MAT6684W: Sieve Methods, Fall 2012

Homework assignment #3

The exercise and theorem numbers refer to the course notes

Exercise (5 marks). Prove Theorem 4.3.6 (without appealing to Theorem 4.3.7):
For $1 \le q \le y \le x$ and $(a,q)=1$, we have that

$$\pi(x;q,a) - \pi(x-y;q,a) \le \frac{2y}{\phi(q)\log(2y/q)}\left(1 + O\left(\frac{\log\log(3y/q)}{\log(2y/q)}\right)\right).$$

Exercise 8.0.4 (7 marks). Show that

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log n} \le \frac{7}{8},$$

using the following argument:
Fix $\delta > 0$. Starting from the formula

$$(1+o(1))N\log N = \sum_{k=1}^{\infty} 2k \sum_{\substack{N<n\le 2N \\ p_{n+1}-p_n=2k}} 1 \quad (N\to\infty),$$

show that

$$\sum_{k\le \frac{(1+\delta)\log N}{2}} ((1+\delta)\log N - 2k) \sum_{\substack{N<n\le 2N \\ p_{n+1}-p_n=2k}} 1 \ge (\delta + o(1))N\log N,$$

as $N\to\infty$. Next, use the sieve to bound $\#\{N<n\le 2N : p_{n+1}-p_n = 2k\}$ from above and deduce the desired result.

Exercise (3 marks). Prove Lemma 8.1.1 (refer to the course notes for the definition of the notation):
We have that

$$\sum_{N<n\le 2N}\left(\sum_{d|(P(z),Q(n))}\lambda_d\right)^2 = N\sum_{m|P(z)}\frac{1}{h(m)}\left(\sum_{\substack{d|P(z) \\ d\equiv 0 \,(\mathrm{mod}\, m)}}\frac{\lambda_d \nu(d)}{d}\right)^2 + O_k\left(M^2D(\log D)^{3k-1}\right).$$

Exercise 8.2.3 (10 marks). For $r\in\mathbb{N}$, set

$$\delta_r = \liminf_{n\to\infty}\frac{p_{n+r}-p_n}{\log n}.$$

Show that $\delta_r \le r-1$ for every integer $r\ge 2$. (Note that when $r=1$, this is Theorem 8.0.2.)