t)\| \leq M(t)$ for all $t \in [0,1]$.

On the other hand, for every solution x of $(\star\star)^\varepsilon_\lambda$, we have a.e. on $\{t \in [0,1] : \|x'(t) - v'(t)\| \geq k\}$,

$$\begin{aligned}
&\frac{\langle x(t)-v(t), x''(t)-v''(t)\rangle + \|x'(t)-v'(t)\|^2}{\|x'(t)-v'(t)\|} \\
&\qquad - \frac{\langle x'(t)-v'(t), x''(t)-v''(t)\rangle\langle x(t)-v(t), x'(t)-v'(t)\rangle}{\|x'(t)-v'(t)\|^3} \\
&\geq \lambda\frac{\langle x(t)-v(t), f(t,x'(t),x'(t))-v''(t)\rangle + \|x'(t)-v'(t)\|^2}{\|x'(t)-v'(t)\|} \\
&\qquad - \lambda\frac{\langle x'(t)-v'(t), f(t,x(t),x'(t))-v''(t)\rangle\langle x(t)-v(t), x'(t)-v'(t)\rangle}{\|x'(t)-v'(t)\|^3} \\
&\qquad + (1-\lambda)\|x'(t)-v'(t)\| - 2M(t)\|v''(t)\|/k.
\end{aligned} \tag{4.1}$$

Inequality (4.1) and (H5) imply that the assumptions of Lemma 3.3 are satisfied. Thus, there exists $K_1 = K_1(\|M\|_0) > 0$ such that $\|x' - v'\|_{L^1[a,b]} \leq K_1$ for any interval $[a,b]$ on which $\|x'(t) - v'(t)\| \geq k$; and $\|x'(t) - v'(t)\| \leq \max\{k, K_1\}$ for some $t \in [0,1]$.

By Lemma 2.4, $x'(t) = \widehat{x'(t)}$ a.e. on $\{t \in [0,1] : \|x - v(t)\| = M(t) > 0\}$. Thus, it follows from (H1) that for almost all t in that set and such that

$$\left(M''(t) + \frac{\langle x(t)-v(t), v''(t)\rangle + M'(t)^2 - \|\widehat{x'(t)} - v'(t)\|^2}{\|x(t)-v(t)\|}\right)^+ > 0,$$

$$\begin{aligned}
&\left\|\left(M''(t) + \frac{\langle x(t)-v(t), v''(t)\rangle + M'(t)^2 - \|\widehat{x'(t)} - v'(t)\|^2}{\|x(t)-v(t)\|}\right)^+ \frac{(x-v(t))}{\|x(t)-v(t)\|}\right\| \\
&\qquad\leq \frac{\langle (x-v(t)), f(t,x(t),\widehat{x'(t)})\rangle}{\|x(t)-v(t)\|} \\
&\qquad\leq \|f(t,x(t),\widehat{x'(t)})\| = \|f(t,x(t),x'(t))\|.
\end{aligned}$$

Hence,

$$\|T^\varepsilon_\lambda(t,x(t),x'(t)) + \varepsilon x(t)\| \leq \|f(t,x(t),x'(t))\| + \varepsilon\|M\|_0.$$

Fix $k_0 = \max\{k, K_1\} + \|v'\|_0$ and $\varepsilon \geq 0$ such that the previously defined operator L_ε is invertible, and

$$\int_{k_0}^{\infty} \frac{ds}{\phi(s) + \varepsilon\|M\|_0} > \|h\|_{L^1} + 1 + K_1 + \|v'\|_{L^1}.$$

Lemma 3.4 applied with $\psi(s) = s(\phi(s) + \varepsilon\|M\|_0)$, $l(t) = h(t) + 1$ gives the existence of K such that $\|x'\|_0 < K$.

By using the multivalued version of Theorem 2.2 with the homotopy given by the operators $L_\varepsilon^{-1} \circ N_{\mathcal{T}^\varepsilon_\lambda}$, and arguing as in the proof of Theorem 3.1, we deduce the existence of a solution to $(\star\star)^\varepsilon_\lambda$. The conclusion follows from the definition of T^ε_1. $\square$

Corollary 4.3. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume* (H1), (H6) *and*

(H7) *there exist $k, \theta > 0$, $h_1 \in L^1([0,1])$ such that for a.e. t and for all (x,p) with $\|x - v(t)\| \leq M(t)$ and $\|p - v'(t)\| \geq k$,*

$$\frac{\langle x - v(t), f(t,x,p) - v''(t)\rangle + \|p - v'(t)\|^2}{\|p - v'(t)\|} - \frac{\langle p - v'(t), f(t,x,p) - v''(t)\rangle\langle x - v(t), p - v'(t)\rangle}{\|p - v'(t)\|^3} \geq \theta\|p - v'(t)\| - h_1(t).$$

Then the problem $(\star)$ has a solution such that $\|x(t) - v(t)\| \leq M(t)$ for all $t \in [0,1]$.

5. Examples

The following problems have a solution.

Example 5.1.

$$x''(t) = \|x'(t)\|\, x'(t) - c$$
$$x(0) = x(1) = (0, \cdots, 0)$$

where $c \in \mathbb{R}^n$ with $\|c\| = 1$. Verify that $v(t) \equiv 0$, $M(t) = t$, any $k > 0$, $\theta = 1$, $h_1(t) = 2t/k$, $h \equiv 0$, $C = D = 1$ satisfy the assumptions of Corollary 3.6. Consequently, this problem has a solution such that $\|x(t)\| \leq t$. Observe that there is no constant M such that $(0, M)$ is a solution-tube to this problem and Hartman's condition (1.2) is not satisfied.

Example 5.2.

$$x''(t) = -4\langle x(t), x'(t)\rangle^2 x(t) + x(t) + c$$
$$x(0) = x(1) = (0, \cdots, 0)$$

where $c \in \mathbb{R}^n$ with $\|c\| = 1$. Verify that $v \equiv 0$, $M \equiv 1$, $\gamma = 1/4$, $\theta = 1$, $m = 4$, any $k > 0$, $h_1 = h_2 \equiv 3/k$, $h \equiv 0$, $C = 4$, $D = 2$ satisfy the assumptions of Theorem 3.1. Consequently, this problem has a solution such that $\|x(t)\| \leq 1$. Observe that Hartman's condition (1.2) and (H4) are not satisfied with this (v, M).

Example 5.3.

$$x''(t) = \|x'(t) - c\,t^{1/2}\|\,(x'(t) - c\,t^{1/2}) + (c\,t^{-1/2})/2$$
$$x(0) = x(1) = (0, \cdots, 0)$$

where $c \in \mathbb{R}^n$. Verify that $v(t) = (2c\,t^{3/2})/3$, $M(t) = 2\|c\|/3$, $\theta = 1$, any $k > 0$, $h_1 \equiv 0$, $h(t) = \|c\|(t^{-1/2} + 2)/2$, $\phi(s) = (s + \|c\|)$ satisfy the assumptions of Corollary 4.3.

Example 5.4.

$$x''(t) = \phi(\|x'(t)\|)x'(t)$$
$$x'(0) = (0, \cdots, 0), \;\; x'(1) = c$$

where $c \in \mathbb{R}^n$, $\phi : [0, \infty) \to (0, \infty)$ is continuous and $\int^\infty ds/\phi(s) = \infty$. Verify that $v(t) \equiv 0$, $M(t) = \|c\|t$, $\theta = 1$, any $k > 0$, $h = h_1 \equiv 0$ satisfy the assumptions of Corollary 4.3. Consequently, this problem has a solution such that $\|x(t)\| \leq \|c\|t$.

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