MAT6684W: Sieve Methods, Fall 2012

Homework assignment #1

The exercise numbers refer to the course notes

Exercise 1.1.2 (5 marks). Use the Eratosthenes-Legendre sieve to show that

$$\#\{n \le x : n \text{ is square-free}\} \sim x \cdot \prod_p \left(1 - \frac{1}{p^2}\right) \quad (x \to \infty).$$

Bonus question (3 marks). Show that

$$\#\{n \le x : n \text{ is square-free}\} = x \cdot \prod_p \left(1 - \frac{1}{p^2}\right) + O(\sqrt{x}) \quad (x \ge 2).$$

Exercise 1.1.4 (5 marks). Find the average value of the greatest common divisor of a and b asymptotically, as a and b range over all integers up to x.

Exercise A.2.2 (5 marks). Let $\sigma(n) = \sum_{d|n} d$. Use the convolution method to show that

$$\sum_{n \le x} \frac{1}{\sigma(n)} \sim c \log x \quad (x \to \infty),$$

for some appropriate constant c.

Exercise A.2.9 (5 marks). Show that

$$\frac{n}{\phi(n)} \asymp \prod_{\substack{p|n \\ p \le y}} \left(1 + \frac{1}{p}\right) \quad (y \ge \sqrt{\log n}).$$

Use this fact to show that

$$\sum_{x-y<n\le x} \frac{n}{\phi(n)} \ll_\epsilon y \quad (x^\epsilon \le y \le x).$$

Exercise B.1.3 (5 marks). Show that the entries of the $N \times N$ multiplication table form a sparse set of the integers in the sense that

$$\lim_{N\to\infty} \frac{1}{N^2} \#\{ab : a \le N,\, b \le N\} = 0$$

Exercise A.2.5 (bonus, 7 marks). Show that

$$\#\{n \le x : n \text{ is square-free}\} = x \cdot \prod_p \left(1 - \frac{1}{p^2}\right) + o(\sqrt{x}) \quad (x \to \infty).$$