MAT6684W: Sieve Methods, Fall 2012

Homework assignment #2

The exercise and theorem numbers refer to the course notes

Exercise 3.1.2 (4 marks). Show that

$$\#\{n \le x : (n, P(z)) = 1\} \ll \frac{x}{\log z} \quad (3/2 \le z \le x)$$

and

$$\#\{n \le x : (n, P(z)) = 1\} \gg \frac{x}{\log z} \quad (3 \le 2z \le x).$$

Bonus question (4 marks). Show the stronger estimates

$$\sum_{\substack{n \le x \\ P^-(n) \ge z}} \frac{n}{\phi(n)} \ll \frac{x}{\log z} \quad (3/2 \le z \le x)$$

and

$$\sum_{\substack{n \le x \\ P^-(n) \ge z}} \frac{\mu^2(n)\phi(n)}{n} \gg \frac{x}{\log z} \quad (3 \le 2z \le x).$$

Exercise 3.1.3 (4 marks). Let $N \ge 2$ be an integer. Show that

$$\#\{(p,q) : p,\, q \,\text{primes},\ p+q = 2N\} \ll \frac{N}{\phi(N)} \cdot \frac{N}{(\log N)^2}.$$

Moreover, if N is large, then prove that

$$\#\{p \le 2N : \Omega(2N-p) \le 8\} \gg \frac{N}{\phi(N)} \cdot \frac{N}{(\log N)^2}.$$

Exercise (5 marks). Prove Theorem 3.1.5:
Let $F_1(x), \ldots, F_r(x)$ be distinct irreducible polynomials over $\mathbb{Z}[x]$ with positive leading coefficient. Suppose that the polynomial $F = F_1 \cdots F_r$ has no fixed prime divisors, i.e. there is no prime p such that $p | F(n)$ for all integers n. Then we have that

$$\#\{n \le x : F_1(n), \ldots, F_r(n) \text{ are all primes}\} \ll_F \frac{x}{(\log x)^r}.$$

Moreover, if x is large enough and d denotes the degree of F, then

$$\#\{n \le x : \Omega(F(n)) \le 4rd\} \gg_F \frac{x}{(\log x)^r}.$$

Hint. If G is an irreducible polynomial over $\mathbb{Z}[x]$ and

$$\nu_G(d) = \#\{m \in \mathbb{Z}/d\mathbb{Z} : G(m) \equiv 0 \,(\mathrm{mod}\, d)\},$$

then there is a constant c_G such that

$$\sum_{p \le x} \frac{\nu_G(p)}{p} = \log\log x + c_G + O_G\left(\frac{1}{\log x}\right) \quad (x \ge 2).$$

Exercise (6 marks). Prove Theorem 3.2.1:
There are absolute constants $A' > 0$ and $B' > 0$ such that, for every $x \geq 3$ and every integer $r \geq 1$, we have that

$$\#\left\{3 < p \leq x : \omega\left(\frac{p-1}{2}\right) = r\right\} \leq \frac{A'x}{\log^2 x}\frac{(\log\log x + B')^{r-1}}{(r-1)!}$$

Hint. Show the stronger statement that

$$\begin{aligned} S_r(x,k) &:= \#\left\{p \leq x : p \equiv 1 \,(\mathrm{mod}\, 2k),\ \omega\left(\frac{p-1}{2k}\right) = r\right\} \\ &\leq \frac{A'x}{\phi(k)\log^2(x/d)}\frac{(\log\log(x/k) + B')^{r-1}}{(r-1)!}, \end{aligned} \tag{1}$$

uniformly in $r \geq 1$ and $1 \leq k \leq x/2$.

Exercise 3.3.2 (6 marks). Show that, for $1 \leq |s| \leq x$ and $x \geq 3$, we have that

$$\sum_{p \leq x} \tau_3(p+s) \asymp x \log x \prod_{p|s}\left(1 - \frac{1}{p}\right)^2$$

Bonus question (8 marks). Let $1 \leq |s| \leq x$ and $x \geq 3$. Show that under the Elliott-Halberstam conjecture, we have that

$$\sum_{p \leq x} \tau_3(p+s) = C(s)x\log x + O(x).$$

for some appropriate constant $C(s)$.