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Systems of first order differential inclusions with maximal monotone terms

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Abstract

In this paper, we establish the existence of solutions to systems of first order differential inclusions with maximal monotone terms satisfying the periodic boundary condition. Our proofs rely on the theory of maximal monotone operators, and the Schauder and the Kakutani fixed point theorems. A notion of solution-tube to these problems is introduced. This notion generalizes the notion of upper and lower solutions of first order differential equations.

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1. Introduction

In this paper, we establish existence results for the following systems of first order differential inclusions:

$$\begin{aligned} &x'(t) \in -A(x(t)) + F(t, x(t)), \quad \text{a.e. } t \in [0, 1],\\ &x(0) = x(1); \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &x'(t) \in A(x(t)) + F(t, x(t)), \quad \text{a.e. } t \in [0, 1],\\ &x(0) = x(1). \end{aligned} \tag{1.2}$$

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Here, $A : \mathrm{dom}(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ is a multi-valued maximal monotone operator, $F : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a multi-valued map with compact values that satisfies a lower semi-continuity or an upper semi-continuity condition. In this last case, F has convex values. We consider the cases where $\mathrm{dom}(A) = \mathbb{R}^n$ and $\mathrm{dom}(A)$ are strictly included in $\mathbb{R}^n$. In this last case, an extra assumption is needed.

In this paper, we introduce the notions of L^p-solution-tube to problems (1.1) and (1.2) similar to the notion of solution-tube introduced in [7] and [10] for first order systems (see also [8]), when $p = 1$. This notion generalizes the notion of upper and lower solutions of first order differential equations; see [9]. Under the assumption of the existence of an L^p-solution-tube, we establish existence results. In particular, our Theorem 3.4 generalizes a result obtained by Montoki [20] in his thesis.

This type of problem was studied by [9,10] and [12] when $A = 0$ and F is single-valued. An important class of those problems appears in particular where $A = \partial\phi$, the subdifferential of a proper convex map ϕ. In [23], Yotsutani studied the problem (1.1) with $A = \partial\phi$ and the periodic boundary condition replaced by the initial value condition. His results generalize results of [21] and [22]. These type of problems were also studied in [18] by Kandilakis and Papageorgiou for a family of problems depending on a parameter, and by Hirano [16] for the periodic problem with $A = \partial\phi$ and F single-valued. Bader [1] considered the case where A is the infinitesimal generator of a C_0-semigroup.

Recently, in a very interesting paper, Bader and Papageorgiou [2] studied the problem (1.1) with $A = \partial\phi$ in the more general context of Hilbert spaces. They also treated the two cases where F satisfies a lower semi-continuity and an upper semi-continuity conditions. In this last case, our condition of existence of an L^2-solution-tube generalizes considerably their condition $\mathrm{H(F)}_2\mathrm{(v)}$. They did not impose this type of condition with the lower semi-continuity condition. In both cases, they assumed a Nagumo-type tangential condition that we do not impose.

We first study Problem (1.1). In Section 3, we state existence results for this problem that we prove in Section 5 after having studied appropriate operators. Finally, existence results for Problem (1.2) are obtained in the last section. Our existence results will rely on the Schauder and the Kakutani fixed point theorems; see [13].

2. Preliminaries

In what follows, we will use the following notations: $I = [0, 1]$, and $C(I, \mathbb{R}^n)$ is the space of continuous functions endowed with the usual norm which we denote $\|\cdot\|_0$. For $p \in [1, \infty]$, $L^p(I, \mathbb{R}^n)$ is the space of L^p-integrable functions with the usual norm $\|\cdot\|_{L^p}$; $W^{1,p}(I, \mathbb{R}^n)$ is the Sobolev space $\{x \in C[0, 1] : x \text{ is absolutely continuous and } x' \in L^p(I, \mathbb{R}^n)\}$ endowed with the usual norm $\|\cdot\|_{1,p}$; and $W_P^{1,p}(I, \mathbb{R}^n)$ is the subset of x in $W^{1,p}(I, \mathbb{R}^n)$ satisfying the periodic boundary condition.

Let X, Y be topological spaces and Ω a measurable space. We say that a multi-valued map $G : \Omega \to X$ is measurable if $\{t \in \Omega : G(t) \cap C \neq \emptyset\}$ is measurable for every closed set $C \subset X$. A multi-valued map $G : X \to Y$ is upper semi-continuous (u.s.c.) (resp. lower semi-continuous (l.s.c.)) if $\{x \in X : G(x) \cap C \neq \emptyset\}$ is closed (resp. open) for every closed (resp. open) set $C \subset Y$; it is continuous if it is lower and upper semi-continuous. Notice that we consider only multi-valued maps with nonempty values. The reader is referred to [3], [6], [15], or [17] for more details on multi-valued maps.

Let H be a Hilbert space and $M : \mathrm{dom}(M) \subset H \to H$ a multi-valued maximal monotone operator. Let us recall that M is a *monotone operator*, if

$$\langle x^* - y^*, x - y\rangle \geq 0 \quad \forall x, y \in \mathrm{dom}(M), \forall x^* \in M(x), \forall y^* \in M(y);$$

and it is *maximal* if

$$\langle x^* - y^*, x - y\rangle \geq 0 \quad \forall y \in \mathrm{dom}(M), \forall y^* \in M(y) \Longrightarrow x \in \mathrm{dom}(M), \text{ and } x^* \in M(x).$$

We recall some results on monotone operators. For their proofs and for more information on monotone operators, the reader is referred to [5,17] or [24].

Lemma 2.1. *A multi-valued monotone map $M : \mathrm{dom}(M) \subset H \to H$ is locally bounded at every point in the interior of its domain.*

Lemma 2.2. *Let $M : \mathrm{dom}(M) \subset H \to H$ be a multi-valued maximal monotone operator. Then M has closed, convex values, and $\mathrm{Gr}(M) := \{(x, x^*) : x^* \in Mx\}$ is sequentially closed in $(H, \mathcal{T}_s) \times (H, \mathcal{T}_w)$ and in $(H, \mathcal{T}_w) \times (H, \mathcal{T}_s)$, where $\mathcal{T}_s$ and $\mathcal{T}_w$ denote, respectively, the strong and the weak topologies of H.*

Lemma 2.3. *Let $M : \mathrm{dom}(M) \subset H \to H$ be a multi-valued monotone operator. Then the following statements are equivalent:*

(a) *M is maximal;*
(b) *$id + M$ is surjective.*

Lemma 2.4. *Let $M : \mathrm{dom}(M) \subset H \to H$ be a multi-valued maximal monotone operator and $N : H \to H$ a single-valued Lipschitzian monotone operator. Then $M+N$ is maximal monotone.*

We can associate with M the operator $\widehat{M} : \mathrm{dom}(\widehat{M}) \subset L^2(I, H) \to L^2(I, H)$ defined by

$$\widehat{M}(x) = \{y \in L^2(I, H) : y(t) \in M(x(t)) \text{ a.e.} t \in I\},$$

where

$$\mathrm{dom}(\widehat{M}) = \{x \in L^2(I, H) : x(t) \in \mathrm{dom}(M) \text{ a.e. } t \in I \text{ and } \\ \exists y \in L^2(I, H) \text{ such that } y(t) \in M(x(t)) \text{ a.e. } t \in I\}.$$

Lemma 2.5. *Let $M : \mathrm{dom}(M) \subset H \to H$ be a multi-valued maximal monotone operator. The operator $\widehat{M}$ is maximal monotone.*

We define, for $\lambda > 0$,

$$J_\lambda = (id + \lambda M)^{-1} \qquad \text{(the } \textit{resolvent} \text{ of } M),$$
$$M_\lambda = \frac{1}{\lambda}(id - J_\lambda) \qquad \text{(the } \textit{Yosida approximation} \text{ of } M).$$

It is well known that $\mathrm{dom}(J_\lambda) = \mathrm{dom}(M_\lambda) = H$, J_λ and M_λ are single-valued, J_λ is non-expansive, M_λ is monotone and Lipschitzian with constant $1/\lambda$, and hence maximal monotone. Moreover, for every $x \in \mathrm{dom}(M)$, $M_\lambda(x) \in M(J_\lambda(x))$, and for all $\lambda > 0$,

$$\|M_\lambda(x)\| \leq \inf\{\|y\| : y \in M(x)\}. \tag{2.1}$$

Moreover, $M_\lambda(x) \to y_0 \in M(x)$ as $\lambda \to 0$, where y_0 is the element of minimal norm in $M(x)$.

Lemma 2.6. *Let* $M : \mathrm{dom}(M) \subset H \to H$ *and* $N : \mathrm{dom}(N) \subset H \to H$ *be multi-valued maximal monotone operators such that* $\mathrm{dom}(M) \cap \mathrm{dom}(N) \neq \emptyset$. *Then*

(a) $M_\lambda + N$ *is maximal for every* $\lambda > 0$*;*
(b) $y \in \mathrm{Im}(id + M + N)$ *if and only if* $\{M_\lambda(x_\lambda)\}$ *is bounded as* $\lambda \to 0^+$*, where* $y = (id + M_\lambda + N)(x_\lambda)$*. Moreover, if those properties hold, then* $x_\lambda \to x$ *and* $M_\lambda(x_\lambda) \to z \in M(x) \cap \{y - x - N(x)\}$.

3. Existence results

Our goal is to establish existence results for the problem (1.1). By a solution, we mean a function $x \in W_p^{1,1}(I, \mathbb{R}^n)$ satisfying (1.1).

We introduce the notion of L^p-solution-tube of the problem (1.1). This notion will play a fundamental role in our existence results.

Definition 3.1. Let $v \in W^{1,p}(I, \mathbb{R}^n)$, and $r \in W^{1,p}(I, \mathbb{R})$ with $p \in [1, \infty]$. We say that (v, r) is an *L^p-solution-tube of* (1.1) if there exists $a \in L^p(I, \mathbb{R}^n)$ such that

(i) $a(t) \in Av(t)$ a.e. $t \in I$;
(ii) for a.e. $t \in I$, and every $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = r(t)$, there exists $y \in F(t, x)$ such that

$$\langle x - v(t), y - a(t) - v'(t)\rangle \le r(t)r'(t);$$

(iii) $v'(t) \in -a(t) + F(t, v(t))$ a.e. on $\{t \in [0, 1] : r(t) = 0\}$;
(iv) $\|v(0) - v(1)\| \le r(0) - r(1)$.

We denote

$$T(v, r) = \{x \in C(I, \mathbb{R}^n) : \|x(t) - v(t)\| \le r(t)\ \forall t \in I\}.$$

Remark 3.2. If $\phi : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $A = \partial\phi$ the subdifferential of ϕ, and (v, r) is an L^p-solution-tube of (1.1), then, for a.e. $t \in I$, and every $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = r(t)$, there exists $y \in F(t, x)$ such that

$$\phi(x) + r(t)r'(t) + \langle x - v(t), v'(t)\rangle \ge \phi(v(t)) + \langle x - v(t), y\rangle.$$

Our results will rely on some of the following assumptions:

(F1-u) $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a multi-valued map with compact convex values such that $t \mapsto F(t, x)$ is measurable for all $x \in \mathbb{R}^n$, and $x \mapsto F(t, x)$ is u.s.c. a.e. $t \in I$;
(F1-l) $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ has compact values and is such that $x \mapsto F(t, x)$ is l.s.c. a.e. $t \in I$, and $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$-measurable (here $I \times \mathbb{R}^n$ is endowed with the σ-algebra generated by subsets $C \times D$, where $C \subset I$ and $D \subset \mathbb{R}^n$ are, respectively, Lebesgue and Borel measurable);
(F2-p) for every $m \ge 0$, there exists $h_m \in L^p(I)$ such that

$$\max\{\|y\| : y \in F(t, x), \|x\| \le m\} \le h_m(t) \qquad \text{a.e. } t \in I;$$

(ST-p) there exists $(v, r) \in W^{1,p}(I, \mathbb{R}^n) \times W^{1,p}(I, [0, \infty[)$ an L^p-solution-tube of (1.1);
(A1) the multi-valued operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is maximal monotone;
(A2) the operator $A : \mathrm{dom}(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ is a multi-valued maximal monotone operator such that $\mathrm{dom}(A) \neq \emptyset$;

(A3) for all bounded set B in $L^2(I,\mathbb{R}^n)\cap \mathrm{dom}(\widehat{A})$,

$$\sup\left\{\inf\left\{\|y\|_{L^2} : y\in \widehat{A}(x)\right\} : x\in B\right\}<\infty.$$

Remark 3.3. Observe that, in what follows, the assumption (F1-l) can be replaced by

(F1-c) $F : I\times\mathbb{R}^n\to\mathbb{R}^n$ is a multi-valued map with compact values such that $t\mapsto F(t,x)$ is measurable for all $x\in\mathbb{R}^n$, and $x\mapsto F(t,x)$ is continuous a.e. $t\in I$.

One aim of this paper is to establish the following existence results:

Theorem 3.4. *Assume* (F1-u), (F2-1), (A1), *and* (ST-1). *Then the problem* (1.1) *has a solution* $x\in W^{1,1}(I,\mathbb{R}^n)\cap T(v,r)$.

We can replace the upper semi-continuity assumption (F1-u) by other continuity conditions such as (F1-l) or (F1-c) (see Remark 3.3). In this case, the values of F do not need to be convex.

Theorem 3.5. *Assume* (F1-l), (F2-1), (A1), *and* (ST-1). *Then the problem* (1.1) *has a solution* $x\in W^{1,1}(I,\mathbb{R}^n)\cap T(v,r)$.

It is also possible to obtain existence results if $\mathrm{dom}(A)\neq\mathbb{R}^n$.

Theorem 3.6. *Assume* (F1-u), (F2-2), (A2), (A3), *and* (ST-2). *Then the problem* (1.1) *has a solution* $x\in W^{1,2}(I,\mathbb{R}^n)\cap T(v,r)$.

We obtain a similar result for F satisfying a lower semi-continuity condition.

Theorem 3.7. *Assume* (F1-l), (F2-2), (A2), (A3), *and* (ST-2). *Then the problem* (1.1) *has a solution* $x\in W^{1,2}(I,\mathbb{R}^n)\cap T(v,r)$.

4. Operators

To prove our existence theorems, we will consider suitable modified problems for which we will establish the existence of solutions. To this aim, we introduce appropriate maps.

Let (v,r) and a be given in (ST-p) (see Definition 3.1). Define

$$F_u : I\times\mathbb{R}^n\to\mathbb{R}^n \qquad \text{by} \qquad F_u=\widetilde{F}\cap G_u,$$

where

$$\widetilde{F}(t,x)=\begin{cases}F(t,\widetilde{x}_t), & \text{if } \|x-v(t)\|>r(t),\\ F(t,x), & \text{if } \|x-v(t)\|\le r(t);\end{cases}$$

$$G_u(t,x)=\begin{cases}v'(t)+a(t), & \text{if } r(t)=0,\\ \mathbb{R}^n, & \text{if } \|x-v(t)\|\le r(t)\\ & \quad\text{and } r(t)>0,\\ \{z:\langle x-v(t),z-a(t)-v'(t)\rangle & \\ \quad\le r'(t)\|x-v(t)\|\}, & \text{otherwise;}\end{cases}$$

with

$$\widetilde{x}_t=v(t)+\frac{r(t)}{\|x-v(t)\|}(x-v(t)). \tag{4.1}$$

Similarly, we define

$$F_l : I\times\mathbb{R}^n\to\mathbb{R}^n \qquad \text{by} \qquad F_l=\widetilde{F}\cap G_l,$$

where

$$G_l(t,x) = \begin{cases} v'(t) + a(t), & \text{if } r(t) = 0, \\ \mathbb{R}^n, & \text{if } \|x - v(t)\| < r(t), \\ \{z : \langle x - v(t), z - a(t) - v'(t)\rangle \\ \qquad \le r'(t)\|x - v(t)\|\}, & \text{otherwise.} \end{cases}$$

Proposition 4.1. *Assume* (F1-u), (F2-1)*, and* (ST-1). *Let* $\mathcal{F}_u : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ *be defined by*

$$\mathcal{F}_u(x) = \{y \in L^1(I, \mathbb{R}^n) : y(t) \in F_u(t, x(t)) + \widetilde{x(t)}_t \text{ a.e.} t \in I\}$$

with $L^1(I, \mathbb{R}^n)$ *endowed with the weak topology. Then* $\mathcal{F}_u$ *is u.s.c. and has compact and convex values. Moreover, there exists* $h \in L^1(I, [0, \infty))$ *such that, for all* $x \in C(I, \mathbb{R}^n)$ *and all* $y \in \mathcal{F}_u(x)$*,* $\|y(t)\| \le h(t)$ *a.e.* $t \in I$.

Proof. It is easy to verify that F_u is measurable in t for all $x \in \mathbb{R}^n$, and u.s.c. in x for almost every $t \in I$. It follows from (ST-1) that $F_u(t, x) \ne \emptyset$ for all $x \in \mathbb{R}^n$ and almost every $t \in I$. Also $F_u(t, x)$ is closed and convex. Therefore, for all $x \in C(I, \mathbb{R}^n)$, $t \mapsto F_u(t, x(t))$ is measurable, and hence has a measurable selection by the Kuratowski, Ryll-Nardzewski selection theorem [19]. So, $\mathcal{F}_u$ has nonempty values. Indeed, from (F2-1) and the fact that $\|\widetilde{x(t)}_t\| \le \|v\|_0 + \|r\|_0$, we deduce that there exists $h \in L^1(I, [0, \infty))$ such that for all $x \in C(I, \mathbb{R}^n)$ and all $y \in \mathcal{F}_u(x)$,

$$\|y(t)\| \le h(t) \qquad \text{a.e. } t \in I. \tag{4.2}$$

It is clear that $\mathcal{F}_u$ has closed, convex, bounded values in $L^1(I, \mathbb{R}^n)$ endowed with the strong topology, and hence has compact convex values in the weak topology by (4.2) and Pettis' theorem.

It is left to show that $\mathcal{F}_u$ is u.s.c. Let E be a weakly closed subset of $L^1(I, \mathbb{R}^n)$ and let $\{x_n\}$ a sequence in $\{x \in C(I, \mathbb{R}^n) : \mathcal{F}_u(x) \cap E \ne \emptyset\}$ converging to x_0. Take $y_n \in \mathcal{F}_u(x_n) \cap E$. By (4.2) and Pettis' theorem, $\{y_n\}$ has a weakly convergent subsequence still denoted $\{y_n\}$. Denote y its weak limit. Notice that $y \in E$, since E is weakly closed. Moreover, there exists $z_n \in \text{co}\{y_n, y_{n+1}, \ldots\}$ such that the sequence $\{z_n\}$ converges strongly to y in $L^1(I, \mathbb{R}^n)$. Without loss of generality, we can assume that $z_n(t) \to y(t)$ almost everywhere in I. Since F_u has convex values and is u.s.c.,

$$\begin{aligned} y(t) &\subset \bigcap_{n\ge1} \overline{\text{co}} \left\{ \bigcup_{m\ge n} y_n(t) \right\} \subset \bigcap_{n\ge1} \overline{\text{co}} \left\{ \bigcup_{m\ge n} F_u(t, x_n(t)) + \widetilde{x_n(t)}_t \right\} \\ &\subset F_u(t, x_0(t)) + \widetilde{x_0(t)}_t \qquad \text{a.e. } t \in I. \end{aligned}$$

Therefore, $y \in \mathcal{F}_u(x_0) \cap E$. □

Remark 4.2. Assume that (F1-u), (F2-2) and (ST-2) hold. Then the conclusion of Proposition 4.1 is true if we replace $L^1(I, \mathbb{R}^n)$ by $L^2(I, \mathbb{R}^n)$.

Proposition 4.3. *Assume* (F1-l), (F2-1), and (ST-1). *Then there exists a continuous single-valued map* $f_l : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ *such that*

$$f_l(x)(t) \in F_l(t, x(t)) + \widetilde{x(t)}_t \qquad a.e.\ t \in I.$$

Moreover, there exists $h \in L^1(I, [0, \infty))$ such that, for all $x \in C(I, \mathbb{R}^n)$, $\|f_l(x)(t)\| \le h(t)$ a.e. $t \in I$. In addition, if (F2-2) *and* (ST-2) *hold, then h can be chosen in $L^2(I, \mathbb{R}^n)$, and hence $f_l(C(I, \mathbb{R}^n)) \subset \{y \in L^2(I, \mathbb{R}^n) : \|y(t)\| \le h(t)$ a.e. $t \in I\}$.*

Proof. Observe that, by (ST-1), F_l has nonempty values. Since F satisfies (F1-l), $(t, x) \mapsto F_l(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$-measurable, and $x \mapsto F_l(t, x)$ is l.s.c. a.e. $t \in I$. Indeed, let $C \subset \mathbb{R}^n$ be open, $E = \{x \in \mathbb{R}^n : F_l(t, x) \cap C \neq \emptyset\}$. If $r(t) = 0$, $F_l(t, x) = v'(t) + a(t)$ for all $x \in \mathbb{R}^n$, and hence E is open. In the case $r(t) > 0$, take $x \in E$. If $\|x - v(t)\| < r(t)$, then there exists $\delta > 0$ such that $\|u - v(t)\| < r(t)$ for all $u \in B(x, \delta)$. Since $F_l(t, u) = F(t, u)$ for all $u \in B(x, \delta)$, the lower semi-continuity of F with respect to its second variable implies that there exists a neighborhood of x in E.

On the other hand, if $\|x - v(t)\| \ge r(t)$, there exists

$$z_0 \in C \cap F(t, \widetilde{x}_t) \cap \{z : \langle x - v(t), z - a(t) - v'(t)\rangle \le r'(t)\|x - v(t)\|\}.$$

Set $\varepsilon > 0$ such that $B(z_0, \varepsilon) \subset C$ and fix $w = -\varepsilon(x - v(t))/(2\|x - v(t)\|)$. For $y \in S^{n-1}$ and $\lambda \ge 0$,

$$\begin{aligned}
&\langle x + \lambda y - v(t), z_0 + w - a(t) - v'(t)\rangle \\
&\quad = \langle x - v(t), z_0 - a(t) - v'(t)\rangle + \lambda\langle y, z_0 + w - a(t) - v'(t)\rangle + \langle x - v(t), w\rangle \\
&\quad \le \|x - v(t)\|r'(t) + \lambda\|z_0 + w - a(t) - v'(t)\| - \frac{\varepsilon}{2}\|x - v(t)\| \\
&\quad \le \|x + \lambda y - v(t)\|r'(t) + \lambda\left(|r'(t)| + \|z_0 + w - a(t) - v'(t)\|\right) - \frac{\varepsilon}{2}\|x - v(t)\|.
\end{aligned}$$

Choose $0 < \lambda_0 < \varepsilon\|x - v(t)\|/(2|r'(t)| + 2\|z_0 + w - a(t) - v'(t)\|)$. So, for every $u \in B(x, \lambda_0)$,

$$z_0 + w \in \{z : \langle u - v(t), z - a(t) - v'(t)\rangle \le r'(t)\|u - v(t)\|\} \cap C.$$

On the other hand, the lower semi-continuity of F with respect to its second variable implies that there exists $\lambda_1 > 0$ such that $B(x, \lambda_1) \subset \{u : F(t, \widetilde{u}_t) \cap C \neq \emptyset\}$. So, for $\lambda_2 = \min\{\lambda_0, \lambda_1\}$,

$$\begin{aligned}
B(x, \lambda_2) \subset \big\{u \in \mathbb{R}^n : C \cap F(t, \widetilde{u}_t) \\
\cap \{z : \langle u - v(t), z - a(t) - v'(t)\rangle \le r'(t)\|u - v(t)\|\} \neq \emptyset\big\} \subset E.
\end{aligned}$$

So, E is open.

Also, by (F2-1) (resp. (F2-2) and (ST-2)), there exists $h \in L^1(I, \mathbb{R}^n)$ (resp. $h \in L^2(I, \mathbb{R}^n)$) such that, for every $x \in C(I, \mathbb{R}^n)$, $F_l(t, x(t)) + \widetilde{x(t)}_t \subset B(0, h(t))$ a.e. $t \in I$, since $\|\widetilde{x(t)}_t\| \le \|v\|_0 + \|r\|_0$. Moreover, we deduce that, for every $x \in C(I, \mathbb{R}^n)$, the map $t \mapsto F_l(t, x(t))$ is measurable with closed nonempty values; see, for example, [14]. By the Kuratowski, Ryll-Nardzewski selection theorem [19], this map has a measurable selection, and hence the map $\mathcal{F}_l : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ defined by

$$\mathcal{F}_l(x) = \{y \in L^1(I, \mathbb{R}^n) : y(t) \in F_l(t, x(t)) + \widetilde{x(t)}_t \text{ a.e. } t \in I\}$$

has bounded nonempty values. It is easy to see that $\mathcal{F}_l$ has closed, decomposable values, i.e. for every $y, z \in \mathcal{F}_l(x)$ and every measurable set $\Omega \subset I$, $y\chi_\Omega + z\chi_{\Omega^c} \in \mathcal{F}_l(x)$.

Now, we show that $\mathcal{F}_l$ is l.s.c. Let $E \subset L^1(I, \mathbb{R}^n)$ be closed and $\{x_n\}$ a sequence in $\{x \in C(I, \mathbb{R}^n) : \mathcal{F}_l(x) \subset E\}$ converging to x_0. Let $y \in \mathcal{F}_l(x_0)$. For every $n \in \mathbb{N}$, there exists $y_n \in \mathcal{F}_l(x_n)$ such that

$$\|y_n(t) - y(t)\| = \operatorname{dist}(y(t), F_l(t, x_n(t)) + \widetilde{x_n(t)}_t);$$

see, for example, [6, Proposition 3.4]. The lower semi-continuity of $x \mapsto F_l(t,x) + \widetilde{x(t)}_t$ implies that

$$\mathrm{dist}(y(t), F_l(t, x_n(t)) + \widetilde{x_n(t)}_t\,) \to 0 \qquad \text{a.e. } t \in I.$$

The Lebesgue dominated convergence theorem implies that $y_n \to y$ in $L^1(I,\mathbb{R}^n)$ and $y \in E$. So, $\mathcal{F}_l(x_0) \subset E$. The conclusion follows from the Fryskowski, Bressan-Colombo selection theorem [4] and [11]. □

We define the multi-valued map $A_* : I \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$A_*(t,x) = \begin{cases} A(\widetilde{x}_t), & \text{if } r(t) > 0, \\ a(t), & \text{if } r(t) = 0, \end{cases}$$

where $\widetilde{x}_t$ is defined in (4.1), and a is given in (ST-1) (see also Definition 3.1).

Proposition 4.4. *Assume* (A1) *and* (ST-1). *Then* $\mathcal{A}_* : C(I,\mathbb{R}^n) \to L^1(I,\mathbb{R}^n)$ *defined by*

$$\mathcal{A}_*(x) = \{y \in L^1(I,\mathbb{R}^n) : y(t) \in A_*(t, x(t)) \ a.e.\ t \in I\}$$

has closed, convex values, and is u.s.c. when $L^1(I,\mathbb{R}^n)$ *is endowed with the weak topology. Moreover there exists* $m > 0$ *such that* $\|y(t)\| \le m$ *a.e. on* $\{t \in I : r(t) > 0\}$ *and for all* $x \in C(I,\mathbb{R}^n)$ *and* $y \in \mathcal{A}_*(x)$.

Proof. By (A1), and Lemmas 2.1 and 2.2, $x \mapsto A(x)$ is u.s.c. with convex, compact values, and there exists $m > 0$ such that $\|z\| \le m$ for all $z \in A(x)$ and all x such that $\|x\| \le \|v\|_0 + \|r\|_0$. So, $x \mapsto A_*(t,x)$ is u.s.c. for all $t \in I$; $t \mapsto A_*(t,x)$ is measurable for all $x \in \mathbb{R}^n$, and $(t,x) \mapsto A_*(t,x)$ has convex compact values. In other words, A_* is a Carathéodory function, i.e. it satisfies (F1-u) and (F2-1). Therefore, in arguing as in Proposition 4.1, we deduce that $\mathcal{A}_*$ has closed convex values and is u.s.c. when $L^1(I,\mathbb{R}^n)$ is endowed with the weak topology. □

Now we want to consider the case where $\mathrm{dom}(A) \ne \mathbb{R}^n$.

Let us define $M_+ : D \subset L^2(I,\mathbb{R}^n) \subset L^2(I,\mathbb{R}^n) \to L^2(I,\mathbb{R}^n)$ by $M_+ = L(x) + \widehat{A}(x)$, where $D = W_P^{1,2}(I,\mathbb{R}^n) \cap \mathrm{dom}(\widehat{A})$ and $L(x) = x'$. Similarly, we define $M_- : D \subset L^2(I,\mathbb{R}^n) \subset L^2(I,\mathbb{R}^n) \to L^2(I,\mathbb{R}^n)$ by $M_-(x) = -L(x) + \widehat{A}(x)$.

Proposition 4.5. *Under* (A2), (A3), M_+ *(resp.* M_-*) is a multi-valued maximal monotone operator.*

Proof. Let us show that M_+ is monotone. First of all, observe that $D \ne \emptyset$. Take $x, y \in D$ and $u \in \widehat{A}(x)$, $w \in \widehat{A}(y)$. Since $\widehat{A}$ is monotone, we have

$$\begin{aligned}\langle x' + u - y' - w, x - y\rangle_{L^2} &\ge \int_I \langle x'(t) - y'(t), x(t) - y(t)\rangle\,\mathrm{d}t \\ &= \frac{1}{2}\left(\|x(1) - y(1)\|^2 - \|x(0) - y(0)\|^2\right) \\ &= 0.\end{aligned}$$

Now, we have to show that M_+ is maximal. By Lemma 2.3, we have to show that $id + M_+$ is surjective. It is well known that $id + L$ is invertible and hence surjective. By Lemma 2.3, L is maximal monotone. Since for $\lambda > 0$, $\widehat{A}_\lambda$ is single-valued, monotone and Lipschitzian, $L + \widehat{A}_\lambda$

is maximal monotone, and hence $id + L + \widehat{A}_\lambda$ is surjective by Lemmas 2.3 and 2.4. So, for $h \in L^2(I, \mathbb{R}^n)$, there exists $x_\lambda \in W_P^{1,2}(I, \mathbb{R}^n)$ such that

$$(id + L + \widehat{A}_\lambda)x_\lambda = h. \tag{4.3}$$

Let $x_0 \in \mathrm{dom}(A)$, and denote

$$z_\lambda = x_0 + A_\lambda(x_0).$$

Using the facts that $x_\lambda(0) = x_\lambda(1)$, $\widehat{A}_\lambda$ is monotone and $\widehat{A}_\lambda(x_0) \equiv A_\lambda(x_0)$, we have that

$$\begin{aligned}
&\int_I \langle h(t) - z_\lambda, x_\lambda(t) - x_0 \rangle \,\mathrm{d}t \\
&\quad = \int_I \langle x_\lambda(t) - x_0 + x_\lambda'(t) + \widehat{A}_\lambda(x_\lambda(t)) - \widehat{A}_\lambda(x_0), x_\lambda(t) - x_0 \rangle \,\mathrm{d}t \\
&\quad \geq \|x_\lambda - x_0\|_{L^2}^2.
\end{aligned}$$

So, $\{x_\lambda\}$ is bounded in $L^2(I, \mathbb{R}^n)$ by a constant c. By (A3) and (2.1),

$$\begin{aligned}
\|\widehat{A}_\lambda(x_\lambda)\|_{L^2} &\leq \inf\{\|y\|_{L^2} : y \in \widehat{A}(x_\lambda)\} \\
&\leq \sup\{\inf\{\|y\|_{L^2} : y \in \widehat{A}(x)\} : \|x\|_{L^2} \leq c\} \\
&< \infty.
\end{aligned}$$

So, $\{\widehat{A}_\lambda(x_\lambda)\}$ is bounded in $L^2(I, \mathbb{R}^n)$ for λ bounded. If follows from Lemma 2.6 that M_+ is maximal. The proof is exactly the same for M_-. □

When M_+ (resp. M_-) is maximal monotone, $id + M_+$ (resp. $id + M_-$) is surjective and invertible, so we denote the resolvent of $M_\pm$ for $\lambda = 1$ by $J_\pm$, which is defined for $x \in L^2(I, \mathbb{R}^n)$ by

$$J_\pm(x) = (id + M_\pm)^{-1}(x) \in W_P^{1,2}(I, \mathbb{R}^n).$$

Proposition 4.6. *Under* (A2) *and* (A3), *the operator* $J_\pm : L^2(I, \mathbb{R}^n) \to W^{1,2}(I, \mathbb{R}^n)$, *where* $W^{1,2}(I, \mathbb{R}^n)$ *is endowed with the topology of* $C(I, \mathbb{R}^n)$, *is completely continuous, and is continuous when* $L^2(I, \mathbb{R}^n)$ *is endowed with the weak topology.*

Proof. We first consider the case of J_+. Since $L^2(I, \mathbb{R}^n)$ is a Hilbert space, it is sufficient to show that, if $y_n \rightharpoonup y$ weakly in $L^2(I, \mathbb{R}^n)$, $x_n = J_+(y_n) \to x = J_+(y)$ in $C(I, \mathbb{R}^n)$. Since $J_+ : L^2(I, \mathbb{R}^N) \to L^2(I, \mathbb{R}^n)$ is nonexpansive, we deduce that $\{x_n\}$ is bounded in $L^2(I, \mathbb{R}^n)$ by some constant $k \geq 0$.

Now, we want to show that $x_n \to x$ in $C(I, \mathbb{R}^n)$. For $\lambda > 0$, let x_n^λ be the unique solution of

$$y_n = x_n^\lambda + (x_n^\lambda)' + \widehat{A}_\lambda(x_n^\lambda).$$

For each $n \in \mathbb{N}$, Lemma 2.6 ensures that $x_n^\lambda \to x_n$, and $\widehat{A}_\lambda(x_n^\lambda) \to u_n \in \widehat{A}(x_n) \cap \{y_n - x_n - x_n'\}$ in $L^2(I, \mathbb{R}^n)$. Observe that, by (A3),

$$\begin{aligned}
\|u_n\| \leq \lim_{\lambda \to 0^+} \|\widehat{A}_\lambda(x_n^\lambda)\|_{L^2} &\leq \lim_{\lambda \to 0^+} \inf\{\|z\|_{L^2} : z \in \widehat{A}(x_n^\lambda)\} \\
&\leq \sup\{\inf\{\|z\|_{L^2} : z \in \widehat{A}(u)\} : \|u\|_{L^2} \leq k + 1\} \\
&< \infty.
\end{aligned}$$

This implies that $\{u_n\}$ is bounded in $L^2(I,\mathbb{R}^n)$, and hence $\{x_n\}$ is bounded in $W^{1,2}(I,\mathbb{R}^n)$. So, there are subsequences still denoted $\{x_n\}$ and $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in $L^2(I,\mathbb{R}^n)$, and $x_n \to x$ strongly in $C(I,\mathbb{R}^n)$ and hence in $L^2(I,\mathbb{R}^n)$, and weakly in $W^{1,2}(I,\mathbb{R}^n)$. Since M_+ is maximal monotone, we deduce that $(x, x'+u) \in \mathrm{Gr}(M_+)$, which is closed in $(L^2(I,\mathbb{R}^n),\mathcal{T}_s)\times(L^2(I,\mathbb{R}^n),\mathcal{T}_w)$. It follows that $x_n = J_+(y_n) \to x = J_+(y)$ strongly in $C(I,\mathbb{R}^n)$. The proof is exactly the same for J_-. □

Remark 4.7. Let $h \in L^2(I,[0,\infty))$ and $E = \{y \in L^2(I,\mathbb{R}^n) : \|y(t)\| \le h(t) \text{ a.e. } t \in I\}$ endowed with the topology of $L^1(I,\mathbb{R}^n)$. It can be shown that $J_\pm : E \to C(I,\mathbb{R}^n)$ is continuous and compact. Indeed, it suffices to argue as in the proof of the previous proposition.

5. Proofs of our existence results

We consider the problems

$$\begin{aligned} &x'(t)+x(t) \in -A_*(t,x(t)) + F_u(t,x(t)) + \widetilde{x(t)}_t \quad \text{a.e. } t\in[0,1],\\ &x(0)=x(1); \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &x'(t)+x(t) \in -A_*(t,x(t)) + F_l(t,x(t)) + \widetilde{x(t)}_t \quad \text{a.e.} t\in[0,1],\\ &x(0)=x(1). \end{aligned} \tag{5.2}$$

A priori bounds can be obtained for the solutions of (5.1) and (5.2).

Proposition 5.1. *Assume that* (A1), (ST-1) *are satisfied. Then every solution* $x \in W_P^{1,1}(I,\mathbb{R}^n)$ *of* (5.1) *or* (5.2) *is such that* $x \in T(v,r)$.

Proof. Without loss of generality, we assume that x is a solution of (5.1). There exists $y, a_x \in L^1(I,\mathbb{R}^n)$ such that $a_x(t)\in A_*(t,x(t))$, $y(t)\in F_u(t,x(t))$, and $x'(t)+x(t) = -a_x(t)+y(t)+\widetilde{x(t)}_t$ a.e. $t\in I$. Since A is a maximal monotone operator and, by (ST-1), we deduce that, for almost every $t \in \{t\in I : \|x(t)-v(t)\| > r(t) > 0\}$,

$$\begin{aligned} &\frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle}{\|x(t)-v(t)\|}\\ &\quad= \frac{\langle x(t)-v(t), -a_x(t)+y(t)+\widetilde{x(t)}_t - x(t) - v'(t)\rangle}{\|x(t)-v(t)\|}\\ &\quad= \frac{\langle \widetilde{x(t)}_t - v(t), y(t)-a(t)-v'(t)\rangle - \langle \widetilde{x(t)}_t - v(t), a_x(t)-a(t)\rangle}{r(t)}\\ &\qquad + r(t) - \|x(t)-v(t)\|\\ &\quad< r'(t). \end{aligned}$$

Also, for almost every $t\in\{t\in I : \|x(t)-v(t)\| > r(t) = 0\}$, we deduce that $y(t) = v'(t)+a(t)$, $x'(t) = v'(t)+\widetilde{x(t)}_t - x(t)$ and

$$\frac{\langle x(t)-v(t), x'(t)-v'(t)\rangle}{\|x(t)-v(t)\|} = -\|x(t)-v(t)\| < 0 = r'(t).$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-v(t)\| < r'(t) \quad \text{a.e. on } \{t\in I : \|x(t)-v(t)\| > r(t)\}. \tag{5.3}$$

Also, since x satisfies the periodic boundary condition and $\|v(0) - v(1)\| \leq r(0) - r(1)$, we deduce that

$$\|x(0) - v(0)\| - r(0) \leq \|x(1) - v(1)\| - r(1). \tag{5.4}$$

The conclusion follows from (5.3) and (5.4). □

Now, we can prove our existence theorems.

Proof of Theorem 3.4. By Propositions 4.1 and 4.4, the operator

$$\mathcal{F}_u - \mathcal{A}_* : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$$

has closed, convex, bounded values and is u.s.c. when $L^1(I, \mathbb{R}^n)$ is endowed with the weak topology. Moreover, there exists $h \in L^1(I, [0, \infty))$ such that

$$\|y(t)\| \leq h(t) \quad \text{a.e. } t \in I \text{ for all } y \in \mathcal{F}_u(x) - \mathcal{A}_*(x) \text{ and } x \in C(I, \mathbb{R}^n). \tag{5.5}$$

Let us define $L : W_P^{1,1}(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ by $L(x) = x'$. It is well known that $L + id$ is linear, continuous and bijective. Therefore, $(L + id)^{-1}$ is continuous and hence $(L + id)^{-1} : (L^1(I, \mathbb{R}^n), \mathcal{T}_w) \to (W^{1,1}(I, \mathbb{R}^n), \mathcal{T}_w)$ is continuous. Let $i : W^{1,1}(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ be the inclusion. Combining the results mentioned above with (5.5) and Pettis' theorem, we deduce that

$$i \circ (L + id)^{-1} \circ (\mathcal{F}_u - \mathcal{A}_*) : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$$

is compact u.s.c. with compact convex values. Therefore, the Kakutani fixed point theorem ensures the existence of a fixed point, and hence a solution x to (5.1). Finally, Proposition 5.1 guaranties that $x \in T(v, r)$, and hence x is a solution to (1.1). □

Proof of Theorem 3.5. Let f_l be the continuous single-valued map given by Proposition 4.3. It follows from Proposition 4.4 that operator $f_l - \mathcal{A}_* : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ has closed, convex, bounded values, and is u.s.c. when $L^1(I, \mathbb{R}^n)$ is endowed with the weak topology. Moreover, there exists $h \in L^1(I, [0, \infty))$ such that

$$\|y(t)\| \leq h(t) \quad \text{a.e. } t \in I \text{ for all } y \in f_l(x) - \mathcal{A}_*(x) \text{ and } x \in C(I, \mathbb{R}^n). \tag{5.6}$$

The rest of the proof is analogous to the proof of Theorem 3.4. □

Now, we want to prove Theorems 3.6 and 3.7. To this aim, we consider the problems

$$\begin{aligned} &x'(t) + x(t) \in -A(x(t)) + F_u(t, x(t)) + \widetilde{x(t)}_t \quad \text{a.e. } t \in [0, 1], \\ &x(0) = x(1); \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} &x'(t) + x(t) \in -A(x(t)) + F_l(t, x(t)) + \widetilde{x(t)}_t \quad \text{a.e. } t \in [0, 1], \\ &x(0) = x(1). \end{aligned} \tag{5.8}$$

Again, solutions to those problems are in $T(v, r)$.

Proposition 5.2. *Assume that* (A2), (A3), (ST-2) *are satisfied. Then every solution* $x \in W_P^{1,2}(I, \mathbb{R}^n)$ *of* (5.7) *or* (5.8) *is such that* $x \in T(v, r)$.

The proof of this proposition is similar to the proof of Proposition 5.1. We are ready to prove our existence results.

Proof of Theorem 3.6. By Proposition 4.1 and Remark 4.2, the operator

$$\mathcal{F}_u : C(I, \mathbb{R}^n) \to L^2(I, \mathbb{R}^n)$$

has closed, convex, bounded values and is u.s.c. when $L^2(I, \mathbb{R}^n)$ is endowed with the weak topology. Moreover, there exists $h \in L^2(I, [0, \infty))$ such that

$$\|y(t)\| \le h(t) \quad \text{a.e. } t \in I \text{ for all } y \in \mathcal{F}_u(x) \text{ and } x \in C(I, \mathbb{R}^n). \tag{5.9}$$

On the other hand, Proposition 4.6 implies that $J_+ \circ \mathcal{F}_u : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ is compact u.s.c. with compact convex values. Moreover, x is a solution of (5.7) if and only if x is a fixed point of $J_+ \circ \mathcal{F}_u$. Therefore, the Kakutani fixed point theorem ensures the existence of a fixed point, and hence a solution x to (5.7). Finally, Proposition 5.2 guarantees that $x \in T(v, r)$, and hence x is a solution to (1.1). □

Proof of Theorem 3.7. Let $f_l : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ be the continuous single-valued map given by Proposition 4.3 such that there exists $h \in L^2(I, [0, \infty))$ satisfying

$$\|y(t)\| \le h(t) \quad \text{a.e. } t \in I \text{ for all } y \in f_l(x) \text{ and } x \in C(I, \mathbb{R}^n). \tag{5.10}$$

It follows from Proposition 4.6 and Remark 4.7 that $J_+ \circ f_l : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ is a single-valued continuous compact operator. It follows from the Schauder fixed point theorem that this operator has a fixed point x, and hence a solution to (5.8). This solution is in $T(v, r)$ by Proposition 5.2. Therefore, x is a solution to (1.1). □

6. Existence results for the problem (1.2)

Now, we consider the problem (1.2)

$$\begin{aligned} &x'(t) \in A(x(t)) + F(t, x(t)), \quad \text{a.e. } t \in [0, 1], \\ &x(0) = x(1). \end{aligned} \tag{6.1}$$

We can obtain existence results analogous to those obtained for the problem (1.1) in introducing a notion of L^p-solution-tube for this problem.

Definition 6.1. Let $v \in W^{1,p}(I, \mathbb{R}^n)$, and $r \in W^{1,p}(I, \mathbb{R})$ with $p \in [1, \infty]$. We say that (v, r) is an *L^p-solution-tube of* (1.2) if there exists $a \in L^p(I, \mathbb{R}^n)$ such that

(i) $a(t) \in Av(t)$ a.e. $t \in I$;
(ii) for a.e. $t \in I$, and every $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = r(t)$, there exists $y \in F(t, x)$ such that
$$\langle x - v(t), y + a(t) - v'(t)\rangle \ge r(t)r'(t);$$
(iii) $v'(t) \in a(t) + F(t, v(t))$ a.e. on $\{t \in [0, 1] : r(t) = 0\}$;
(iv) $\|v(0) - v(1)\| \le r(1) - r(0)$.

We denote

$$T(v, r) = \{x \in C(I, \mathbb{R}^n) : \|x(t) - v(t)\| \le r(t) \ \forall t \in I\}.$$

We consider the following condition:

(ST-p)* there exists $(v, r) \in W^{1,p}(I, \mathbb{R}^n) \times W^{1,p}(I, [0, \infty[)$ an L^p-solution-tube of (1.2).

We obtain similar existence results to those obtained above.

Theorem 6.2. *Assume* (F2-1), (ST-1)*, (A1), *and* (F1-u) *or* (F1-l). *Then the problem* (1.2) *has a solution* $x \in W^{1,1}(I, \mathbb{R}^n) \cap T(v, r)$.

Theorem 6.3. *Assume* (F2-2), (ST-2)*, (A2), (A3), *and* (F1-u) *or* (F1-l). *Then the problem* (1.2) *has a solution* $x \in W^{1,2}(I, \mathbb{R}^n) \cap T(v, r)$.

The proofs are similar to those presented in the previous section when we consider the following auxiliary problems:

$$\begin{aligned} &x'(t) - x(t) \in A_*(t, x(t)) + F^*_\square(t, x(t)) - \widetilde{x(t)}_t \quad \text{a.e.} t \in [0, 1],\\ &x(0) = x(1); \end{aligned}$$

and

$$\begin{aligned} &x'(t) - x(t) \in A(x(t)) + F^*_\square(t, x(t)) - \widetilde{x(t)}_t \quad \text{a.e. } t \in [0, 1],\\ &x(0) = x(1); \end{aligned}$$

where $\square = u$ or l, and $F^*_\square = \tilde{F} \cap G^*_\square$ with

$$G^*_u(t, x) = \begin{cases} v'(t) + a(t), & \text{if } r(t) = 0,\\ \mathbb{R}^n, & \text{if } \|x - v(t)\| \le r(t) \\ & \quad \text{and } r(t) > 0,\\ \{z : \langle x - v(t), z - a(t) - v'(t)\rangle & \\ \qquad \ge r'(t)\|x - v(t)\|\}, & \text{otherwise;} \end{cases}$$

and

$$G^*_l(t, x) = \begin{cases} v'(t) + a(t), & \text{if } r(t) = 0,\\ \mathbb{R}^n, & \text{if } \|x - v(t)\| < r(t),\\ \{z : \langle x - v(t), z - a(t) - v'(t)\rangle & \\ \qquad \ge r'(t)\|x - v(t)\|\}, & \text{otherwise.} \end{cases}$$

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