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BOUNDARY AND PERIODIC VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, using the Schauder Fixed Point Theorem, we establish some existence results to boundary and periodic value problems for systems of nonlinear second order differential equations. Also, the notion of upper and lower solutions to a differential equation is generalized in a natural way to systems of differential equations.

1. Introduction

In this paper, we consider the boundary and periodic value problem for systems of nonlinear second order differential equations

$$(\star)\quad \begin{cases} x''(t) = f(t, x(t), x'(t)) \text{ a.e. } t \in [0,1] \\ \quad x \in BC \end{cases}$$

where $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, is a Carathéodory function and BC denotes a boundary condition such as non-homogeneous Dirichlet, Neumann, Sturm-Liouville conditions, or the periodic condition:

$$(P)\quad \begin{cases} x(0) = x(1), \\ x'(0) = x'(1); \end{cases}$$

$$(SL)\quad \begin{cases} A_0 x(0) - \beta_0 x'(0) = r_0, \\ A_1 x(1) + \beta_1 x'(1) = r_1; \end{cases}$$

where A_i is a $n \times n$ matrix (possibly nonsymmetric) for which there exists $\alpha_i \geq 0$ such that $\langle x, A_i x\rangle \geq \alpha_i \|x\|^2$ for all x in $\mathbb{R}^n$; $\beta_i = 0,1$; $\alpha_i + \beta_i > 0$; $i = 0,1$.

The literature on this problem is voluminuous, and we refer to [1–4,7–11] and the references therein. Recall that in the scalar case ($n = 1$), many results rely on an assumption of the following form:

(1.1) $xf(t,x,0) \geq 0 \quad \text{for} \quad |x| = M$.

This assumption was generalized by one of existence of upper and lower solutions:

(1.2) there exist $\phi \leq \psi \in W^{2,1}([0,1],\mathbb{R})$ such that
$\phi''(t) \geq f(t,\phi(t),\phi'(t))$, and $\psi''(t) \leq f(t,\psi(t),\psi'(t))$ a.e. $t \in [0,1]$.

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The condition (1.1) was generalized for systems of differential equations by the following assumption from which arose many results (see [1,3,8,10]):

(1.3) there exists a constant $M > 0$ such that $\langle x, f(t,x,p)\rangle + \|p\|^2 \geq 0$ for $\|x\| = M$ and $\langle x, p\rangle = 0$.

On the other hand, the assumption (1.2) was generalized in a number of ways, of which we mention the following two:

(1.4) there exist $\phi \leq \psi \in W^{2,1}([0,1],\mathbb{R}^n)$ such that
$\phi_i'' \geq f_i(t, x_1, \ldots, x_{i-1}, \phi_i, x_{i+1}, \ldots, x_n, p_1, \ldots, p_{i-1}, \phi_i', p_{i+1}, \ldots, p_n)$,
$\psi_i'' \leq f_i(t, x_1, \ldots, x_{i-1}, \psi_i, x_{i+1}, \ldots, x_n, p_1, \ldots, p_{i-1}, \psi_i', p_{i+1}, \ldots, p_n)$
for $\phi_j(t) \leq x_j \leq \psi_j(t)$, $-c_j \leq p_j \leq c_j$ for $j \neq i$, and c being any vector satisfying $|\phi_i'(t)|, |\psi_i'(t)| < c_i$ (see [2]);

(1.5) there exist $\phi \leq \psi \in W^{2,1}([0,1],\mathbb{R}^n)$ such that $\phi''(t) \geq f(t,\phi(t))$, and $\psi''(t) \leq f(t,\psi(t))$ a.e. $t \in [0,1]$.

This last assumption was given in the case where the considered problem was the periodic problem and the function f did not depend on the derivative x' (see [11]).

I recall also that those assumptions came with some other assumptions related to the boundary conditions. In particular, the assumptions (1.1) and (1.3) are deficient for the non-homogeneous Neumann problem.

Note also that in the scalar case, the assumption of existence of upper and lower solutions (1.2) generalizes the assumption (1.1). This is not the case for systems; that is, the assumptions (1.4) and (1.5) don't generalize the assumption (1.3).

In this paper, we introduce a new notion which is a natural generalization of the assumption (1.3). Moreover, in the scalar case, this notion is equivalent to the notion of upper and lower solutions. Furthermore, using this notion, we obtain existence results for problems with periodic, or non-homogeneous Dirichlet, Neumann, or Sturm-Liouville boundary conditions. Our Theorems 4.1 and 4.2 are generalizations of some results obtained by Bebernes and Schmitt [1], Fabry and Habets [4], and Hartman [8]. As we mentioned at the beginning, our results are given in the Carathéodory context. The proofs rely on the Schauder Fixed Point Theorem.

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2. Preliminaries

In this section, we establish notations, definitions, and results which are used throughout this paper. We denote $\langle\, ,\rangle$ the scalar product, and $\|\cdot\|$ the Euclidian norm in $\mathbb{R}^n$. The Banach space of the k-times continuously differentiable functions x is denoted by $C^k([0,1],\mathbb{R}^n)$ with the norm: $\|x\|_k = \max\{\|x\|_0, \|x'\|_0, \ldots, \|x^{(k)}\|_0\}$, where $\|x\|_0 = \max\{\|x(t)\| : t \in [0,1]\}$. The Sobolev space of functions in $C^1([0,1],\mathbb{R}^n)$ with the derivative being absolutely continuous is denoted by $W^{2,1}([0,1],\mathbb{R}^n)$. We define $C_0([0,1],\mathbb{R}^n) = \{x \in C([0,1],\mathbb{R}^n) : x(0) = 0\}$, and $C^k_B([0,1],\mathbb{R}^n)$, (resp. $W^{2,1}_B([0,1],\mathbb{R}^n)$) the set of functions $x \in C^k([0,1],\mathbb{R}^n)$ (resp. $W^{2,1}([0,1],\mathbb{R}^n)$) satisfying the boundary condition $x \in BC$. Let $L^1([0,1],\mathbb{R}^n)$ denote the space of integrable functions, with the usual norm $\|\cdot\|_{L^1}$.

Let $\varepsilon \geq 0$, we define the operator $L_\varepsilon : C^1_B([0,1],\mathbb{R}^n) \to C_0([0,1],\mathbb{R}^n)$ by:

$$L_\varepsilon(x)(t) = x'(t) - x'(0) - \varepsilon \int_0^t x(s)ds.$$

A function $F : C^1([0,1],\mathbb{R}^n) \to L^1([0,1],\mathbb{R}^n)$ is said *integrably bounded* if there exists an integrable function h in $L^1([0,1],[0,\infty))$ such that

$$\|F(x)(t)\| \le h(t) \quad \text{a.e.} \quad t \in [0,1], \text{ and for every } x \text{ in } C^1([0,1],\mathbb{R}^n).$$

We associate to F an operator $N_F : C^1([0,1],\mathbb{R}^n) \to C_0([0,1],\mathbb{R}^n)$ defined by:

$$N_F(x)(t) = \int_0^t F(x)(s)ds.$$

We recall the following result (see [7]).

Lemma 2.1. *If* $F : C^1([0,1],\mathbb{R}^n) \to L^1([0,1],\mathbb{R}^n)$ *is a continuous and integrably bounded function, then the associated operator* N_F *is continuous and compact.*

We say that a function $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a *Carathéodory function* if: (i) for every (x,p) in $\mathbb{R}^{2n}$, the function $t \mapsto f(t,x,p)$ is measurable; (ii) the function $(x,p) \mapsto f(t,x,p)$ is continuous for almost every t in $[0,1]$; (iii) for every $k > 0$, there exists a function h_k in $L^1([0,1],[0,\infty))$ such that $\|f(t,x,p)\| \le h_k(t)$ a.e. $t \in [0,1]$, for all $\|x\| \le k$ and $\|p\| \le k$.

For sake of completeness, we state the following results which will be used later in this paper.

Lemma 2.2. *Let* $u : [0,1] \to \mathbb{R}^n$ *be an absolutely continuous function and let* E *be a negligeable set in* $\mathbb{R}^n$*, then* $meas\{t \in [0,1] : u(t) \in E \text{ and } u'(t) \neq 0\} = 0$.

Lemma 2.3. *Let* $u \in W^{2,1}([0,1],\mathbb{R})$ *and* $\varepsilon > 0$. *Assume one of the following properties is satisfied:*

(i) $u''(t) \ge 0$ *a.e.* $t \in [0,1]$,
$a_0u(0) - b_0u'(0) \le 0$, $a_1u(1) + b_1u'(1) \le 0$,
where $a_i, b_i \ge 0$, *and* $\max\{a_i, b_i\} > 0$, $\max\{a_0, a_1\} > 0$;

(ii) $u''(t) - \varepsilon u(t) \ge 0$ *a.e.* $t \in [0,1]$,
$a_0u(0) - b_0u'(0) \le 0$, $a_1u(1) + b_1u'(1) \le 0$,
where $a_i, b_i \ge 0$, *and* $\max\{a_i, b_i\} > 0$;

(iii) $u''(t) - \varepsilon u(t) \ge 0$ *a.e.* $t \in [0,1]$,
$u(0) = u(1)$, $u(0) \le 0$ *or* $u'(1) - u'(0) \le 0$.

Then $u(t) \le 0$ *for all* $t \in [0,1]$.

Let us consider the following problem:

$$(\star) \begin{cases} x''(t) = f(t,x(t),x'(t)) \text{ a.e. } t \in [0,1] \\ \quad x \in BC \end{cases}$$

where $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a Carathéodory function and BC denotes one of the following boundary conditions:

$$(P) \begin{cases} x(0) = x(1), \\ x'(0) = x'(1); \end{cases}$$

$$(SL) \begin{cases} A_0x(0) - \beta_0x'(0) = r_0, \\ A_1x(1) + \beta_1x'(1) = r_1; \end{cases}$$

where A_i is a $n \times n$ matrix (possibly nonsymmetric) for which there exists $\alpha_i \geq 0$ such that $\langle x, A_i x\rangle \geq \alpha_i \|x\|^2$ for all x in $\mathbb{R}^n$; $\beta_i = 0, 1$; $\alpha_i + \beta_i > 0$, $i = 0, 1$. In particular, (SL) includes non-homogeneous Dirichlet and Neumann boundary conditions. A *solution* to $(\star)$ is a function $x \in W^{2,1}_B([0,1], \mathbb{R}^n)$ satisfying $(\star)$.

Now, we introduce the notion of *solution-tube* to the problem $(\star)$. This notion will play an essential role in our existence results.

Definition 2.4. A *solution-tube* to the problem $(\star)$ is a couple (v, M) where M is a non-negative function in $W^{2,1}([0,1], \mathbb{R})$, and v is a function in $W^{2,1}([0,1], \mathbb{R}^n)$ such that:

(i) $\langle x - v(t), f(t, x, p) - v''(t)\rangle + \|p - v'(t)\|^2 \geq M(t)M''(t) + (M'(t))^2$ a.e. $t \in [0,1]$ and for all $(x, p) \in \mathbb{R}^{2n}$ such that $\|x - v(t)\| = M(t)$, and $\langle x - v(t), p - v'(t)\rangle = M(t)M'(t)$;
and $v''(t) = f(t, v(t), v'(t))$ a.e. on $\{t \in [0,1] : M(t) = 0\}$;

(ii) if BC denotes (SL), $\|r_0 - (A_0 v(0) - \beta_0 v'(0))\| \leq \alpha_0 M(0) - \beta_0 M'(0)$, $\|r_1 - (A_1 v(1) + \beta_1 v'(1))\| \leq \alpha_1 M(1) + \beta_1 M'(1)$;
and if BC denotes (P), $v(0) = v(1)$, $\|v'(1) - v'(0)\| \leq M'(1) - M'(0)$, and $M(0) = M(1)$.

Remark that if BC denotes the homogeneous boundary condition (SL) (i.e. $r_0 = r_1 = 0$) or the periodic condition (P), to say that $(0, M)$ is a solution-tube to $(\star)$ with $M > 0$ being a constant, is equivalent to have:

$$\langle x, f(t, x, p)\rangle + \|p\|^2 \geq 0 \text{ a.e. } t \in [0,1] \text{ and for all } (x, p) \in \mathbb{R}^{2n}$$
$$\text{with } \|x\| = M \text{ and } \langle x, p\rangle = 0.$$

This condition was considered by many authors, we mention [1,3,8,11].

Remark also that in the scalar case, the notion of upper and lower solutions to $(\star)$ is equivalent to the notion of solution-tube to $(\star)$. Indeed, if $\phi \leq \psi \in W^{2,1}([0,1], \mathbb{R})$ are respectively lower and upper solutions to $(\star)$, then $((\phi + \psi)/2, (\psi - \phi)/2)$ is a solution-tube to $(\star)$. Conversely, if (v, M) is a solution-tube to $(\star)$, then $v - M$, and $v + M$ are respectively lower and upper solutions to $(\star)$.

3. Main Theorem

Before the statement of the main exitence result for the problem $(\star)$, we introduce some notations.

Let $v \in W^{2,1}([0,1], \mathbb{R}^n)$, and $M \in W^{2,1}([0,1], [0, \infty))$. Define

$$p_i = \begin{cases} \|r_i\| + \|A_i\| \, (M(i) + \|v(i)\|), & \text{if } BC = (SL), \text{ and } \beta_i \neq 0, \\ \infty, & \text{otherwise} \end{cases}$$

$i = 0, 1$;

$$p_2 = \begin{cases} \min\{|M'(t)| : M(t) = 0\}, & \text{if } \{t \in [0,1] : M(t) = 0\} \neq \emptyset \\ \infty, & \text{otherwise;} \end{cases}$$

and

$$c = c(BC, v, M) = \min\{p_0, p_1, p_2\}.$$

We may now state our main Theorem.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume there exists (v, M) a solution-tube to $(\star)$ such that $c = c(BC, v, M) < \infty$. In addition, suppose there exist $\gamma \in L^1([0,1],[0,\infty))$ and a Borel measurable function $\phi : [0,\infty) \to (0,\infty)$ such that*

$$\|f(t,x,p)\| \leq \gamma(t)\,\phi(\|p\|) \text{ a.e. } t \text{ and for all } (x,p) \text{ with } \|x - v(t)\| \leq M(t);$$

and

$$\int_c^\infty \frac{ds}{\phi(s)} > \|\gamma\|_{L^1},$$

Then the problem $(\star)$ has a solution such that $\|x(t)-v(t)\| \leq M(t)$ for all $t \in [0,1]$.

To prove this theorem, we will modify the function f. To this modified function, we will associate a problem for which we will deduce the existence of a solution. Finally, we will observe that this solution is in fact a solution to our original problem $(\star)$. In order to do that, we need to introduce some notations. Before that, we give some examples.

Examples 3.2. (1) The following problem has a solution.

$$\begin{cases} x''(t) = x'(t) - \|x(t)\|\, x(t) + (1, 0, \cdots, 0) \\ x(0) = x(1) = (0, \cdots, 0) \end{cases}$$

Verify that $v(t) \equiv 0$, $M(t) = t - t^2$, $\phi(s) = s + 17/16$, $\gamma(t) \equiv 1$, satisfy the assumptions of Theorem 3.1. Consequently, this problem has a solution such that $\|x(t)\| \leq t - t^2$. Obseve that there is no constant M such that $(0, M)$ is a solution-tube to this problem.

(2) The following problem has a solution.

$$\begin{cases} x''(t) = (x'(t) - (t, \cdots, t))\,(\|x(t)\| + 2) \\ x(0) = (0, \cdots, 0),\ x(1) = (1, \cdots, 1) \end{cases}$$

Verify that $v(t) = (\frac{t^2}{2}, \cdots, \frac{t^2}{2})$, $M(t) = \frac{t\sqrt{n}}{2}$, $\phi(s) = (s + \sqrt{n})(2 + \sqrt{n})$, $\gamma(t) \equiv 1$, satisfy the assumptions of Theorem 3.1. Consequently, this problem has a solution such that $\|x(t) - (\frac{t^2}{2}, \cdots, \frac{t^2}{2})\| \leq \frac{t\sqrt{n}}{2}$. Observe that if we look for a solution-tube of the form $(0, M(t))$, then we must have $\|x(1) - (0, \cdots, 0)\| = \|(1, \cdots, 1)\| = \sqrt{n} \leq M(1)$. Therefore $(0, M(t))$ gives a worse approximation of the solution. Note also that there is no solution-tube $(0, M)$ with M a constant.

Let (v, M) be the solution-tube to $(\star)$ given in Theorem 3.1, and let K be a positive constant which will be determined later. To $(t, x, p) \in [0,1] \times \mathbb{R}^{2n}$, we associate $\widetilde{x}, \widetilde{p}$, and $\hat{p}$ given by:

$$\widetilde{x} = \begin{cases} \frac{M(t)}{\|x - v(t)\|}(x - v(t)) + v(t), & \text{if } \|x - v(t)\| > M(t), \\ x, & \text{otherwise;} \end{cases}$$

$$\widetilde{p} = \begin{cases} \frac{K}{\|p - v'(t)\|}(p - v'(t)) + v'(t), & \text{if } \|p - v'(t)\| > K, \\ p, & \text{otherwise;} \end{cases}$$

and

$$\hat{p} = \begin{cases} \widetilde{p} + (x - v(t)) \times \\ \quad \left(\frac{M'(t)}{\|x-v(t)\|} - \frac{\langle x-v(t), \tilde{p}-v'(t)\rangle}{\|x-v(t)\|^2} \right), & \text{if } \|x - v(t)\| > M(t), \\ \widetilde{p} + \left(1 - \frac{K}{\|p-v'(t)\|}\right) \frac{M'(t)}{M(t)} (x - v(t)), & \text{if } M(t) > 0, \|x - v(t)\| \leq M(t), \\ & \quad \text{and } \|p - v'(t)\| > K, \\ p, & \text{otherwise.} \end{cases}$$

Observe that $\|\widetilde{x}\|, \|\widetilde{p}\|, \|\hat{p}\|$ are bounded independently of (t, x, p).

Remark 3.3. If $\|x - v(t)\| > M(t)$ then

(i) $\|\widetilde{x} - v(t)\| = M(t)$,
(ii) $\langle \widetilde{x} - v(t), \hat{p} - v'(t)\rangle = M(t)M'(t)$,
(iii) $\|\hat{p} - v'(t)\|^2 = \|\widetilde{p} - v'(t)\|^2 + (M'(t))^2 - \frac{\langle x-v(t), \tilde{p}-v'(t)\rangle^2}{\|x-v(t)\|^2}$,
(iv) if $K \geq 2\,\|M'\|_0$, then there exists a constant K_0 depending only on v' and M' such that $\|\hat{p}\|^2 \leq \|p\|^2 + K_0$.

Let $\varepsilon \geq 0$, we define the functions $f_1, f_2 : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ by:

$$f_1(t, x, p) = \begin{cases} \frac{M(t)}{\|x-v(t)\|} f(t, \widetilde{x}, \hat{p}) + \left(1 - \frac{M(t)}{\|x-v(t)\|}\right) \times \\ \qquad \left(v''(t) + \frac{M''(t)}{\|x-v(t)\|}(x - v(t))\right), & \text{if } \|x - v(t)\| > M(t), \\ f(t, x, \hat{p}), & \text{otherwise;} \end{cases}$$

$$f_2(t, x, p) = f_1(t, x, p) - \varepsilon\widetilde{x}.$$

Observe that $f_1(t, x, p) = f(t, x, p)$, $f_2(t, x, p) = f(t, x, p) - \varepsilon x$ on $\{(t, x, p) : \|x - v(t)\| \leq M(t),\ \|p - v'(t)\| \leq K\}$, and there exists h in $L^1([0,1], [0,\infty))$ such that $\|f_i(t, x, p)\| \leq h(t)$ a.e. t, and for all $(x, p) \in \mathbb{R}^{2n}$, $i = 1, 2$.

To the function f_i, we associate $F_i : C^1([0,1], \mathbb{R}^n) \to L^1([0,1], \mathbb{R}^n)$, $(i = 1, 2)$, an operator defined by:

$$F_i(x)(t) = f_i(t, x(t), x'(t)).$$

The function f_1, and consequently f_2 are not necessarily Carathéodory functions, but we have the following result:

Proposition 3.4. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function and let (v, M) be a solution-tube to $(\star)$. Then the previously defined operator $F_1 : C^1([0,1], \mathbb{R}^n) \to L^1([0,1], \mathbb{R}^n)$, is continuous and integrably bounded.*

Proof. Obviously, F_1 is integrably bounded. Therefore, it is sufficient to show that if $x_n \to x$ in $C^1([0,1], \mathbb{R}^n)$, then

$$f_1(t, x_n(t), {x'}_n(t)) \to f_1(t, x(t), x'(t)) \quad \text{a.e.} \quad t \in [0,1]. \tag{3.1}$$

The conclusion will follow from the Lebesgue Dominated Convergence Theorem.

Since f is a Carathéodory function, it is clear from the definition of f_1 that the relation (3.1) holds almost everywhere on $\{t \in [0,1] : \|x(t) - v(t)\| \neq M(t)\}$. On the other hand, by Lemma 2.2, we have

$$\langle x(t) - v(t), x'(t) - v'(t)\rangle = M(t)M'(t) \quad \text{a.e.}$$

on $\{t \in [0,1] : \|x(t) - v(t)\| = M(t) > 0\}$. Therefore, it is easy to verify that almost everywhere on that set,

$$\hat{x}'_n(t) \to \hat{x}'(t).$$

Thus, the relation (3.1) is satisfied almost everywhere on that set.

Finally, on $\{t \in [0,1] : \|x(t) - v(t)\| = 0 = M(t)\}$, $x(t) = v(t)$, $x'(t) = v'(t)$, $M'(t) = 0$, $M''(t) = 0$ a.e.; so, $f_1(t, x(t), x'(t)) = f(t, x(t), x'(t)) = f(t, v(t), v'(t)) = v''(t)$ a.e. Observe that, on that set, $f_1(t, y, p) = v''(t)$ a.e., for all p, and $y \neq v(t)$. This completes the proof. □

Corollary 3.5. *Under the assumptions of Proposition* 3.4*, the operator* $F_2 : C^1([0,1], \mathbb{R}^n) \to L^1([0,1], \mathbb{R}^n)$ *previously defined, is continuous, and integrably bounded.*

Now, we consider the associated problems:

$$(\star)_1 \quad \begin{cases} x''(t) = f_1(t, x(t), x'(t)) \quad \text{a.e.} \quad t \in [0,1] \\ x \in BC \end{cases}$$

$$(\star)_2 \quad \begin{cases} x''(t) - \varepsilon\, x(t) = f_2(t, x(t), x'(t)) \quad \text{a.e.} \quad t \in [0,1] \\ x \in BC \end{cases}$$

Fix $\varepsilon \geq 0$ such that the operator $L_\varepsilon : C^1_B([0,1], \mathbb{R}^n) \to C_0([0,1], \mathbb{R}^n)$ defined in §2 is invertible. In particular, if BC denotes (SL) with $\max\{\alpha_0, \alpha_1\} > 0$, then we can take $\varepsilon = 0$ (see [6]).

The following result gives a priori bounds on the solutions to the problem $(\star)_2$.

Lemma 3.6. *Let f be a Carathéodory function and (v, M) a solution-tube to $(\star)$. Then every solution to $(\star)_2$ satisfies $\|x(t) - v(t)\| \leq M(t)$ for every $t \in [0,1]$.*

Proof. First of all, we remark that

$$(3.2) \quad \langle x - v(t), f_1(t, x, p) - v''(t)\rangle + \|p - v'(t)\|^2 \geq M''(t)\,\|x - v(t)\| + \frac{\langle x - v(t), p - v'(t)\rangle^2}{\|x - v(t)\|^2}$$

a.e. $t \in [0,1]$, and for every $(x, p) \in \mathbb{R}^{2n}$, with $\|x - v(t)\| > M(t)$.

Indeed, if $\|x - v(t)\| > M(t)$, using Definition 2.4 and Remark 3.3 give

$$\begin{aligned}
&\langle x - v(t), f_1(t,x,p) - v''(t)\rangle + \|p - v'(t)\|^2 \\
&\quad = \langle \widetilde{x} - v(t), f(t,\widetilde{x},\hat{p}) - v''(t)\rangle + M''(t)\ (\|x - v(t)\| - M(t)) + \|p - v'(t)\|^2 \\
&\quad \geq M(t)M''(t) + (M'(t))^2 + M''(t)\ (\|x - v(t)\| - M(t)) \\
&\qquad\quad + \|p - v'(t)\|^2 - \|\hat{p} - v'(t)\|^2 \\
&\quad = M''(t)\,\|x - v(t)\| + \|p - v'(t)\|^2 - \|\widetilde{p} - v'(t)\|^2 + \frac{\langle x - v(t), \widetilde{p} - v'(t)\rangle^2}{\|x - v(t)\|^2} \\
&\quad = \begin{cases} M''(t)\,\|x - v(t)\| + \frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^2}, & \text{if } \|p - v'(t)\| \leq K \\ M''(t)\,\|x - v(t)\| + \frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^2} & \\ \quad + \left(1 - \frac{K^2}{\|p-v'(t)\|^2}\right) \times & \\ \qquad \left(\|p - v'(t)\|^2 - \frac{\langle x-v(t), p-v'(t)\rangle^2}{\|x-v(t)\|^2}\right), & \text{otherwise} \end{cases} \\
&\quad \geq M''(t)\,\|x - v(t)\| + \frac{\langle x - v(t), p - v'(t)\rangle^2}{\|x - v(t)\|^2}.
\end{aligned}$$

On the other hand, let x be a solution to $(\star)_2$. So,

$$x''(t) = f_2(t, x(t), x'(t)) + \varepsilon x(t) = f_1(t, x(t), x'(t)) + \varepsilon(x(t) - \widetilde{x}(t)). \tag{3.3}$$

On the set $\{t \in [0,1] : \|x(t) - v(t)\| > M(t)\}$, we have

$$\|x(t) - v(t)\|' = \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle}{\|x(t) - v(t)\|}$$

which exists for all t, and

$$\|x(t) - v(t)\|'' = \frac{\langle x(t) - v(t), x''(t) - v''(t)\rangle + \|x'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3}. \tag{3.4}$$

Fix $\delta > 0$, and let $E_\delta = \{t \in [0,1] : \|x(t) - v(t)\| > M(t) + \delta\}$. The function $\|x(t) - v(t)\|$ belongs to the space $W^{2,1}(E_\delta, \mathbb{R})$. Therefore, if we note $w(t) = \|x(t) - v(t)\| - (M(t) + \delta)$, then, using the relations (3.2), (3.3) and (3.4), we verify that:

$$\begin{aligned}
&w''(t) - \varepsilon w(t) \\
&\quad = \frac{\left\langle x(t) - v(t), f_1(t, x(t), x'(t)) - v''(t) + \varepsilon(1 - \frac{M(t)}{\|x(t)-v(t)\|})(x(t) - v(t))\right\rangle}{\|x(t) - v(t)\|} \\
&\qquad + \frac{\|x'(t) - v'(t)\|^2}{\|x(t) - v(t)\|} - \frac{\langle x(t) - v(t), x'(t) - v'(t)\rangle^2}{\|x(t) - v(t)\|^3} \\
&\qquad - M''(t) - \varepsilon w(t) \\
&\quad \geq \varepsilon\delta \\
&\quad \geq 0.
\end{aligned}$$

In order to apply the maximum principle (Lemma 2.3), we need to verify some boundary conditions. If BC denotes (SL) then, either

$$w(0) \le 0 \quad \text{or} \quad \alpha_0 w(0) - \beta_0 w'(0) \le 0.$$

Indeed,

$$\begin{aligned}\|x(0)-v(0)\| &(\alpha_0\|x(0)-v(0)\| - \beta_0\|x(0)-v(0)\|') \\ &\le \langle x(0)-v(0), A_0(x(0)-v(0)) - \beta_0(x'(0)-v'(0))\rangle \\ &\le \|x(0)-v(0)\|\,\|r_0 - (A_0 v(0) - \beta_0 v'(0))\| \\ &\le \|x(0)-v(0)\|(\alpha_0 M(0) - \beta_0 M'(0)) \\ &\le \|x(0)-v(0)\|(\alpha_0(M(0)+\delta) - \beta_0(M+\delta)'(0)).\end{aligned}$$

Similarly, either

$$w(1) \le 0 \quad \text{or} \quad \alpha_1 w(1) + \beta_1 w'(1) \le 0.$$

On the other hand, if BC denotes the periodic boundary condition (P),

$$\|x(0)-v(0)\| = \|x(1)-v(1)\|, \quad M(0) = M(1),$$

and, either

$$\|x(0)-v(0)\| \le 0 \quad \text{or} \quad w'(1) - w'(0) \le 0.$$

Indeed,

$$\begin{aligned}\|x(1)-v(1)\|' - \|x(0)-v(0)\|' &= \frac{\langle x(0)-v(0), v'(0)-v'(1)\rangle}{\|x(0)-v(0)\|} \le \\ \|v'(1)-v'(0)\| &\le M'(1) - M'(0) = (M+\delta)'(1) - (M+\delta)'(0).\end{aligned}$$

By Lemma 2.3 applied to w, we deduce that $\|x(t)-v(t)\| \le M(t)+\delta$. But this inequality holds for every $\delta > 0$; therefore, $\|x(t)-v(t)\| \le M(t)$ for every $t \in [0,1]$. This completes the proof. □

Now, we can prove our main Theorem.

Proof of Theorem 3.1. To prove this theorem, the constant K will be chosen appropriately, and we will show that the problem $(\star)_2$ has a solution x satisfying $\|x(t)-v(t)\| \le M(t)$, and $\|x'(t)-v'(t)\| \le K$. Thus, using the definition of f_2, this solution will be a solution to our original problem $(\star)$.

By Lemma 3.6, we know that every solution to $(\star)_2$ satisfies $\|x(t) - v(t)\| \le M(t)$. Now, we will determine K in order that $\|x'(t)-v'(t)\| \le K$ for every solution to $(\star)_2$.

Let x be a solution to $(\star)_2$ and $c = c(BC, v, M)$ be the constant previously defined. By assumption, $c < \infty$. Using the boundary condition, we can show that there exists $t_0 \in [0,1]$ such that $\|x'(t_0)\| \le c$. Fix $\widetilde{K} > c$ such that

$$\|\gamma\|_{L^1} < \int_c^{\widetilde{K}} \frac{ds}{\phi(s)}, \tag{3.5}$$

and choose K such that

$$\|p\| \le \widetilde{K} \quad \text{implies} \quad \|p - v'(t)\| \le K \quad \text{for all} \quad t \in [0,1]. \tag{3.6}$$

We claim that $\|x'(t)\| < \widetilde{K}$ for all $t \in [0,1]$. Suppose that $\|x'(t_1)\| \ge \widetilde{K}$ for some $t_1 \in [0,1]$. Then, there exist $t_2, t_3 \in [0,1]$ such that $\|x'(t_2)\| = c$, $\|x'(t_3)\| = \widetilde{K}$, and $c < \|x'(t)\| \le \widetilde{K}$ for all t between t_2 and t_3. Without loss of generality, assume that $t_2 < t_3$, then

$$\|x'(t)\|' = \frac{\langle x'(t), x''(t)\rangle}{\|x'(t)\|}$$

which exists for all $t \in (t_2, t_3]$, and

$$x''(t) = f(t, x(t), x'(t)) \quad \text{a.e.} \quad t \in [t_2, t_3]$$

by the definition of f_2. Thus,

$$\|x'(t)\|' \le \|x''(t)\| \le \gamma(t)\,\phi(\|x'(t)\|) \quad \text{a.e.} \quad t \in (t_2, t_3].$$

Dividing by ϕ, integrating from t_2 to t_3, we obtain:

$$\int_{t_2}^{t_3} \frac{\|x'(t)\|'}{\phi(\|x'(t)\|)}\,dt \le \|\gamma\|_{L^1}.$$

By the inequality (3.5) and the change of variables formula (see [5]), we get

$$\|\gamma\|_{L^1} < \int_c^{\widetilde{K}} \frac{ds}{\phi(s)} = \int_{t_2}^{t_3} \frac{\|x'(t)\|'}{\phi(\|x'(t)\|)}\,dt \le \|\gamma\|_{L^1}.$$

This leads to a contradiction. In consequence, $\|x'(t)\| < \widetilde{K}$ for all $t \in [0,1]$, and the relation (3.6) gives

$$\|x'(t) - v'(t)\| \le K \quad \text{for all } t \in [0,1]. \tag{3.7}$$

On the other hand, a solution to $(\star)_2$ is a fixed point for the operator $L_\varepsilon^{-1} \circ N_{F_2} : C_B^1([0,1],\mathbb{R}^n) \to C_B^1([0,1],\mathbb{R}^n)$, where L_ε and N_{F_2} are defined in §2. Using Lemma 2.1 and Corollary 3.5, we deduce the compacity of this operator. The Schauder Fixed Point Theorem gives the existence of a fixed point to $L_\varepsilon^{-1} \circ N_{F_2}$, and then a solution to $(\star)_2$. Using Lemma 3.6 and the relation (3.7), we get the conclusion. □

4. Other existence results

In what follows, we will generalize some results given by Hartmann [8,9], Bebernes and Schmitt [1], Fabry and Habets [4], for the periodic or the Dirichlet problem. In those results, the function f was continuous, and they assumed the existence of what we call a solution-tube of the form $(0, M)$ with M a positve constant, as in [1,8,9], or a positive function in $C^2([0,1],\mathbb{R})$, as in [4]. As we mentioned before, we will obtain results not only for the Dirichlet or periodic boundary conditions, but also for the non-homogeneous Neumann and Sturm-Liouville boundary conditions.

Theorem 4.1. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume there exists (v, M) a solution-tube to the problem $(\star)$. In addition, assume there exist a constant $k \geq 0$, a function $h \in L^1([0,1],[0,\infty))$, and a Borel measurable function $\phi : [0,\infty) \to (0,\infty)$ such that*

$$\int^{\infty} \frac{s}{\phi(s)}\, ds = \infty,$$

and the two following properties are satisfied a.e. $t \in [0,1]$, and for $(x,p) \in \mathbb{R}^{2n}$ with $\|x - v(t)\| \leq M(t)$:

(i) $\|f(t,x,p)\| \leq 2k\left(\langle x, f(t,x,p)\rangle + \|p\|^2\right) + h(t)$;
(ii) $|\langle p, f(t,x,p)\rangle| \leq \|p\|\, \phi(\|p\|)$.

Then the system $(\star)$ has a solution such that $\|x(t) - v(t)\| \leq M(t)$.

Proof. We will show that every solution to the problem $(\star)_2$ satisfies $\|x(t) - v(t)\| \leq M(t)$, and $\|x'(t) - v'(t)\| \leq K$ where K will be an appropriate constant which will be chosen later.

Let K_0 be the constant given in Remark 3.3(iv). Let $M_1 = \|M\|_0 + \|v\|_0$, $K_1 = 4(1 + kM_1)M_1 + \|h\|_{L^1} + k\,K_0/2$, and take $K_2 > K_1$ such that

$$\int_{K_1}^{K_2} \frac{s}{\phi(s)} ds > K_1 + 2kM_1^2; \tag{4.1}$$

and choose $K > 2\,\|M'\|_0$ such that

$$\|p\| \leq K_2 \quad \text{implies} \quad \|p - v'(t)\| \leq K \ \text{ for all } \ t \in [0,1]. \tag{4.2}$$

We will show that every solution to $(\star)_2$ satisfies $\|x'(t)\| \leq K_2$, hence

$$\|x'(t) - v'(t)\| \leq K \text{ for all } \ t \in [0,1].$$

Let x be a solution to $(\star)_2$. By Lemma 3.6, we already know that x satisfies $\|x(t) - v(t)\| \leq M(t)$, and thus

$$\|x(t)\| \leq M_1. \tag{4.3}$$

The assumption (i), the inequality (4.3), Remark 3.3(iv), and the following relation

$$x(t+1/2) - x(t) - \frac{x'(t)}{2} = \int_t^{t+1/2} (t + 1/2 - s)\, x''(s) ds, \quad 0 \leq t \leq 1/2$$

imply that

$$\|x'(t)\| \leq 4M_1 + 4kM_1^2 + \|h\|_{L^1} - 2k(\|x(t)\|^2)' + \frac{k\,K_0}{2}, \quad \text{for } \ 0 \leq t \leq 1/2. \tag{4.4}$$

Similarly, the assumption (i), the inequality (4.3), Remark 3.3(iv), and the following relation

$$x(t) - x(t-1/2) - \frac{x'(t)}{2} = \int_{t-1/2}^{t} (t - 1/2 - s)\, x''(s) ds, \quad 1/2 \leq t \leq 1$$

leads to

$$\text{(4.5)}\quad \|x'(t)\| \le 4M_1 + 4kM_1^2 + \|h\|_{L^1} + 2k(\|x(t)\|^2)' + \frac{k\,K_0}{2}, \quad \text{for } \ 1/2 \le t \le 1.$$

Adding (4.4) and (4.5) gives

$$\|x'(1/2)\| \le 4M_1 + 4kM_1^2 + \|h\|_{L^1} + k\,K_0/2 = K_1.$$

Now, suppose there exists $t_0 \in [0,1]$ such that $\|x'(t_0)\| \ge K_2$. Then there exist t_1 and $t_2 \in [0,1]$ such that $\|x'(t_1)\| = K_1,\ \|x'(t_2)\| = K_2$, and $K_1 < \|x'(t)\| < K_2$ for t between t_1 and t_2. Without loss of generality, assume $1/2 \le t_1 < t_2$. Then the assumption (ii) and the inequality (4.5) imply that

$$\frac{\langle x'(t), x''(t)\rangle}{\phi(\|x'(t)\|)} \le \frac{|\langle x'(t), x''(t)\rangle|}{\phi(\|x'(t)\|)} \le \|x'(t)\| \le K_1 + 2k(\|x(t)\|^2)'.$$

Integrating from t_1 to t_2, and using the change of variables formula and the inequality (4.1) give

$$\begin{aligned}\int_{K_1}^{K_2} \frac{s}{\phi(s)}\, ds = \int_{t_1}^{t_2} \frac{\langle x'(t), x''(t)\rangle}{\phi(\|x'(t)\|)}\, dt &\le K_1 + 2kM_1^2 \\ &< \int_{K_1}^{K_2} \frac{s}{\phi(s)}\, ds.\end{aligned}$$

This is a contradiction. Therefore, $\|x'(t)\| \le K_2$ for all $t \in [0,1]$ and then $\|x'(t) - v'(t)\| \le K$. The rest of the proof follows as in the proof of Theorem 3.1, and we get the existence of a solution to the problem $(\star)$. □

The following theorem generalizes a result given by Fabry and Habets [4], and obtained for the classical homogeneous Dirichlet problem. They assumed $M(t) > 0$ for all $t \in [0,1]$, which is not the case here.

Theorem 4.2. *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be a Carathéodory function. Assume there exists (v, M) a solution-tube to the problem $(\star)$. In addition, assume there exist a constant $k \in [0,1)$, a function $h \in L^1([0,1],[0,\infty))$, and a Borel measurable function $\phi : [0,\infty) \to (0,\infty)$ such that*

$$\int^{\infty} \frac{s^2}{\phi(s)}\, ds = \infty,$$

and the two following properties are satisfied a.e. $t \in [0,1]$, and for $(x,p) \in \mathbb{R}^{2n}$ with $\|x - v(t)\| \le M(t)$:

(i) $0 \le \langle x, f(t,x,p)\rangle + k\|p\|^2 + h(t)$;
(ii) $|\langle p, f(t,x,p)\rangle| \le \|p\|\,\phi(\|p\|)$.

Let BC denote the boundary condition (P), or (SL) with $(1-\beta_i)r_iM(i) = 0$, $i = 0,1$. Then the system $(\star)$ has a solution such that $\|x(t) - v(t)\| \le M(t)$.

Proof. As before, we will show that the problem $(\star)_2$ has a solution satisfying $\|x(t) - v(t)\| \le M(t)$, and $\|x'(t) - v'(t)\| \le K$, where K is a constant which will be determined later.

First of all, taking into account the boundary condition and Lemma 3.6, we note that there exists a constant $k_0 = k_0(BC, v, M)$ such that every solution to $(\star)_2$ is such that

$$\langle x(1), x'(1)\rangle - \langle x(0), x'(0)\rangle \le k_0. \tag{4.6}$$

Let $k_1 = \left(\frac{1}{1-k}(\|h\|_{L^1} + k_0 + k\,K_0)\right)^{1/2}$, where K_0 is the constant given in Remark 3.3(iv). Let k_2 be such that

$$\int_{k_1}^{k_2} \frac{s^2}{\phi(s)}\,ds > k_1^2, \tag{4.7}$$

and take $K > 2\,\|M'\|_0$ such that

$$\|p\| \le k_2, \quad \text{implies that} \quad \|p - v'(t)\| \le K. \tag{4.8}$$

Let x be a solution to $(\star)_2$. By Lemma 3.6,

$$\|x(t) - v(t)\| \le M(t). \tag{4.9}$$

By using Remark 3.3(iv), the relation (4.9), and the assumption (i), we get

$$\begin{aligned}\langle x(t), x''(t)\rangle = \langle x(t), f_1(t, x(t), x'(t))\rangle &= \langle x(t), f(t, x(t), \hat{x}'(t))\rangle \\ &\ge -k\|x'(t)\|^2 - h(t) - k\,K_0.\end{aligned}$$

Integrating by parts gives

$$\langle x(1), x'(1)\rangle - \langle x(0), x'(0)\rangle - \int_0^1 \|x'(t)\|^2\,dt \ge -k\int_0^1 \|x'(t)\|^2\,dt - \|h\|_{L^1} - k\,K_0.$$

It follows that

$$\int_0^1 \|x'(t)\|^2\,dt \le \frac{1}{1-k}(\|h\|_{L^1} + k_0 + k\,K_0) = k_1^2.$$

Now, suppose there exists $t_0 \in [0,1]$ such that $\|x'(t_0)\| \ge k_2$. Then there exist t_1 and t_2 in $[0,1]$ such that $\|x'(t_1)\| = k_1$, $\|x'(t_2)\| = k_2$, and $k_1 < \|x'(t)\| \le k_2$, and then $\|x'(t) - v'(t)\| \le K$ for t between t_1 and t_2. Without loss of generality, assume $t_1 < t_2$, then the assumption (ii) implies that

$$\frac{\|x'(t)\|\,\langle x'(t), x''(t)\rangle}{\phi(\|x'(t)\|)} \le \|x'(t)\|^2 \quad \text{a.e.} \quad t \in (t_1, t_2).$$

Integrating from t_1 to t_2, and using the change of variables formula give

$$\begin{aligned}\int_{k_1}^{k_2} \frac{s^2}{\phi(s)}\,ds = \int_{t_1}^{t_2} &\frac{\|x'(t)\|\,\langle x'(t), x''(t)\rangle}{\phi(\|x'(t)\|)}\,dt \\ &\le \int_{t_1}^{t_2} \|x'(t)\|^2\,dt \le k_1^2 < \int_{k_1}^{k_2} \frac{s^2}{\phi(s)} ds.\end{aligned}$$

We get a contradiction. Therefore, $\|x'(t)\| < k_2$, and then $\|x'(t) - v'(t)\| \le K$ for all $t \in [0,1]$. This completes the proof. □

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